PARAMETER SHIFT IN NORMAL GENERALIZED HYPERGEOMETRIC SYSTEMS

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Abstract. We treat the problem of shifting parameters of the generalized hypergeometric systems defined by Gelfand when their associated toric varieties are normal. In this context we define and determine the Bernstein-Sato polynomials for the natural morphisms of shifting parameters. We also give some examples.

Let \( A = \{ \chi_1, \ldots, \chi_N \} \subset \mathbb{Z}^n \) be a finite subset with certain properties. In [G], [GGZ], [GZK1], [GZK2], [GKZ] and so on, Gelfand and his collaborators defined and studied generalized hypergeometric systems \( M_\alpha \) associated to \( A \) with parameter \( \alpha \). Aomoto defined and studied a broader class of systems (cf. [A1]-[A4]). Generalized hypergeometric systems of this kind were also defined in [KKM] and [H], where they were named canonical systems. For \( 1 \leq j \leq N \), there exists a natural morphism \( f_{x_j}: M_{x_j} \rightarrow M_\alpha \), which corresponds to the differentiation of solutions. In this paper, we treat the problem of determining when \( f_{x_j} \) becomes isomorphic under the condition that a certain associated affine toric variety is normal.

In §1 and §2, we define the system \( M_\alpha \) and the natural morphism \( f_{x_j} \), and give a necessary condition (Theorem 2.3) for the morphism \( f_{x_j} \) to be an isomorphism. In §3, we introduce an assumption, which we call the normality and keep throughout this paper. In §4, §5, and §6, we define an ideal \( B(\chi_j) \) of the \( b \)-functions for the morphism \( f_{x_j} \), and obtain a sufficient condition in terms of the \( b \)-functions (Corollary 5.4) for the morphism \( f_{x_j} \) to be isomorphic. The ideal \( B(\chi_j) \) turns out to be singly generated by a certain polynomial (Theorem 6.4). In §7, some examples are given.

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1. Generalized hypergeometric systems. First of all, we recall the definition of generalized hypergeometric systems following Gelfand et al. (cf. [GGZ]). Suppose we are given \( N \) integral vectors \( \chi_j = (\chi_{1j}, \ldots, \chi_{nj}) \in \mathbb{Z}^n \) \( (j = 1, \ldots, N) \) satisfying two conditions.

(1) The vectors \( \chi_1, \ldots, \chi_N \) generate the lattice \( \mathbb{Z}^n \).
(2) All the vectors $\chi_j$ lie on some affine hyperplane $\sum_{i=1}^n c_i x_i = 1$ in $\mathbb{R}^n$, where $c_i \in \mathbb{Z}$.

We denote by $L$ the subgroup in $\mathbb{Z}^n$ consisting of those $a=(a_j)_{j=1}^N$ satisfying $\sum_{j=1}^N a_j x_j = 0$. Let $(v_1, \ldots, v_N)$ be a coordinate system on $V = \mathbb{C}^N$. Let $W = W_V$ denote the Weyl algebra on $V$, i.e.,

$$W = W_V = C[v_1, \ldots, v_N, D_1, \ldots, D_N]$$

where $D_j = \partial/\partial v_j$ for $j = 1, \ldots, N$. We put for $a \in L$

$$\Box_a = \prod_{a_j > 0} D_j^{a_j} - \prod_{a_j < 0} D_j^{-a_j}.$$ 

For a parameter $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ we define a generalized hypergeometric system $M_\alpha$ on $V$ as a $W$-module to be $W$ modulo the left $W$-module generated by $\prod_\alpha (aeL)$, i.e.,

$$M_\alpha = W\left/ \left( \sum_{i=1}^n W \left( \sum_{j=1}^n \chi_\alpha \theta_j - \chi_i \right) + \sum_{a \in L} W \Box_a \right) \right..$$

Here $\theta_j = v_j D_j$ for $j = 1, \ldots, N$, and $\sum_{a \in L} W \Box_a$ denotes the left $W$-submodule of $W$ consisting of all sums $\sum_{a \in L} w_a \Box a$ with $w_a \in W$ such that only finitely many $w_a$ are not zero. We denote by $Q$ the Newton polyhedron, i.e., $Q$ is the convex hull in $\mathbb{R}^n$ of the points $\chi_1, \ldots, \chi_N$, by $\Lambda$ the semigroup $\mathbb{Z}_0 \chi_1 + \cdots + \mathbb{Z}_0 \chi_N$, and by $R$ the semigroup ring $C[\Lambda]$ regarded as a $\mathbb{Z}^n$-graded ring in an obvious way.

2. Saturated subsets. We now define saturated subsets of $\{1, \ldots, N\}$, which later turn out to correspond to faces of the polyhedron $Q$. Here the empty set $\emptyset$ is regarded as a face of the polyhedron $Q$. One might refer to [D] or [O] for the theory of toric varieties.

**Definition.** Let $I$ be a subset of $\{1, \ldots, N\}$. We call $I$ a saturated subset when for any $a \in L$ either $I \cap \{i|a_i \neq 0\} = \emptyset$ or there exist $i, j \in I$ such that $a_i > 0$ and $a_j < 0$.

