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GROUPS GRADED BY FINITE ROOT SYSTEMS[1]

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Abstract. A Steinberg group $\mathrm{St}(\Delta, R)$ is defined by the data of a ring R and a root system Δ. This paper aims to study the relationship between the group-theoretic structure of a Steinberg group and the associated ring. We introduce graded groups which are groups satisfying some axioms that are basic properties of $\mathrm{St}(\Delta, R)$, and then show that these properties suffice to determine the structures of graded groups, *by constructing a ring out of a graded group*. Also the central extensions of graded groups are studied.

Introduction. In this paper, the groups graded by finite root systems Δ, or Δ-graded groups, are introduced. These are analogues of Lie algebras graded by finite root systems which are studied by Berman and Moody [1]. The background is the structures of Steinberg groups and Chevalley groups. The connection among Δ-graded groups, Steinberg groups and central extensions can be seen throughout the article.

Assume that our rings are always associative and with the identity element denoted by 1. For each $l \geq 1$, all $(l+1)\times(l+1)$ invertible matrices over R form the general linear group $GL_{l+1}(R)$. Let E_{ij} be the (i, j) matrix unit of $GL_{l+1}(R)$. Then the elementary group $E_{l+1}(R)$, the subgroup of $GL_{l+1}(R)$ generated by $I+rE_{ij}$ for $r \in R$ and $i \neq j$, models the definition of the Steinberg group $\mathrm{St}(A_l, R)$, where A_l is a type of root systems. Both $\mathrm{St}(A_l, R)$ and E_{l+1} can be assigned a grading by the root system of Type A_l in terms of the group commutators. Now the question is: without given a ring in advance, would the graded property will determine the structure of such a group? This motivates our definition for a Δ-graded group (cf. Definition (2.1)), where we assume that the root system Δ is always one of the types A_l, $l \geq 3$, D_l, $l \geq 4$ and E_l, $l=6, 7, 8$, unless otherwise stated. We have:

(2.3) THEOREM. *Let G be a group graded by Δ. Then there is an associative ring R with* 1, *such that G is a homomorphic image of the Steinberg group* $\mathrm{St}(\Delta, R)$. *Moreover, R is commutative if Δ is of Type D_l or E_l.*

Note that here all associative rings fit in here. For the proof, the critical point is to define the ring R out of such a group. The main theme of the proof is set in [1] on the Lie algebra level.

Then for each Δ-graded group, we may attach a ring R. A Δ-homomorphism of Δ-graded groups is naturally understood to be a group homomorphism which preserves

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the Δ-grading. So in the category of Δ-graded groups, the morphisms involved are Δ-homomorphisms.

Considering the central extensions of groups, we have:

(3.2) THEOREM. *Let Δ be of Type A_l, $l \geq 4$, E_l, $l=6, 7, 8$ or D_l, $l \geq 5$. Any covering (U, ψ) of a Δ-graded group G is also Δ-graded and ψ is a Δ-homomorphism. Moreover, there is a surjective homomorphism Ψ from* $\mathrm{St}(\Delta, R)$ *onto U such that*

$$\begin{tikzcd} \mathrm{St}(\Delta, R) \arrow[r, "\Psi"] \arrow[dr] & U \arrow[d, "\psi"] \\ & G \end{tikzcd}$$

is commutative, where R is the ring attached to G.

(3.3) THEOREM. *Let Δ be of Type A_l, $l \geq 4$; D_l, $l \geq 5$; or E_l, $l=6, 7, 8$. Let G and G' be perfect and G Δ-graded. If there is a group which is a covering for both G and G', then G' is also Δ-graded in such a way that G and G' are Δ-homomorphic images of the same Steinberg group* $\mathrm{St}(\Delta, R)$.

The paper is organized as follows. In §1, we present some preliminary notation and define a set $\mathfrak{S}$ whose elements act as a model for graded groups. Then we show that for any element $(\dot{G}, \dot{\phi}) \in \mathfrak{S}$, the Weyl group of Δ is a subquotient group $\dot{G}$. In §2, we define groups graded by finite root systems and prove Theorem (2.3). We show Theorems (3.2) and (3.3) in §3.

CONVENTIONS. In a group G, write $a^b := bab^{-1}$ and the commutator $(a, b) := aba^{-1}b^{-1}$, and denote by $\mathrm{Int}\, b$ the conjugation by b, i.e. $\mathrm{Int}\, b.a := a^b$. Write $H < G$ if H is a subgroup of G. $\langle \cdots \rangle$ means a (sub)group generated by $\cdots$.

The following formulas on commutators will be used later on.

(0.1) $(a, b) = (b, a)^{-1}$

(0.2) $(ab, c) = (b, c)^a (a, c)$

$(a, bc) = (a, b)(a, c)^b$

(0.3) $(ab, cd) = (b, c)^a (b, d)^{ac} (a, c)(a, d)^c$

(0.4) $(a^c, (b, c))(c^b, (a, b))(b^a, (c, a)) = 1$

(0.5) $(a, (b, c)) = ((a, b), c)$, if $(a, c) = 1$, $((a, b), (b, c)) = 1$, and $((b, c), c) = 1$.

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1. Preliminaries. In §1 and §2, we assume that Δ is a finite indecomposable simply-laced root system of rank $l \geq 3$, i.e. Δ is of Type A_l, $l \geq 3$, D_l, $l \geq 4$, E_6, E_7, or E_8. Let Q be the root lattice spanned by Δ. The Weyl group invariant bilinear form on Q, so normalized that $(\alpha|\alpha)=2$ for all $\alpha \in \Delta$, will be denoted by $(\cdot|\cdot)$. This form is positive definite. If $\alpha, \beta \in \Delta$, then $(\alpha|\beta)$ takes the values ± 2, ± 1 and 0, respectively, if and only if $\alpha = \pm\beta$, $\alpha \mp \beta \in \Delta$ and $\alpha \pm \beta \notin \Delta \cup \{0\}$, respectively. For each root $\alpha \in \Delta$, the reflection in α is the linear map r_α: $\lambda \mapsto \lambda - (\lambda|\alpha)\alpha$ on Q. Then the Weyl group, denoted by W, is generated by all the reflections r_α. In particular, W is generated by all the simple reflections r_{α_i} where $\Pi := \{\alpha_1, \ldots, \alpha_l\}$ is a base for the root system Δ.