We can regard $R$ as the quotient of $C[D_1, \ldots, D_N]$ by the $C[D_1, \ldots, D_n]$-submodule generated by $\Box_a$ ($a \in L$). Let $R_\lambda$ ($\lambda \in \Lambda$) denote the subspace of $R$ generated by the image of $D_1^{b_1} \cdots D_N^{b_N}$ with $b_j \in \mathbb{Z}_0$ ($1 \leq j \leq N$) satisfying $\lambda = \sum_{j=1}^N b_j \chi_j$. Then we have

$$R = C[D_1, \ldots, D_N] / \sum_{a \in L} C[D_1, \ldots, D_N] \Box_a = \bigoplus_{\lambda \in \Lambda} R_\lambda.$$ 

Here $\sum_{a \in L} C[D_1, \ldots, D_N] \Box_a$ denotes the ideal of $C[D_1, \ldots, D_N]$ consisting of all sums $\sum_{a \in L} p_a \Box a$ with $p_a \in C[D_1, \ldots, D_N]$ such that only finitely many $p_a$ are not zero. Clearly the images of $D_1^{b_1} \cdots D_N^{b_N}$ and $D_1^{b_1} \cdots D_N^{b_N}$ in $R$ coincide if $\sum_{j=1}^N b_j \chi_j = \sum_{j=1}^N b_j \chi_j$. Hence the subspace $R_\lambda$ of $R$ is one-dimensional. Elements in $R_\lambda$ are said to be
\(\Lambda\)-homogeneous, and the ideals generated by \(\Lambda\)-homogeneous elements are also said to be \(\Lambda\)-homogeneous. For a saturated subset \(I\), we denote by \(P(I)\) the \(\Lambda\)-homogeneous ideal of \(R\) generated by all \(D_i\) for \(i \in I\), where we use the same letter \(D_i\) for its image in \(R\).

**Lemma 2.1.** \(\{P(I) \mid I \text{ is saturated}\}\) is the set of \(\Lambda\)-homogeneous prime ideals of \(R\).

**Proof.** We first prove that \(P(I)\) is prime. Since \(\dim R_\lambda = 1\) for all \(\lambda \in \Lambda\), it is enough to show that \(m_2 \in P(I)\) if \(m_1 \notin P(I)\) and \(m = m_1 m_2 \in P(I)\) for two monomials \(m_1, m_2\). Set \(m_1 = \prod_{j=1}^{N} D_i^{j_{1i}}, m_2 = \prod_{j=1}^{N} D_i^{j_{2i}}\) and \(m = \sum_{j=1}^{N} D_i^{j_1j_{2i}}\). Then we have \(\prod_{j=1}^{N} D_i^{j_{2i}} = \prod_{j=1}^{N} D_i^{c_{1i} + c_{2i}}\), and there exists \(i \in I\) such that \(b_i > 0\). Since \(I\) is saturated and \(b_i > 0\), there exists \(i' \in I\) such that \(c_{1i'} + c_{2i'} > 0\). Since \(m_1 \notin P(I)\), we have \(c_{1i'} = 0\). Thus we obtain \(c_{2i'} > 0\) and \(m_2 \in P(I)\).

We next assume \(P\) to be a \(\Lambda\)-homogeneous prime ideal. Denote \(I(P) := \{1 \leq i \leq N \mid D_i \in P\}\). Since \(\dim R_\lambda = 1\) for all \(\lambda \in \Lambda\), the \(\Lambda\)-homogeneous ideal \(P\) is generated by some monomials. Moreover, since \(P\) is prime, we see that \(P\) is generated by \(\{D_i \mid i \in I(P)\}\). For \(i \in I(P)\) and \(\alpha \in L\) such that \(\alpha > 0\), we see that \(\prod_{\alpha_j > 0} D_j^{\alpha_j} \in P\). Since \(\prod_{\alpha_j > 0} D_j^{\alpha_j} = \prod_{\alpha_j < 0} D_j^{-\alpha_j}\) and \(P\) is prime, there exists \(k\) such that \(a_k < 0\) and \(D_k \notin P\). We have thus proved \(I(P)\) to be saturated.

Let \(\Gamma\) be a face of \(Q\). We denote by \(P(\Gamma)\) the ideal of \(R\) generated by all \(D_j\) for \(j \notin \Gamma\).

**Lemma 2.2 (cf. [1]).** \(\{P(\Gamma) \mid \Gamma\text{ is a face of }Q\}\) is the set of \(\Lambda\)-homogeneous prime ideals of \(R\).

As a result, for a saturated subset \(I\), the \(\chi_j\) \((j \notin I)\) span a face of \(Q\). Conversely, for a face \(\Gamma\), \(I(\Gamma) = \{1 \leq j \leq N \mid \chi_j \notin \Gamma\}\) is a saturated subset. In particular, the set of nonempty minimal saturated subsets bijectively corresponds to the set of faces of codimension one. For a face \(\Gamma\) of \(Q\) of codimension one we denote by \(F_\Gamma\) the linear form for the hyperplane spanned by \(\Gamma\) such that the coefficients of \(F_\Gamma\) are integers, that their greatest common divisor is one, and that \(F_\Gamma(\chi) > 0\) for any \(\chi \in \Lambda\).

**Definition.** We call a point \(l = (l_1, \ldots, l_N) \in (\mathbb{Z}_{>0})^N\) a quotient point associated to a saturated subset \(I\) when \(I = \{j \mid l_j \neq 0\}\) and for any \(a \in L\) either \(I \cap \{i \mid a_i \neq 0\} = \emptyset\) or there exist \(i, j \in I\) such that \(0 < l_i \leq a_i\) and \(0 > -l_j \geq a_j\).

For \(\chi = \sum_{j=1}^{N} b_j \chi_j\) such that each \(b_j\) is a nonnegative integer, we denote by \(D^\chi\) the operator \(\prod_{j=1}^{N} D_i^{b_j}\). Since \(\sum_{j=1}^{N} \chi_i \theta_j - a_i)D^\chi = D^\chi(\sum_{j=1}^{N} \chi_i \theta_j - a_i - \sum_{j=1}^{N} b_j \theta_j)\), we have a natural morphism \(f_\chi : M_\alpha \to M_\alpha\) by multiplying \(D^\chi\) from the right.