Let $\mathfrak{g}$ be the simple Lie algebra over the complex field $\boldsymbol{C}$ associated with Δ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Then

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}^\alpha ,$$

where $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \,|\, [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h} = \mathfrak{g}^0\}$ and $\mathfrak{g}^\alpha \neq (0)$ if and only if $\alpha \in \Delta \cup \{0\}$.

Let $\{E_\alpha, H_i \colon \alpha \in \Delta,\ i=1, \ldots, l\}$ be a Chevalley basis of $\mathfrak{g}$ (see [8, §1]). If $\alpha, \beta, \alpha+\beta \in \Delta$, then $[E_\alpha, E_\beta] = c_{\alpha,\beta} E_{\alpha+\beta}$ for some $c_{\alpha,\beta} \in \{\pm 1\}$. From the skew-symmetry of the Lie bracket $[\cdot,\cdot]$ and the application of the canonical anti-involution of $\mathfrak{g}$, we have formulas

(1.1) $$c_{\alpha,\beta} = -c_{\beta,a} ,$$

(1.2) $$c_{\alpha,\beta} = -c_{-\alpha,-\beta} .$$

We will see and use more formulas about $c_{\alpha,\beta}$'s later on. We fix a choice of a Chevalley basis throughout this paper. In the case $\Delta = A_l$, a Chevalley basis is chosen as in the following example.

(1.3) Example. The description of the root system Δ of Type A_l is

$$\{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i, j \leq l+1\} ,$$

where $\{\varepsilon_1, \ldots, \varepsilon_{l+1}\}$ is an orthonormal basis of $\boldsymbol{R}^{l+1}$. Let $\Pi = \{\alpha_i \colon i=1, \ldots, l\}$ be a base for Δ with $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. The Weyl group is the symmetric group S_{l+1}. The corresponding simple Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}_{l+1}(\boldsymbol{C})$. The set $\{E_{ij}, i \neq j;\ H_i = E_{ii} - E_{i+1,i+1}, i=1, \ldots, l\}$ is a Chevalley basis of $\mathfrak{sl}_{l+1}(\boldsymbol{C})$ where E_{ij} are the standard matrix units.

Now we give Definition (1.4) and Lemma (1.5) which are taken from [1].

(1.4) Definition. An ordered pair $(\beta, \gamma) \in \Delta \times \Delta$ is an *A_2-pair* if $(\beta|\gamma) = -1$. Thus (β, γ) is an A_2-pair if and only if it is a base for an A_2 subroot system of Δ. Two A_2-pairs (β, γ), (β', γ') are *equivalent*, and written $(\beta, \gamma) \sim (\beta', \gamma')$, if there is an element w of the Weyl group W of Δ such that $\beta' = w\beta$, $\gamma' = w\gamma$. The equivalence class of (β, γ) is denoted by $[(\beta, \gamma)]$. Also an ordered triple (α, β, γ) is called an *A_3-triple*, if $\{\alpha, \beta, \gamma\}$ forms a base of an A_3 root system such that $(\alpha|\beta) = (\beta|\gamma) = -1$, and $(\alpha|\gamma) = 0$. We define an ordered quadruple $(\alpha, \beta, \gamma, \delta)$ to be *A_4-quadruple* in a similar way.

(1.5) LEMMA. (i) *If Δ is of Type D_l or E_l then there is only one equivalence class of A_2-pairs.*

(ii) *If Δ is of Type A_l there are exactly two equivalence classes of A_2-pairs, which are (cf. Example (1.3))*

$$[(\alpha_1, \alpha_2)] = \{(\varepsilon_i - \varepsilon_j, \varepsilon_j - \varepsilon_k) \mid i, j, k \text{ distinct}\},$$

$$[(\alpha_2, \alpha_1)] = \{(\varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_i) \mid i, j, k \text{ distinct}\}.$$

We call $[(\alpha_1, \alpha_2)]$ the class of positive A_2-pairs. Furthermore if (β, γ) is an A_2-pair then

$$(\beta, \gamma) \sim (-\gamma, -\beta), \quad (\beta, \gamma) \not\sim (\gamma, \beta).$$

(iii) *In all cases if (β, γ) and (γ, δ) are A_2-pairs with $(\beta \mid \delta) = 0$ then*

$$(\beta, \gamma) \sim (\gamma, \delta) \sim (\beta, \gamma + \delta) \sim (\beta + \gamma, \delta).$$

The unique equivalence class of A_2-pair for Δ of Types D_l, E_l are said to be *positive*.

(1.6) DEFINITION. Assume R is an associative ring with the identity 1. In the cases where the root system Δ is of Type D_l or E_l, R is further assumed to be commutative. The *Steinberg group* is the abstract group with the following presentation:

generators: $\hat{x}_\alpha(r)$; $\alpha \in \Delta$, $r \in R$.

relations:

(R1) $\hat{x}_\alpha(r)\hat{x}_\alpha(s) = \hat{x}_\alpha(r+s)$,

(R2) $(\hat{x}_\alpha(r), \hat{x}_\beta(s)) = \begin{cases} 1, & \text{if } \alpha + \beta \notin \Delta \cup \{0\}, \\ \hat{x}_{\alpha+\beta}(c_{\alpha,\beta} rs), & \text{if } (\alpha, \beta) \text{ is a positive } A_2\text{-pair}, \end{cases}$

where $c_{\alpha,\beta}$ is given by a fixed Chevalley basis.

(1.7) REMARKS. (i) The above definition is the same as that in [4], [5], [7], or [8].