**Theorem 2.3.** For \(j_0 \in \{1, \ldots, N\}\), the morphism \(f_{\chi_{j_0}}\) is not isomorphic if there exist a face \(\Gamma\) of codimension \(d\) and a quotient point \(l\) associated to \(I(\Gamma)\) such that \(\Gamma\) does not contain \(\chi_{j_0}\), and \(F_{\Gamma}(\chi) = \sum_{j \in I(\Gamma) - (j_0)} (l_j - 1)F_{\Gamma}(\chi)\) for \(k = 1, \ldots, d\), where \(\Gamma = \Gamma_1 \cap \cdots \cap \Gamma_d\) and the codimension of each \(\Gamma_k\) is one.
PROOF. Suppose that there exist a face \( \Gamma = \Gamma_1 \cap \cdots \cap \Gamma_d \) and a quotient point \( l \) associated to \( I(\Gamma) \supseteq J_0 \) such that \( F^{\Gamma_k}(x) = \sum_{j \in I(\Gamma) - \{j_0\}} (l_j - 1)F^{\Gamma_k}(x_j) \) for \( k = 1, \ldots, d \). Let \( J \) be the complement of \( I(\Gamma) \). Let \( C^{I(\Gamma)} = \{ (v_i) \mid i \in I(\Gamma) \} \), \( C^l = \{ (v_j) \mid j \in J \} \) and \( L_j := \{ a \in L \mid a_i = 0 \text{ for all } i \in I(\Gamma) \} \). Consider the quotient

\[
M' = \text{Coker}(f_{x_l}) \left/ \left( \sum_{j \in I(\Gamma) - \{j_0\}} W_{\chi_j} D_j^l + \sum_{j \in I(\Gamma) - \{j_0\}} W_{\chi_j} (l_j - 1) \right) \right.
\]

\[
= W_{\chi_l} \left/ \left( W_{\chi_j} D_{j_0} + \sum_{i=1}^n W_{\chi_j} \left( \sum_{j=1}^N \chi_{ij} \theta_j - \alpha_i \right) + \sum_{a \in L_j} W_{\chi_j} D_j^l \right. \right.
\]

\[
+ \sum_{j \in I(\Gamma) - \{j_0\}} W_{\chi_j} (l_j - 1) + \sum_{a \in L_j} W_{\chi_j} \right. \right.
\]

\[
= W_{C^l} \left/ \left( \sum_{i=1}^n W_{C^l} \sum_{j \in J} (\chi_{ij} \theta_j - \beta_i) + \sum_{a \in L_j} W_{C^l} \right) \right. \right.
\]

\[
\left( W_{C^{I(\Gamma)}} D_{j_0} + \sum_{j \in I(\Gamma) - \{j_0\}} W_{C^{I(\Gamma)}} D_j^l + \sum_{j \in I(\Gamma) - \{j_0\}} W_{C^{I(\Gamma)}} (l_j - 1) \right),
\]

where \( \beta_i = \alpha_i - \sum_{j \in I(\Gamma) - \{j_0\}} (l_j - 1) \chi_{ij} \). We have \( F^{\Gamma_k}(\beta) = 0 \) for any \( k \) and the module

\[
W_{C^l} \left/ \left( \sum_{i=1}^n W_{C^l} \sum_{j \in J} (\chi_{ij} \theta_j - \beta_j) + \sum_{a \in L_j} W_{C^l} \right) \right. \right.
\]

is a generalized hypergeometric system on \( C^l \) with respect to \( \chi_j \) \((j \in J)\).

Furthermore, the module

\[
W_{C^{I(\Gamma)}} \left/ \left( W_{C^{I(\Gamma)}} D_{j_0} + \sum_{j \in I(\Gamma) - \{j_0\}} W_{C^{I(\Gamma)}} D_j^l + \sum_{j \in I(\Gamma) - \{j_0\}} W_{C^{I(\Gamma)}} (l_j - 1) \right) \right.
\]

\[
= W_{C^{I(\Gamma)}} \prod_{j \in I(\Gamma) - \{j_0\}} v_j^{l_j - 1} = C[v_i]_{i \in I(\Gamma)}
\]

is not zero. We thus deduce that \( M' \), hence accordingly \( \text{Coker}(f_{x_l}) \) is not zero.

3. Normality assumption. For a \( \mathbb{Z}^n \)-graded \( R \)-module \( M \) we define a subset \( \Lambda(M) \subset \mathbb{Z}^n \) by \( \Lambda(M) := \{ \lambda \in \mathbb{Z}^n \mid M_{\lambda} \neq 0 \} \), when \( M = \bigoplus_{\lambda \in \mathbb{Z}^n} M_{\lambda} \). Since we have

\[
R_{\geq 0} \chi_1 + \cdots + R_{\geq 0} \chi_n = \bigcap_{\ell} \{ \chi \in \mathbb{R}^n \mid F_\ell(\chi) \geq 0 \},
\]
where \( \Gamma \) runs through the faces of codimension one, the following is the normality condition, i.e., the condition for the ring \( R \) to be normal (see, e.g., [S1]).

**Normality Condition.**

\[
\bigcap_{\Gamma} \{ \chi \in \mathbb{R}^n \mid F_{\Gamma}(\chi) \geq 0 \} \cap \mathbb{Z}^n = \Lambda,
\]

where \( \Gamma \) runs through the faces of codimension one. From now on, we always assume the normality.

**Lemma 3.1.** Let \( \chi_0 \in \Lambda \), and let \( (D^{x_0}) \) be the ideal of \( R \) generated by \( D^{x_0} \). Then we have

\[
\Lambda((D^{x_0})) = \mathbb{Z}^n \cap \bigcap_{\Gamma} \{ \chi \in \mathbb{R}^n \mid F_{\Gamma}(\chi) \geq F_{\Gamma}(\chi_0) \}.
\]

**Proof.** Suppose that \( \chi \in \mathbb{Z}^n \) and \( F_{\Gamma}(\chi) \geq F_{\Gamma}(\chi_0) \) for any \( \Gamma \) of codimension one. Let \( \chi' = \chi - \chi_0 \in \mathbb{Z}^n \). Then we have \( F_{\Gamma}(\chi') \geq 0 \) for any \( \Gamma \). By the normality we see that \( \chi' \in \Lambda \). Therefore \( \chi \in \chi_0 + \Lambda = \Lambda((D^{x_0})) \). The other inclusion is clear.

**4. Decomposition of ideals.** Let \( (\Gamma, \chi_0) \) be a pair of a face \( \Gamma \) of codimension one and \( \chi_0 \in \Lambda \). To such a pair \( (\Gamma, \chi_0) \) we associate an ideal \( D(\Gamma, \chi_0) \) of \( R \) defined as the one generated by all \( \Gamma > \chi_0 \) such that \( F_{\Gamma}(\chi) \geq F_{\Gamma}(\chi_0) \).

**Proposition 4.1.** We have the following decomposition of the ideal \( (D^{x_0}) \):

\[
(D^{x_0}) = \bigcap_{\Gamma} D(\Gamma, \chi_0).
\]

**Proof.** Since \( D^{x_0} \) belongs to \( D(\Gamma, \chi_0) \) for any pair \( (\Gamma, \chi_0) \), it is clear that \( (D^{x_0}) \) is contained in the intersection \( \bigcap_{\Gamma} D(\Gamma, \chi_0) \). In order to show the other inclusion, it is enough to verify that the intersection \( \bigcap_{\Gamma} \Lambda(D(\Gamma, \chi_0)) \) is a subset of \( \Lambda((D^{x_0})) \). Suppose that \( \chi \in \mathbb{Z}^n \) does not belong to \( \Lambda((D^{x_0})) \). By Lemma 3.1 there exists a face \( \Gamma \) of codimension one such that \( F_{\Gamma}(\chi) < F_{\Gamma}(\chi_0) \). By the definition of the ideal \( D(\Gamma, \chi_0) \) we see that \( \chi \) does not belong to \( \Lambda(D(\Gamma, \chi_0)) \).

Let \( \Gamma' \) denote the left ideal of \( W \) generated by all \( \mathbb{R}^n \), \( \Gamma'(\chi_0) \) the one generated by \( \Gamma' \) and \( D^{x_0} \), and \( \Gamma'(\chi_0) \) the one generated by \( \Gamma' \) and all \( \prod_{b_j \geq 0} D^{b_j} \) such that \( \sum_{b_j \geq 0} F_{\Gamma}(\chi_j) \geq F_{\Gamma}(\chi_0) \). For a left ideal \( J \) of \( W \) we denote by \( \overline{J} \) the graded ideal with respect to the order filtration in \( W \).

**Lemma 4.2.** (1) Let \( J \) be a left ideal of \( W \) generated by homogeneous operators \( P_1, \ldots, P_s \) in \( C[D_1, \ldots, D_N] \). Then the graded ideal \( \overline{J} \) is generated by \( \overline{P_1}, \ldots, \overline{P_s} \) in the graded ring \( \overline{W} \), where \( \overline{P_j} \) is the image of \( P_j \) in \( \overline{W} \) for any \( j \).

(2) Let \( J \) and \( J' \) be two left ideals of the algebra \( W \). Suppose that \( J \subseteq J' \) and
\(J = J\). Then \(J\) coincides with \(J\).

The proof is straightforward.

**Proposition 4.3.** We have the following decomposition of the left ideal \(I'(\chi_0)\):

\[
I'(\chi_0) = \bigcap_I I'(\Gamma, \chi_0).
\]

**Proof.** Clearly \(I'(\chi_0)\) is contained in \(\bigcap_I I'(\Gamma, \chi_0)\). We thus have \(I'(\chi_0) = (\bigcap_I I'(\Gamma, \chi_0))^{-1} \subseteq \bigcap_I I'(\Gamma, \chi_0)^{-1}\). By Proposition 4.1 and Lemma 4.2 (1), we see that \((I'(\chi_0))^{-1} = \bigcap_I (I'(\Gamma, \chi_0))^{-1}\) in \(W\). We thus conclude that \(I'(\chi_0) = \bigcap_I I'(\Gamma, \chi_0)\) from Lemma 4.2 (2).

We denote by \(W[s]\) the noncommutative ring \(C[s_1, \ldots, s_n] \otimes C W\), where each \(s_i\) is an indeterminate central element. Let \(I\) be the left ideal of \(W[s]\) generated by \(\sum_{j=1}^N x_i \beta_j - s_i^2 (i = 1, \ldots, n)\) and \(\square_a (a \in L)\). We denote by \(M[s]\) the quotient \(W[s]/I\). Let \(I(\chi_0)\) be the left ideal of \(W[s]\) generated by \(I\) and \(D^{\chi_0}\), and \(I(\Gamma, \chi_0)\) the one generated by \(I\) and all \(\prod b_{j \geq 0} D_{j}^{b_{j}}\) such that \(\sum_{b_{j \geq 0}} b_{j} F_{j}(\chi) \geq F_{j}(\chi_0)\). To \(P = \sum_{c e W[s]}, \text{where } P e W\) and \(c = (c_1, \ldots, c_n) \in (Z_{>0})^n\) is a multi-index, we associate the element \(P' : = \sum_{c} P(\sum_{j=1}^N x_i \beta_j)^{c_1} \cdots (\sum_{j=1}^N x_i \beta_j)^{c_n} e W\).

**Proposition 4.4.** We have the following decomposition of the left ideal \(I(\chi_0)\):

\[
I(\chi_0) = \bigcap_I I(\Gamma, \chi_0).
\]

**Proof.** Clearly \(I(\chi_0)\) is contained in \(\bigcap_I I(\Gamma, \chi_0)\). Suppose that \(P\) belongs to \(\bigcap_I I(\Gamma, \chi_0)\). Since we have \([\sum_{j=1}^N x_i \beta_j \prod b_{j \geq 0} D_{j}^{b_{j}}] = (-\sum_{a_{j \geq 0}^r a_{j} \beta_{j}) \prod b_{j \geq 0} D_{j}^{b_{j}}\) and \([\sum_{j=1}^N x_i \beta_j \square a] = (-\sum_{a_{j \geq 0}^r a_{j} \beta_{j}) \square a, P e I(\Gamma, \chi_0)\) implies that \(P' e I'(\Gamma, \chi_0)\) for any \(\Gamma\). We thus see that \(P'\) belongs to \(I'(\chi_0)\) and accordingly \(P\) to \(I(\chi_0)\).