(ii) Although $c_{\alpha,\beta}$ depends on the choice of a Chevalley basis, the Steinberg groups do not (up to isomorphism).

For $\alpha \in \Delta$ and $u \in R^\times$ (the units group of R), let

$$\hat{n}_\alpha(u) := \hat{x}_\alpha(u)\hat{x}_{-\alpha}(-u^{-1})\hat{x}_\alpha(u), \quad \hat{h}_\alpha(u) := \hat{n}_\alpha(u)\hat{n}_\alpha(-1).$$

Then from [4], [5], [7] and [8], we have

(R3) $\hat{n}_\alpha(u)\hat{x}_\beta(r)\hat{n}_\alpha(u)^{-1} = \hat{x}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)} r)$,

(R4) $\hat{n}_\alpha(u)\hat{n}_\beta(v)\hat{n}_\alpha(u)^{-1} = \hat{n}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)} v)$; $n_\alpha(u) = n_{-\alpha}(-u^{-1})$,

(R5) $\hat{n}_\alpha(u)\hat{h}_\beta(v)\hat{n}_\alpha(u)^{-1} = \hat{h}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)} v)\hat{h}_{r_\alpha\beta}(\eta u^{-(\beta|\alpha)})^{-1}$,

(R6) $\hat{h}_\alpha(u)\hat{x}_\beta(r)\hat{h}_\alpha(u)^{-1} = \hat{x}_\beta(u^{(\beta|\alpha)} r)$,

(R7) $\hat{h}_\alpha(u)\hat{n}_\beta(v)\hat{h}_\alpha(u)^{-1} = \hat{n}_\beta(u^{(\beta|\alpha)} v)$; $\hat{h}_\alpha(u)^{-1} = \hat{h}_{-\alpha}(u)$,

(R8) $\hat{h}_\alpha(u)\hat{h}_\beta(v)\hat{h}_\alpha(u)^{-1} = \hat{h}_\beta(u^{(\beta|\alpha)} v)\hat{h}_\beta(u^{(\beta|\alpha)})^{-1}$,

where $u, v \in R^\times$; $r \in R$ and the elements in R appearing commute with each other, and where $\alpha, \beta \in \Delta$ and $\eta = \eta(\alpha, \beta)$ is such that

$$\eta(\alpha, \beta) = \begin{cases} c_{\alpha, \pm\beta}, & \text{if } (\alpha \mid \beta) = \mp 1, \\ -1, & \text{if } (\alpha \mid \beta) = \pm 2, \\ 1, & \text{if } (\alpha \mid \beta) = 0. \end{cases}$$

The following are important subgroups of $\mathrm{St}(\Delta, R)$.

$$\hat{N} := \langle \hat{n}_\alpha(u) \mid u \in R^\times, \alpha \in \Delta \rangle$$

$$\hat{H} := \langle \hat{h}_\alpha(u) \mid u \in R^\times, \alpha \in \Delta \rangle$$

$$\chi^\alpha := \langle \hat{x}_\alpha(r) \mid r \in R \rangle$$

$$\chi^\pm := \chi^\pm(\Delta_+) := \langle \chi^\alpha \mid \alpha \in \Delta_\pm \rangle \quad \text{for a positive system } \Delta_+.$$

Let $\boldsymbol{K}$ be any commutative ring. We consider the Steinberg group $\mathrm{St}(\Delta, \boldsymbol{K})$ in the rest of this section.

Take a pair (G, ϕ) where ϕ is a surjective homomorphism from $\mathrm{St}(\Delta, \boldsymbol{K})$ onto a group G and $\phi|_{\chi^+}$ is one-to-one for some positive system Δ_+. Let $\mathfrak{S} = \mathfrak{S}(\Delta, \boldsymbol{K})$ be the collection of all such pairs.

For $(G, \phi) \in \mathfrak{S}$, denote $\phi(\hat{x}_\alpha(a))$, $\phi(\hat{n}_\alpha(u))$, $\phi(\hat{h}_\alpha(u))$, $\phi(\hat{N})$, $\phi(\hat{H})$ by $x_\alpha(a)$, $n_\alpha(u)$, $h_\alpha(u)$, N, H, respectively. Denote $G^\alpha = \phi(\chi^\alpha)$, $G^\pm = \phi(\chi^\pm)$. From (R3) and (R6), we have

$$n_\alpha(u) G^\beta n_\alpha(u)^{-1} = G^{r_\alpha \beta}. \tag{1.8}$$

$$h_\alpha(u) G^\beta h_\alpha(u)^{-1} = G^\beta. \tag{1.9}$$

(1.10) LEMMA. *Let $(G, \phi) \in \mathfrak{S}$. The restriction of ϕ to $\tilde{\chi}^+ := \chi^+(\tilde{\Delta}_+)$ relative to any positive system $\tilde{\Delta}_+$ is one-to-one.*

PROOF. By definition, there is a positive system Δ_+ of Δ such that ϕ is one-to-one on χ^+ which corresponds to Δ_+. Suppose $\tilde{\Delta}_+$ is another positive system of Δ. We need to show that ϕ is one-to-one on $\tilde{\chi}^+$ which corresponds to $\tilde{\Delta}_+$. Recall that there is an element $w \in W$, such that $w(\Delta_+) = \tilde{\Delta}_+$ (cf. [3]). Take a preimage $\hat{n} \in \hat{N}$ of w. Then $\hat{n}\hat{x}\hat{n}^{-1} \in \tilde{\chi}^+$, for any $\hat{x} \in \chi^+$. Now the lemma follows from the fact that $\phi(\hat{x}) = 1$ if and only if $\phi(\hat{n}\hat{x}\hat{n}^{-1}) = 1$. $\square$

Recall that in the Steinberg group $\mathrm{St}(\Delta, R)$, $\chi^+ := \chi^+(\Delta_+)$ has a unique decomposition $\chi^+ = \prod_{\alpha \in \Delta_+} \chi^\alpha$, for an arbitrarily chosen linear order on Δ_+ and each χ^α is isomorphic to the additive group $(R, +)$ (cf. [4] and [8]). Then in the case $R = \boldsymbol{K}$, these facts can be passed onto G^+ for $(G, \phi) \in \mathfrak{S}$. Since the Weyl group W of Δ is a Coxeter group, the map

$$r_\alpha \mapsto \hat{n}_\alpha(u)\hat{H} \tag{1.11}$$

defines a homomorphism from W onto $\hat{N}/\hat{H}$. Moreover this is an isomorphism. By means of it we will identify these two groups.

(1.12) Lemma. *Let* $(G, \phi)\in\mathfrak{S}$. *If* $\alpha\neq\beta$, *then* $G^{\alpha}\cap G^{\beta}=1$.

Proof. Choose a positive system Δ_+ for Δ for which α is simple. If $\beta\in\Delta_+$, then we are done by Lemma (1.10). If $\beta\in\Delta_-\setminus\{-\alpha\}$, then α, β are in the positive system $r_\alpha(-\Delta_+)$ and we are done too. It remains to show $G^{\alpha}\cap G^{-\alpha}=1$. Since χ^{α} is isomorphic to the additive group $(R, +)$, we have $x_\alpha(r)=1$ if and only if $r=0$. Suppose $x_\alpha(r)=x_{-\alpha}(s)$. Take $\gamma\in\Delta$ with $(\alpha\,|\,\gamma)=-1$. Then $1=(x_{-\alpha}(s), x_\gamma(1))=(x_\alpha(r), x_\gamma(1))=x_{\alpha+\gamma}(c_{\alpha,\gamma}r)$ by (R2), and hence $r=0$, and so $s=0$. □

(1.13) Lemma. *Let* $(G, \phi)\in\mathfrak{S}$, *and we keep the above notation. Then* $H\lhd N$, *and* N/H *is isomorphic to the Weyl group* W.

Proof. Since $\hat{H}\lhd\hat{N}$, we have $H\lhd N$. There is a homomorphism $\psi: \hat{N}/\hat{H}\to N/H$ which factors through the composite map $\hat{N}\xrightarrow{\phi}N\to N/H$. Clearly, ψ is surjective and $\psi(\hat{n}_\alpha(r)\hat{H})=n_\alpha(r)H$. View ψ as the map from W onto N/H. Suppose $\psi(w)=\bar{1}=H$ *for* $w\in W$. Express $w=r_{\beta_1}r_{\beta_2}\cdots r_{\beta_k}$ as a product of reflections. Then $\psi(\hat{n}_{\beta_1}(1)\hat{n}_{\beta_2}(1)\cdots\hat{n}_{\beta_k}(1)\hat{H})=\bar{1}$. So, $h:=n_{\beta_1}(1)n_{\beta_2}(1)\cdots n_{\beta_k}(1)\in H$. And by (1.9) and (1.8),

$$G^{\alpha}=hG^{\alpha}h^{-1}=n_{\beta_1}(1)n_{\beta_2}(1)\cdots n_{\beta_k}(1)G^{\alpha}(n_{\beta_1}(1)n_{\beta_2}(1)\cdots n_{\beta_k}(1))^{-1}=G^{w\alpha}.$$

So $G^{w\alpha}=G^{\alpha}$, for each $\alpha\in\Delta$. Then $w\alpha=\alpha$ by the above lemma. Then $w\doteq 1$, hence ψ is an injection and an isomorphism. □

In the Steinberg group $\mathrm{St}(\Delta, \boldsymbol{K})$, let

$$\hat{N}_0:=\langle\hat{n}_\alpha(1)\,|\,\alpha\in\Delta\rangle, \tag{1.14}$$

$$\hat{H}_0:=\langle\hat{h}_\alpha(-1)\,|\,\alpha\in\Delta\rangle. \tag{1.15}$$

(1.16) Lemma. *Let* $(G, \phi)\in\mathfrak{S}$ *and* $N_0:=\phi(\hat{N}_0)$, $H_0:=\phi(\hat{H}_0)$. *Then* $\hat{N}_0/\hat{H}_0\cong N_0/H_0\cong W$.

Proof. $\hat{H}_0$ and H_0 are normal subgroups of $\hat{N}_0$ and N_0, respectively, by (R5). As in (1.11), $r_\alpha\mapsto\hat{n}_\alpha(u)\hat{H}_0\mapsto n_\alpha(u)H_0$ defines a homomorphism from W onto N_0/H_0. It is an isomorphism by the same proof as that of Lemma (1.13). □

(1.17) Corollary. $\hat{H}_0=\hat{N}_0\cap\hat{H}$.

Proof. By the second isomorphism theorem of groups, we have

$$\hat{N}_0/(\hat{N}_0\cap\hat{H})\cong\hat{N}_0\hat{H}/\hat{H}=\hat{N}/\hat{H}\cong W\cong\hat{N}_0/\hat{H}_0.$$

Since $\hat{H}_0\subseteq\hat{N}_0\cap\hat{H}$, we have $\hat{H}_0=\hat{N}_0\cap\hat{H}$. □

2. Groups graded by finite root systems. Let $\boldsymbol{K}$ be a commutative ring. We maintain all previous notation and terminology for $\mathfrak{S}=\mathfrak{S}(\Delta, \boldsymbol{K})$ and elements in $\mathfrak{S}$ (usually with overdots).