**5. \(b\)-functions.** Let \(B(\chi_0)\) be the kernel of the natural morphism \(C[s] \rightarrow W[s]/I(\chi_0)\). We call a nonzero element of \(B(\chi_0)\) a \(b\)-function of \(M[s]\) with respect to \(\chi_0\).

**Proposition 5.1.** For a polynomial \(b(s) e B(\chi_0)\) there exists an operator \(Q e W\) such that \(b(s) = QD^{\chi_0}\) in \(M[s]\).

The proof is clear. In the situation of Proposition 5.1, we have \(b(x) = QD^{\chi_0}\) in \(M_\alpha\) for any \(\alpha \in C^n\).

**Lemma 5.2.** For \(d, e \in Z_{\geq 0}\) and any \(1 \leq j \leq N\), we have in \(W\)

\[
D_j^d v_j = \sum_{k=0}^{\min(d, e)} \binom{d}{k} \binom{k-1}{j-1} (e-r)^j D_j^{d-k},
\]

where \(v_j = a_{j, 0}^\alpha\).
and

\[ \sum_{k=0}^{\min(d,e)} \binom{d}{k} \left( \prod_{r=0}^{k-1} (e-r) \right) \left( \prod_{q=0}^{e-k-1} (\theta_j-q) \right) = \prod_{r=0}^{e-1} (\theta_j+d-r). \]

The proof is omitted.

**Proposition 5.3.** Let \( d_1, \ldots, d_N \in \mathbb{Z}_{\geq 0} \), \( Q \in W \), and \( P \in C[\theta_1, \ldots, \theta_N] \). Suppose that we have in \( M[s] \)

\[ QD_1^{d_1} \cdots D_N^{d_N} = P(\theta_1, \ldots, \theta_N). \]

Then we have in \( M[s] \)

\[ D_1^{d_1} \cdots D_N^{d_N} Q = P(\theta_1 + d_1, \ldots, \theta_N + d_N). \]

**Proof.** Let \( e_1, \ldots, e_{2N} \in \mathbb{Z}_{\geq 0} \) satisfy \( \sum_{j=1}^{N} e_j = \sum_{j=1}^{N} (e_{N+j} + d_j) \). Then we have in \( M[s] \)

\[ v_1^{e_1} \cdots v_N^{e_N} D_1^{e_1} \cdots D_N^{e_N} = \prod_{j=1}^{N} \prod_{r_j=0}^{e_j-1} (\theta_j + d_j - r_j). \]

By Lemma 5.2, we see in \( M[s] \)

\[ D_1^{d_1} \cdots D_N^{d_N} v_1^{e_1} \cdots v_N^{e_N} D_1^{e_1} \cdots D_N^{e_N} = \prod_{j=1}^{N} \prod_{r_j=0}^{e_j-1} (\theta_j + d_j - r_j). \]

Since \( Q \) is a linear sum of terms of the form \( v_1^{e_1} \cdots v_N^{e_N} D_1^{e_1} \cdots D_N^{e_N} \) with the relation \( \sum_{j=1}^{N} e_j = \sum_{j=1}^{N} (e_{N+j} + d_j) \), we reach the assertion.

**Corollary 5.4.** Suppose that there exists a polynomial \( b(s) \in B(\chi_0) \) such that \( b(\alpha) \neq 0 \). Then the morphism \( f_{\chi_0} : M_\alpha \to M_\alpha \) is isomorphic.

**Proof.** Let \( \chi_0 = \sum_{j=1}^{N} d_j \chi_j \) with \( d_j \in \mathbb{Z}_{\geq 0} \) \((j = 1, \ldots, N)\). In this case, there exists an operator \( Q \in W \) such that

\[ QD^{\chi_0} = QD_1^{d_1} \cdots D_N^{d_N} = b(s) = b(s_1, \ldots, s_n) = b\left( \sum_{j=1}^{N} \chi_1 \theta_j, \ldots, \sum_{j=1}^{N} \chi_N \theta_j \right) \]

is \( M[s] \). By Proposition 5.3, we see that

\[ D_1^{d_1} \cdots D_N^{d_N} Q = b\left( \sum_{j=1}^{N} \chi_1 \theta_j + d_j, \ldots, \sum_{j=1}^{N} \chi_N \theta_j + d_j \right) = b(s + \chi_0) \]

in \( M[s] \). Hence we obtain \( QD^{\chi_0} = b(\alpha) \neq 0 \) in \( M_\alpha \), and \( D^{\chi_0} Q = b(\alpha - \chi_0 + \chi_0) = b(\alpha) \neq 0 \) in \( M_\alpha \). Therefore the morphism \( f_{\chi_0} \) is bijective.

Let \( B(\Gamma, \chi_0) \) be the kernel of the natural morphism \( C[s] \to W[s]/I(\Gamma, \chi_0) \). Since we have \( I(\chi_0) = \bigcap_{\Gamma} I(\Gamma, \chi_0) \), we obtain:
LEMMA 5.5. 

\[ B(\chi_0) = \bigcap_\Gamma B(\Gamma, \chi_0) . \]

We remark that \( B(\Gamma, \chi_0) = C_\phi \) for \( \chi_0 \in \mathbb{Z}_{\geq 0} \). Suppose that \( \chi_0 \) does not belong to \( \mathbb{Z}_{\geq 0} \). For \( \chi_0 \in \mathbb{Z}_{\geq 0} \), we denote by \( \Theta(\Gamma, \chi_0) \) the ideal of \( C_\phi \) generated by all \( \prod_{j > 0} \theta_j (\theta_j - 1) \cdots (\theta_j - b_j + 1) \) for \( \sum_{j > 0} b_j F_\Gamma(\chi_j) \geq m \). Clearly \( \Theta(\Gamma, F_\Gamma(\chi_0)) \) is contained in \( I(\Gamma, \chi_0) \). For \( \chi_0 \notin \mathbb{Z}_{\geq 0} \), there exists an integer \( c > 0 \) such that \( \chi_j F_\Gamma(\chi_j) \geq m \), and thus \( \theta_j (\theta_j - 1) \cdots (\theta_j - c_j + 1) \) belongs to \( \Theta(\Gamma, \chi_0) \). Consequently, we see that the zero set \( V(\Theta(\Gamma, \chi_0)) \) is a finite set contained in \( \mathbb{Z}_{\geq 0} \), and the multiplicity of \( C_\phi \) at each point of \( V(\Theta(\Gamma, \chi_0)) \) is one. Therefore \( \Theta(\Gamma, \chi_0) \) is a radical ideal. We define a finite subset \( \mathcal{Z}(\Gamma, \chi_0) \) of \( \mathbb{Z}_{\geq 0} \) by

\[ \mathcal{Z}(\Gamma, \chi_0) := \left\{ \sum_{j > 0} v_j F_\Gamma(\chi_j) \in \mathbb{Z}_{\geq 0} \mid v \in V(\Theta(\Gamma, \chi_0)) \right\} . \]