(2.1) DEFINITION. A group G is said to be *graded* by a (finite) root system Δ (of Type A_l, $l \geq 3$; D_l, $l \geq 4$; or E_6, E_7, E_8) or Δ-graded if there are subgroups G^α, for all $\alpha \in \Delta$ and an element $(\dot{G}, \dot{\phi}) \in \mathfrak{S}$ such that

(Gr1) $G = \langle G^\alpha \mid \alpha \in \Delta \rangle$,

(Gr2) $\dot{G}^\alpha \subseteq G^\alpha$, for $\alpha \in \Delta$,

(Gr3) $(G^\alpha, G^\beta) \subseteq \begin{cases} \{1\}, & \text{if } \alpha+\beta \notin \Delta \cup \{0\}, \\ G^{\alpha+\beta}, & \text{if } \alpha+\beta \in \Delta, \end{cases}$

(Gr4) $G^\alpha \cap G^\beta = \{1\}$ if $\alpha \neq \beta$,

(Gr5) $n_\alpha(1) G^\beta n_\alpha(1)^{-1} = G^{r_\alpha \beta}$, for $\alpha, \beta \in \Delta$ with $(\alpha \mid \beta) = -1$,

where $n_\alpha(1) = \dot{\phi}(\hat{n}_\alpha(1)) \in \dot{G}$.

(2.2) EXAMPLE. Let R be an associative ring with the identity 1. When the root system Δ is of Type D_l or E_l, assume further that R is commutative. Then $\mathrm{St}(\Delta, R)$ and the Chevalley group are Δ-graded.

Our main result (see the restatement at the end of this section) is:

(2.3) THEOREM. *Let G be a group graded by Δ. Then there is an associative ring R with* 1, *containing* $\boldsymbol{K}$ *as a subring, such that G is a homomorphic image of the Steinberg group* $\mathrm{St}(\Delta, R)$. *Moreover, R is commutative if Δ is of Type D_l or E_l.*

An outline of the proof is as follows. Fix a root $\alpha \in \Delta$. Then G^α is abelian. Let $R = G^\alpha$, so R has an additive structure. For $r \in R$, write $x_\alpha(r)$ to be the corresponding element in G^α. Elements $x_\beta(r)$ for other roots β, can be defined since G^α and G^β are isomorphic as abelian groups. The multiplication in R comes from the commutator relations (R2) and (Gr3). Such process will make $\{x_\beta(r), \beta \in \Delta, r \in R\}$ satisfy the relations (R1) and (R2). Then (Gr1) makes sure that G is a homomorphic image of $\mathrm{St}(\Delta, R)$.

For each root $\alpha \in \Delta$, take a set A^α having the same cardinality as G^α. Fix a bijective map $\log_\alpha$ from G^α onto A^α. By (Gr3), G^α is abelian, thus A^α carries an additive abelian group structure by making $\log_\alpha$ into an isomorphism. So, for $x, y \in G^\alpha$, $0 := \log_\alpha 1$, $\log_\alpha x + \log_\alpha y := \log_\alpha xy$. Let

$$1_\alpha := \log_\alpha x_\alpha(1),$$

where $x_\alpha(1) = \dot{\phi}(\hat{x}_\alpha(1)) \in \dot{G}^\alpha$.

Set $\dot{N}_0 = \dot{\phi}(\hat{N}_0)$, $\dot{H}_0 = \dot{\phi}(\hat{H}_0)$ (see (1.13) and (1.14)). By Lemma (1.16), the map $n_\alpha(u)\dot{H}_0 \mapsto r_\alpha$, gives an isomorphism from $\dot{N}_0/\dot{H}_0$ onto W. Let π be the composite map from $\dot{N}_0 \to \dot{N}_0/\dot{H}_0 \to W$.

(2.4) LEMMA. *Let $\alpha, \beta \in \Delta$. Then*

(i) $G^{\alpha+\beta} = (G^\alpha, G^\beta)$, *if* $(\alpha \mid \beta) = -1$.

(ii) $nG^\beta n^{-1} = G^{w\beta}$, *where $n \in \dot{N}_0$ with $\pi(n) = w \in W$.*

(iii) $h_\alpha(-1) x h_\alpha(-1)^{-1} = x^{(-1)^{(\alpha \mid \beta)}}$ *for $x \in G^\beta$.*

PROOF. In this proof, (Gr3) is widely used. Suppose $(\alpha \mid \beta) = -1$, i.e. $\alpha + \beta \in \Delta$

and $x\in G^{\beta}$. Since $n_\alpha(1)=n_{-\alpha}(-1)=x_{-\alpha}(-1)x_\alpha(1)x_{-\alpha}(-1)$, so by (Gr3),

$$x^{n_\alpha(1)}=x^{n_{-\alpha}(-1)}=x(x_{-\alpha}(-1),(x_\alpha(1),x))(x_\alpha(1),x)\,.$$

The right hand side belongs to $G_{\alpha+\beta}$ by (Gr5). By (Gr4),

$$x(x_{-\alpha}(-1),(x_\alpha(1),x))=1\,, \tag{2.5}$$

$$x^{n_\alpha(1)}=(x_\alpha(1),x)\,. \tag{2.6}$$

(2.5) implies $x\in(G^{\alpha+\beta},G^{-\alpha})$. This proves (i).

Similarly, we have $x^{n_\alpha(-1)}=x^{n_{-\alpha}(1)}=x(x_{-\alpha}(1),(x_\alpha(-1),x))(x_\alpha(-1),x)$. Applying (Gr3) to this equality, we get

$$x(x_{-\alpha}(1),(x_\alpha(-1),x))=1\,, \tag{2.5$'$}$$

$$x^{n_\alpha(-1)}=(x_\alpha(-1),x)\,. \tag{2.6$'$}$$

So $n_\alpha(-1)G^{\beta}n_\alpha(1)\subset G^{\alpha+\beta}$. Putting this together with (Gr5), we get $n_\alpha(\varepsilon)G^{\beta}n_\alpha(-\varepsilon)=G^{r_\alpha\beta}$ for all $\alpha,\beta\in\Delta$ with $(\alpha|\beta)=\pm1$ and $\varepsilon=\pm1$. Hence to show (ii), it suffices to show $n_\beta(\varepsilon)G^{\beta}n_\beta(-\varepsilon)=G^{-\beta}$ for $\varepsilon=\pm1$. By (i), $G^{\beta}=(G^{\alpha+\beta},G^{-\alpha})$ for some $\alpha\in\Delta$. Then applying the conjugation with respect to $n_\beta(\varepsilon)$ we get the result.