PROPOSITION 5.6. The polynomial \( b(\Gamma, \chi_0) \in C_\phi \) defined by

\[ b(\Gamma, \chi_0) := \prod_{v \in \mathcal{Z}(\Gamma, \chi_0)} (F_\Gamma(s) - z) \]

belongs to \( B(\Gamma, \chi_0) \).

PROOF. We denote by \( b(\theta) \) the polynomial \( \prod_{v \in \mathcal{Z}(\Gamma, \chi_0)} (F_\Gamma(\chi_0) \theta - z) \) in \( C_\phi \). Then we see that \( b(v) = 0 \) for all \( v \in V(\Theta(\Gamma, F_\Gamma(\chi_0))) \). Since \( \Theta(\Gamma, F_\Gamma(\chi_0)) \) is a radical ideal, the polynomial \( b(\theta) \) belongs to \( \Theta(\Gamma, F_\Gamma(\chi_0)) \), in particular, to \( I(\Gamma, \chi_0) \). Since \( b(\Gamma, \chi_0) = b(\theta) \) in \( M_\phi \), we conclude that \( b(\Gamma, \chi_0) \in B(\Gamma, \chi_0) \).

COROLLARY 5.7. We define a polynomial \( b_{\chi_0} \in C_\phi \) by \( b_{\chi_0} := \prod_\Gamma b(\Gamma, \chi_0) \). Then the polynomial \( b_{\chi_0} \) belongs to \( B(\chi_0) \).

The proof is clear.

COROLLARY 5.8. Let \( j_0 \in \{ 1, \ldots, N \} \). Assume that for any \( a \in L \) and any face \( \Gamma \) of codimension one not containing \( j_0 \), we have either \( \sum_{a_j > 0} a_j F_\Gamma(\chi_j) = 0 \) or \( \sum_{a_j > 0} a_j F_\Gamma(\chi_j) \geq F_\Gamma(\chi_{j_0}) \). Then the morphism \( f_{\chi_{j_0}} : M_{\chi_{j_0}} \to M_{\chi} \) is isomorphic if and only if \( b_{\chi_{j_0}}(a) \neq 0 \).

PROOF. Suppose that \( b_{\chi_{j_0}}(a) = 0 \). Then there exists a face \( \Gamma \) of \( Q \) of codimension one not containing \( j_0 \) with \( b(\Gamma, \chi_{j_0})(a) = 0 \). Hence there exists \( v \in Z(\Gamma, F_\Gamma(\chi_{j_0})) \) such that \( F_\Gamma(\chi_{j_0}) = v \). In other words, there exists \( v = (v_j)_{j \in I(\Gamma)} \in V(\Theta(\Gamma, F_\Gamma(\chi_{j_0}))) \) such that \( F_\Gamma(\chi_{j_0}) = \sum_{j \in I(\Gamma)} v_j F_\Gamma(\chi_j) \). Define \( v' = (v'_j)_{j \in I(\Gamma)} = (v'_j)_{j \in I(\Gamma)} \) for \( j \notin I(\Gamma) \). Under the assumption, the condition \( v \in V(\Theta(\Gamma, F_\Gamma(\chi_{j_0}))) \) implies that \( v' \) is a quotient point associated to \( I(\Gamma) \). By Theorem 2.3, the morphism \( f_{\chi_{j_0}} \) is not isomorphic.

When \( b_{\chi_{j_0}}(a) \neq 0 \), the morphism \( f_{\chi_{j_0}} \) is isomorphic by Corollary 5.4 and Corol-
6. The set \( Z(\Gamma, m) \).

**Lemma 6.1.** The set \( Z(\Gamma, m) \) is contained in \( \{0, 1, \ldots, m-1\} \).

**Proof.** We use induction on \( m \). When \( m = 1 \), it is clear that \( \Theta(\Gamma, 1) \) contains \( \theta_i \) for any \( i \in I(\Gamma) \). We thus see that \( V(\Theta(\Gamma, 1)) = \{(0,\ldots,0)\} \) and \( Z(\Gamma, 1) = \{0\} \).

Let \( v^0 \in V(\Theta(\Gamma, m)) \) belong to \( \Theta(\Gamma, m) \). Suppose that \( v^0 \) is not in \( \Theta(\Gamma, m) \). We define \( v' \in V(\Theta(\Gamma, m)) \) by \( v'_i = 0 \) and \( v'_i = v_i \) for all \( i \in I(\Gamma) \). If \( F_\Gamma(\sum_{i \in I(\Gamma)} v^0_i - v'_i) \geq m - v^0_i F_\Gamma(\chi_i) \), then \( F_\Gamma(\sum_{i \in I(\Gamma)} v^0_i - v'_i) \geq m \), and thus \( \theta_{i_0} (\theta_{i_0} - 1) \cdot \ldots \cdot (\theta_{i_0} - v^0_{i_0} + 1) \times \prod_{i \in I(\Gamma) - \{i_0\}} \theta_i (\theta_i - 1) \cdot \ldots \cdot (\theta_i - b_i + 1) \) belongs to \( \Theta(\Gamma, m) \). Hence we obtain \( \prod_{i \in I(\Gamma) - \{i_0\}} v'_i (v'_i - 1) \cdot \ldots \cdot (v'_i - b_i + 1) = 0 \). We thus see that \( v' \in V(\Theta(\Gamma, m)) \) belongs to \( \Theta(\Gamma, m-v^0_i F_\Gamma(\chi_i)) \). By the induction hypothesis, \( \sum_{i \neq i_0} v'_i F_\Gamma(\chi_i) \) belongs to \( \{(0,\ldots,0), (v^0_i F_\Gamma(\chi_i), v^0_i F_\Gamma(\chi_i) + 1,\ldots, m-1)\} \). Therefore the sum \( \sum_{i \neq i_0} v'_i F_\Gamma(\chi_i) \) belongs to \( \{(0,\ldots,0), (v^0_i F_\Gamma(\chi_i), v^0_i F_\Gamma(\chi_i) + 1,\ldots, m-1)\} \).

**Lemma 6.2.** Fix a face \( \Gamma \) of codimension one. Then there exists \( k \in \{1, \ldots, N\} \) such that \( F_\Gamma(\chi_k) = 1 \).

**Proof.** Since the greatest common divisor of the coefficients of \( F_\Gamma \) is one, there exists \( \chi \in \mathbb{Z}^n \) such that \( F_\Gamma(\chi) = 1 \). If necessary, translate \( \chi \) by an element of \( \mathbb{Z}^n \) such that \( F_\Gamma(\chi) = 1 \). By the normality assumption, we conclude that there exists \( k \in \{1, \ldots, N\} \) such that \( F_\Gamma(\chi_k) = 1 \).

**Lemma 6.3.** \( Z(\Gamma, m) = \{0, 1, \ldots, m-1\} \).

**Proof.** Suppose that \( F_\Gamma(\chi_k) = 1 \) and \( j \in \{0, 1, \ldots, m-1\} \). Define \( v \in (\mathbb{Z}^n)_{1(I)} \) by \( v_k = j \) and \( v_i = 0 \) for all \( i \in I(\Gamma) - \{k\} \). Then \( v \in V(\Theta(\Gamma, m)) \). Hence \( j \) belongs to the set \( Z(\Gamma, m) \).

**Theorem 6.4.** The ideal \( B(\chi_0) \) is singly generated by the polynomial \( b_{\chi_0} \).

**Proof.** Let \( \chi \in \mathbb{C}^n \) satisfy \( F_\Gamma(\chi) \notin \mathbb{Z}^n \) for any face \( \Gamma' \) of codimension one different from \( \Gamma \). Suppose that \( F_\Gamma(\chi) = 1 \). Since \( F_\Gamma(\chi_0 - F_\Gamma(\chi_0) \chi_k) = 0 \), we see that \( \chi_0 - F_\Gamma(\chi_0) \chi_k \) belongs to \( Z \Gamma \). Hence the morphism \( f_{\chi_0} : M_\chi - \chi_0 \rightarrow M_\chi \) is isomorphic if and only if \( F_\Gamma(\chi) \neq 0, 1, \ldots, F_\Gamma(\chi_0) - 1 \).

**Remark (cf. [S2]).** When we are given an example explicitly, we can calculate not only the \( b \)-functions but also operators \( Q \) in the notation of Proposition 5.1. This calculation gives us the contiguity relations which generalize the relations of the following type:
(c-a)F(a-1, b; c; x) = \left\{ x(1-x) \frac{d}{dx} - bx + c - a \right\}F(a, b; c; x),

where F is the classical hypergeometric function.

7. Examples. All of the following examples satisfy the normality assumption (see [S1]). We denote \( f_j \) (resp. \( b_j \)) instead of \( f_{x_j} \) (resp. \( b_{x_j} \)).

EXAMPLE 1. Let \( V = C^{2p} \), and

\[
M_{ab} = W \left( \sum_{i=1}^{p} W(\theta_i + \theta_{2p} - \alpha_i) + \sum_{i=1}^{p-1} W(\theta_{p+i} - \theta_{2p} - \beta_i) \right).
\]

(1) Let \( 1 \leq i \leq p \). Then \( b_i(\alpha, \beta) = \alpha_i(\alpha_i + \beta_1)(\alpha_i + \beta_2) \cdots (\alpha_i + \beta_{p-1}) \), and \( f_i \) is isomorphic if and only if \( \alpha_i \neq 0, \alpha_i + \beta_1 \neq 0, \ldots, \alpha_i + \beta_{p-1} \neq 0 \).

(2) Let \( 1 \leq i \leq p-1 \). Then \( b_{p+i}(\alpha, \beta) = (\alpha_1 + \beta_i)(\alpha_2 + \beta_i) \cdots (\alpha_p + \beta_i) \), and \( f_{p+i} \) is isomorphic if and only if \( \alpha_1 + \beta_i \neq 0, \ldots, \alpha_p + \beta_i \neq 0 \).

(3) \( b_{2p}(\alpha, \beta) = \alpha_1 \alpha_2 \cdots \alpha_p \) and \( f_{2p} \) is isomorphic if and only if \( \alpha_1 \neq 0, \ldots, \alpha_p \neq 0 \).

EXAMPLE 2. Let \( V = C^{(k+1)l} = \{(v_i) | 1 \leq i \leq l, 0 \leq j \leq k \} \) and

\[
M_{ab} = W \left( \sum_{j=1}^{k} \left( \sum_{i=1}^{l} \theta_{ij} - \alpha_i \right) + \sum_{i=1}^{l} W(\sum_{j=0}^{k} \theta_{ij} - \beta_i) + \sum_{i \neq j, j \neq j'} W(D_{ij}D_{ij'} - D_{ij}D_{ij'}) \right).
\]

We put \( \alpha = \sum_{j=1}^{k} \beta_i - \sum_{j=1}^{k} \alpha_i \). Then \( b_i(\alpha, \beta) = \alpha_i \beta_i \), and \( f_{ij} \) is isomorphic if and only if \( \alpha_i \neq 0 \) and \( \beta_i \neq 0 \).