(iii) holds if $(\alpha|\beta)=0$. Suppose $(\alpha|\beta)=-1$ and $x\in G^{\beta}$. So by (2.6′) and (2.5′) $x^{h_\alpha(-1)}=x^{n_\alpha(-1)n_\alpha(-1)}=(x_\alpha(-1),x)^{n_\alpha(-1)}=(x_{-\alpha}(1),x^{n_\alpha(-1)})=(x_{-\alpha}(1),(x_\alpha(-1),x))=x^{-1}$. Since $n_\alpha(\varepsilon)=n_{-\alpha}(-\varepsilon)$ by (R4) and $h_\alpha(\varepsilon)=h_{-\alpha}(\varepsilon)^{-1}$ by (R7) for $\varepsilon=\pm1$, (iii) holds for $(\alpha|\beta)=\pm1$. Finally it suffices to prove $x^{h_\beta(-1)}=x$ for $x\in G^{\beta}$. Take $\alpha\in\Delta$ such that $\alpha+\beta\in\Delta$. Then by (2.5) and the above step, we have

$$\begin{aligned}x^{h_\beta(-1)}&=((x_\alpha(1),x),x_{-\alpha}(-1))^{h_\beta(-1)}=((x_\alpha(1),x)^{-1},x_{-\alpha}(-1)^{-1})\\&=((x_\alpha(1),x),x_{-\alpha}(-1))=x\,,\end{aligned}$$

where the second last equality follows from the identity $(y,z)=(y^{-1},z^{-1})$ for $y\in G^{\gamma}$ and $z\in G^{\delta}$ with $(\gamma|\delta)=-1$. □

For any $\alpha\in\Delta$, let W^{α} be the stabilizer of α in W and $\dot N_0^{\alpha}:=\pi^{-1}(W^{\alpha})$. Then $W^{\alpha}=\langle r_\beta\,|\,\beta\in\Delta,(\beta|\alpha)=0\rangle$ and

$$\dot N_0^{\alpha}=\langle n_\beta(1)\,|\,\beta\in\Delta,(\beta|\alpha)=0\rangle\cdot\dot H_0\,. \tag{2.7}$$

Take $n\in\dot N_0$. If $n=\prod_{i=1}^{k}n_{\beta_i}(\varepsilon_i)$ and $w=\pi(n)$, $\varepsilon_i=\pm1$, then from (R3) we have

$$nx_\alpha(1)n^{-1}=\mathrm{Int}\,n.x_\alpha(1)=x_{w\alpha}(\varepsilon)\qquad\text{for some}\quad\varepsilon=\varepsilon(n,\alpha)\in\{\pm1\}\,. \tag{2.8}$$

Since $x_\beta(r)=1$ in $\dot G$ for $r\in\boldsymbol{K}$ implies $r=0$, ε is uniquely determined by n and α.

For any $\alpha,\beta\in\Delta$, choose an element $n\in\dot N_0$ such that $\pi(n)\alpha=\beta$. Assume that ε is uniquely given by n and α according to (2.8). Define

$$\lambda_{\beta,\alpha}:=\varepsilon^{-1}\log_\beta.\mathrm{Int}\,n.\log_\alpha^{-1}\,. \tag{2.9}$$

Then by Lemma (2.4ii), $\lambda_{\beta,\alpha}$ is an isomorphism from A^α onto A^β such that the diagram

(2.10)
$$\begin{array}{ccc} G^\alpha & \xrightarrow{\operatorname{Int} n} & G^\beta \\ {\scriptstyle \log_\alpha}\downarrow & & \downarrow{\scriptstyle \varepsilon^{-1}\log_\beta} \\ A^\alpha & \xrightarrow{\lambda_{\beta,\alpha}} & A^\beta \end{array}$$

commutes.

(2.11) LEMMA. *For any $\alpha, \beta \in \Delta$, there is a unique isomorphism $\lambda_{\beta,\alpha}$ from A^α to A^β, given by* (2.9). *In other words, $\lambda_{\beta,\alpha}$ is independent of the choice of $n \in \dot{N}_0$.*

PROOF. Let $n' \in \dot{N}_0$ be another element with $\pi(n')\alpha = \beta$, and ε' be determined by n' as in (2.8). Note that $\pi(n^{-1}n')\alpha = \alpha$, and hence $\pi(n^{-1}n') \in W^\alpha$. So, $n^{-1}n' \in \dot{N}_0^\alpha$. Hence by (2.7) there are $n'' \in \langle n_\gamma(1) \mid \gamma \in \Delta, (\gamma \mid \alpha) = 0\rangle$ and $h \in \dot{H}_0$ such that $n^{-1}n' = n''h$. By Lemma (2.4ii), there is $c \in \{\pm 1\}$, satisfying $\operatorname{Int} h.x = x^c$ for any $x \in G^\alpha$. Note that $\operatorname{Int} n'' = 1$ on G^α, since n'' commutes with G^α by (Gr3). Thus for any $r \in A^\alpha$,

$$\begin{aligned} \lambda'_{\beta,\alpha}(r) &:= \varepsilon'^{-1} \log_\beta.\operatorname{Int} n'.\log_\alpha^{-1}(r) = \varepsilon'^{-1} \log_\beta.\operatorname{Int} n.\operatorname{Int} n''.\operatorname{Int} h.\log_\alpha^{-1}(r) \\ &= \varepsilon'^{-1} \log_\beta.\operatorname{Int} n.(\log_\alpha^{-1}(r))^c = \varepsilon'^{-1}\varepsilon c \lambda_{\beta,\alpha}(r)\,. \end{aligned}$$

Now $\lambda'_{\beta,\alpha}(1_\alpha) = \varepsilon'^{-1} \log_\beta.\operatorname{Int} n'.x_\alpha(1) = \log_\beta.x_\beta(1) = 1_\beta$. Also $\lambda_{\beta,\alpha}(1_\alpha) = 1_\beta$. Thus $\varepsilon'^{-1}\varepsilon c = 1$, and hence $\lambda'_{\beta,\alpha} = \lambda_{\beta,\alpha}$. □

Note that the sign ε^{-1} in (2.9) is also uniquely determined by the fact $\lambda_{\beta,\alpha}(1_\alpha) = 1_\beta$. The following corollary is a direct consequence of the above lemma.

(2.12) COROLLARY. (i) $\lambda_{\alpha,\beta} = \lambda_{\beta,\alpha}^{-1}$, (ii) $\lambda_{\gamma,\beta}\lambda_{\beta,\alpha} = \lambda_{\gamma,\alpha}$, (iii) $\lambda_{\alpha,\alpha} = \mathrm{Id}$.

PROOF. The maps on both sides of all three equalities are of the form as in (2.9) with possible different signs. The result follows from the application on 1_α. □

Now let us fix a root $\alpha \in \Delta$. Let $R := A^\alpha$. Since for each $\beta \in \Delta$, the map $t \mapsto x_\beta(t) \in \dot{G}^\beta \subseteq G^\beta$ is an injection from $(\boldsymbol{K}, +)$ into G^β, the map ι_β defined by $t \mapsto x_\beta(t) \mapsto \log_\beta x_\beta(t)$ is an injection from $(\boldsymbol{K}, +)$ into A^β. We identify $\boldsymbol{K}$ with its image inside A^β via ι_β. Then

(2.13)
$$\lambda_{\beta,\alpha}|_{\boldsymbol{K}} = \mathrm{Id}\,.$$

In fact, for any $t \in \boldsymbol{K}$,

$$t \mapsto \iota_\alpha(t) = \log_\alpha x_\alpha(t) \xmapsto{\lambda_{\beta,\alpha}} \varepsilon^{-1} \log_\beta.\operatorname{Int} n.x_\alpha(t) = \varepsilon^{-1} \log_\beta x_\beta(\varepsilon t) = \iota_\beta(t) \longleftarrow\!\!\dashv t\,,$$

since $nx_\alpha(t)n^{-1} = x_\beta(\varepsilon t)$, where $n \in \dot{N}_0$ is so chosen that $\pi(n)\alpha = \beta$ and ε is determined by n in (2.8).

For any $r \in R$, let (for the fixed root α)

(2.14)
$$x_\alpha(r) := \log_\alpha^{-1} r$$

and for any root β,

(2.15)
$$x_\beta(r) := \log_\beta^{-1}(\lambda_{\beta,\alpha}(r)) .$$

Since $\lambda_{\alpha,\alpha} = \mathrm{Id}$, the definition of $x_\beta(r)$ is consistent with the cases where $\beta = \alpha$ and $r \in \boldsymbol{K}$. Then

$$\begin{aligned} x_\beta(r+s) &= \log_\beta^{-1}(\lambda_{\beta,\alpha}(r+s)) = \log_\beta^{-1}(\lambda_{\beta,\alpha}(r) + \lambda_{\beta,\alpha}(s)) \\ &= \log_\beta^{-1}(\lambda_{\beta,\alpha}(r)) \cdot \log_\beta^{-1}(\lambda_{\beta,\alpha}(s)) = x_\beta(r) x_\beta(s) , \end{aligned}$$

that is,

(2.16)
$$x_\beta(r+s) = x_\beta(r) x_\beta(s) .$$

Consequently $x_\beta(-r) = x_\beta(r)^{-1}$.

We are ready to define a multiplication for R. For any given A_2-pair (β, γ), define a multiplication $m_{(\beta,\gamma)}$: $R \times R \to R$ on R by

(2.17)
$$(x_\beta(r), x_\gamma(s)) = x_{\beta+\gamma}(c_{\beta,\gamma} m_{(\beta,\gamma)}(r, s)) .$$

This definition is motivated by (R2). Note that $m_{(\beta,\gamma)}$ restricted to $\boldsymbol{K} \times \boldsymbol{K}$ is the usual multiplication in $\boldsymbol{K}$.

(2.18) LEMMA. *Let (β, γ) be an A_2-pair. Take $n \in \dot{N}_0$ so that $\pi(\beta) = \gamma$. Let $\varepsilon = \varepsilon(n, \beta)$ be determined by n as in* (2.8). *Then for $r \in R$,*

$$n x_\beta(r) n^{-1} = x_\gamma(\varepsilon r) .$$

PROOF. Note both $x_\gamma(\varepsilon r)$ and $n x_\beta(r) n^{-1}$ are in G^γ by (R3). Then we have

$$\begin{aligned} n x_\beta(r) n^{-1} &= \mathrm{Int}\, n . \log_\beta^{-1} . \lambda_{\beta,\alpha}(r) && \text{from (2.15)} \\ &= \log_\gamma^{-1} . \varepsilon \lambda_{\gamma,\beta} . \lambda_{\beta,\alpha}(r) && \text{from (2.9)} \\ &= \log_\gamma^{-1} . \varepsilon \lambda_{\gamma,\alpha}(r) && \text{from (2.12)} \\ &= \log_\gamma^{-1} . \lambda_{\gamma,\alpha}(\varepsilon r) && \\ &= x_\gamma(\varepsilon r) && \text{from (2.15)} . \end{aligned}$$

□

(2.19) LEMMA. *Let (β, γ) be an A_2-pair. Then $m_{(\beta,\gamma)}$ is biadditive, i.e. for $r, s, t \in R$ and $m := m_{(\beta,\gamma)}$, we have $m(r+s, t) = m(r, t) + m(s, t)$ and $m(r, s+t) = m(r, s) + m(r, t)$.*