EXAMPLE 3. Let \( V = C^{n(n-1)/2} = \{(v_{ij}) | 1 \leq i < j \leq n \} \) (\( n \geq 4 \)), and

\[
M_{a} = W \left( \sum_{k=1}^{n} W(\sum_{i=1}^{k-1} \theta_{ik} + \sum_{j=k+1}^{n} \theta_{kj} - \alpha_k) + \sum_{1 \leq i < j < k \leq n} W(D_{ij}D_{kl} - D_{ik}D_{jl}) \right)
+ \sum_{1 \leq i < j < k \leq n} W(D_{ij}D_{jl} - D_{ij}D_{kj}) + \sum_{1 \leq i < j < k \leq n} W(D_{ij}D_{kl} - D_{ik}D_{jk}).
\]

Then \( 2^{n-2} : b_{s}(\alpha) = \alpha_{s} \alpha_{t} \prod_{k \neq s, t} (\sum_{i \neq k} \alpha_i - \alpha_k) \). \( f_{st} \) is isomorphic if and only if \( \alpha_s \neq 0, \alpha_t \neq 0 \) and \( \sum_{i \neq k} \alpha_i - \alpha_k \neq 0 \) for any \( k \neq s, t \).

EXAMPLE 4. Let \( V = C^{n(n+1)/2} = \{(v_{ij}) | 1 \leq i \leq j \leq n \} \) (\( n \geq 2 \)), and

\[
M_{a} = W \left( \sum_{k=1}^{n} W(\sum_{i=1}^{k-1} \theta_{ik} + \sum_{j=k+1}^{n} \theta_{kj} - \alpha_k) + \sum_{1 \leq i < j < k \leq n} W(D_{ij}D_{kk} - D_{ik}D_{jk}) \right)
+ \sum_{1 \leq i < j < k \leq n} W(D_{ij}D_{kk} - D_{ik}D_{jk}) + \sum_{1 \leq i < j < k \leq n} W(D_{ij}D_{kk} - D_{ik}D_{jk}).
\]
\[
\sum_{1 \leq i < j < k \leq n} W(D_{ij}D_{jk} - D_{ik}) + \sum_{1 \leq i < k < l \leq n} W(D_{ik}D_{jl} - D_{jk}D_{il}).
\]

(1) \(b_{ss}(x) = \alpha_s(x_{s-1})\), and \(f_{ss}\) is isomorphic if \(\alpha_x \neq 0, 1\), and not isomorphic if \(\alpha_s = 0\).

(2) \(b_{ss}(x) = \alpha_s \alpha_t\) for \(s < t\), and \(f_{ss}\) is isomorphic if and only if \(\alpha_s, \alpha_t \neq 0\).

**Example 5.** Let \(V = C^{2n-2} = \{(v_i) | i = \pm 1, \pm 2, \ldots, \pm (n-1)\}\) (\(n \geq 4\)) and

\[
M_\alpha = W\left(\sum_{i=1}^{n-1} W(\theta_i - \theta_{i-1} - \alpha) + W\left(\sum_{i=1}^{n-1} (\theta_i + \theta_{i-1} - \alpha_n) + W(D_{i}D_{i-1} - D_{j}D_{j-1})\right)\right).
\]

For a subset \(I\) of \(\{1, 2, \ldots, n-1\}\), we denote by \(I'\) the complement of \(I\).

(1) \(2^{n-2}. b_0(\alpha) = \prod_{I \neq I'}(\alpha_n + \sum_{i \in I'} \alpha_i - \sum_{i \in I} \alpha_i)\) for \(s > 0, f_s\) (\(s > 0\)) is isomorphic if and only if \(\alpha_n + \sum_{i \in I} \alpha_i - \sum_{i \in I'} \alpha_i \neq 0\) for any \(I \neq I'\).

(2) \(2^{n-2}. b_{-s}(\alpha) = \prod_{I \neq I'}(\alpha_n + \sum_{i \in I'} \alpha_i - \sum_{i \in I} \alpha_i)\) for \(s > 0, f_{-s}\) (\(s > 0\)) is isomorphic if and only if \(\alpha_n + \sum_{i \in I} \alpha_i - \sum_{i \in I'} \alpha_i \neq 0\) for any \(I \neq I'\).

**Example 6.** Let \(V = C^{2n-1} = \{(v_i) | -(n-1) \leq i \leq -(n-1)\}\) (\(n \geq 2\)) and

\[
M_\alpha = W\left(\sum_{i=1}^{n-1} W(\theta_i - \theta_{i-1} - \alpha) + W\left(\sum_{i=1}^{-(n-1)} \theta_i - \alpha_n + \sum_{i=1}^{n-1} W(D_{i}D_{i-1} - D_{j}D_{j-1})\right)\right).
\]

As in Example 5, \(I'\) denotes the complement of \(I\) in \(\{1, 2, \ldots, n-1\}\).

(1) \(b_0(\alpha) = \prod_{I} (\alpha_n + \sum_{i \in I} \alpha_i - \sum_{i \in I'} \alpha_i)\), and \(f_0\) is isomorphic if and only if \(\alpha_n + \sum_{i \in I} \alpha_i - \sum_{i \in I'} \alpha_i \neq 0\) for any subset \(I\) of \(\{1, \ldots, n-1\}\).

(2) \(b_0(\alpha) = \prod_{I \neq I'}(\alpha_n + \sum_{i \in I'} \alpha_i - \sum_{i \in I} \alpha_i)\) for \(s > 0, f_s\) (\(s > 0\)) is isomorphic if and only if \(\alpha_n + \sum_{i \in I} \alpha_i - \sum_{i \in I'} \alpha_i \neq 0\) for any \(I \neq I'\).

(3) \(b_{-s}(\alpha) = \prod_{I \neq I'}(\alpha_n + \sum_{i \in I'} \alpha_i - \sum_{i \in I} \alpha_i)\) for \(s > 0, f_{-s}\) (\(s > 0\)) is isomorphic if and only if \(\alpha_n + \sum_{i \in I} \alpha_i - \sum_{i \in I'} \alpha_i \neq 0\) for any \(I \neq I'\).

**References**


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