PROOF. From (2.17), (2.16) and (Gr3), we have

$$\begin{aligned} x_{\beta+\gamma}(c_{\beta,\gamma}m(r+s,t)) &= (x_\beta(r+s), x_\gamma(t)) = (x_\beta(r)x_\beta(s), x_\gamma(t)) \\ &= x_\beta(r)(x_\beta(s), x_\gamma(t))x_\beta(r)^{-1}(x_\beta(r), x_\gamma(t)) \qquad \text{by (0.2)} \\ &= x_\beta(r)x_{\beta+\gamma}(c_{\beta,\gamma}m(s,t))x_\beta(r)^{-1}x_{\beta+\gamma}(c_{\beta,\gamma}m(r,t)) \\ &= x_{\beta+\gamma}(c_{\beta,\gamma}m(r,t)+c_{\beta,\gamma}m(s,t)) . \end{aligned}$$

So $m(r+s,t)=m(r,t)+m(s,t)$ and similarly $m(r,s+t)=m(r,s)+m(r,t)$. □

(2.20) LEMMA. *If (β,γ) and (β',γ') are equivalent A_2-pairs of Δ, then $m_{(\beta,\gamma)}=m_{(\beta',\gamma')}$.*

PROOF. Put $m=m_{(\beta,\gamma)}$ and $m'=m_{(\beta',\gamma')}$. Choose $n\in\dot{N}_0$ so that $\pi(n)=w\in W$ and $w\beta=\beta'$, $w\gamma=\gamma'$. Suppose that

$$\operatorname{Int} n.x_\beta(1)=x_{\beta'}(a), \quad \operatorname{Int} n.x_\gamma(1)=x_{\gamma'}(b), \quad \operatorname{Int} n.x_{\beta+\gamma}(1)=x_{\beta'+\gamma'}(c),$$

where $a=\varepsilon(n,\beta)$, $b=\varepsilon(n,\gamma)$, $c=\varepsilon(n,\beta+\gamma)\in\{\pm 1\}$ (cf. (2.8)). Let $\varepsilon_1=c_{\beta,\gamma}$, $\varepsilon_2=c_{\beta',\gamma'}$. Then by calculating the equality $\operatorname{Int} n.(x_\beta(1), x_\gamma(1))=(\operatorname{Int} n.x_\beta(1), \operatorname{Int} n.x_\gamma(1))$, we have $x_{\beta'+\gamma'}(\varepsilon_1 c)=x_{\beta'+\gamma'}(ab\varepsilon_2)$. Hence

$$\varepsilon_2=\varepsilon_1 ca^{-1}b^{-1} . \tag{2.21}$$

Then

$$\begin{aligned} x_{\beta'+\gamma'}(\varepsilon_2 m'(r,s)) &= (x_{\beta'}(r), x_{\gamma'}(s)) \qquad \text{by the definition of } m' \\ &= (\log_{\beta'}^{-1}.\lambda_{\beta',\alpha}(r), \log_{\gamma'}^{-1}.\lambda_{\gamma',\alpha}(s)) \qquad \text{by (2.15)} \\ &= (\log_{\beta'}^{-1}.\lambda_{\beta',\beta}.\lambda_{\beta,\alpha}(r), \log_{\gamma'}^{-1}\lambda_{\gamma',\gamma}\lambda_{\gamma,\alpha}(s)) \qquad \text{by (2.12)} \\ &= (\operatorname{Int} n.\log_\beta^{-1}(a^{-1}\lambda_{\beta,\alpha}(r)), \operatorname{Int} n.\log_\gamma^{-1}(b^{-1}\lambda_{\gamma,\alpha}(r))) \qquad \text{by (2.9)} \\ &= (\operatorname{Int} n.x_\beta(a^{-1}r), \operatorname{Int} n.x_\gamma(b^{-1}s)) \qquad \text{by (2.15)} \\ &= \operatorname{Int} n.(x_\beta(a^{-1}r), x_\gamma(b^{-1}s)) \\ &= \operatorname{Int} n.x_{\beta+\gamma}(c_{\beta,\gamma}m(a^{-1}r, b^{-1}s)) \qquad \text{by (2.17)} \\ &= x_{\beta'+\gamma'}(c\varepsilon_1 a^{-1}b^{-1}m(r,s)) \qquad \text{by (2.19) and (2.18)} \\ &= x_{\beta'+\gamma'}(\varepsilon_2 m(r,s)) \qquad \text{by (2.21)} . \end{aligned}$$

Then $m'(r,s)=m(r,s)$ and $m_{(\beta,\gamma)}=m_{(\beta',\gamma')}$. □

(2.22) LEMMA. *Let (β,γ) be an A_2-pair. Then (γ,β) defines the opposite multiplication of $m_{(\beta,\gamma)}$. In particular, if Δ is of Type D_l or E_l, then $m_{(\beta,\gamma)}$ is commutative.*

PROOF. Let $m=m_{(\beta,\gamma)}$ and $m'=m_{(\gamma,\beta)}$. For $r,s\in R$,

$$\begin{aligned} x_{\beta+\gamma}(c_{\beta,\gamma}m(r,s)) &= (x_\beta(r), x_\gamma(s)) = (x_\gamma(s), x_\beta(r))^{-1} \\ &= x_{\beta+\gamma}(-c_{\gamma,\beta}m'(s,r)) = x_{\beta+\gamma}(c_{\beta,\gamma}m'(s,r)) , \end{aligned}$$

where we note $-c_{\gamma,\beta}=c_{\beta,\gamma}$ from (1.1). Then $m(r,s)=m'(s,r)$. When Δ is of Type D_l or E_l, there is only one equivalence class of A_2-pairs by Lemma (1.5), hence $m_{(\beta,\gamma)}$ is commutative. □

(2.23) LEMMA. *Let (β,γ) be an A_2-pair. With respect to the multiplication $m=m_{(\beta,\gamma)}$, R is associative and with the unit element $1=1_\alpha$.*

PROOF. Associativity. Since rank $\Delta=l\geq 3$, there is a root δ such that (γ,δ) is an A_2-pair and $(\beta\,|\,\delta)=0$. By Lemma (2.20) and Lemma (1.4), $m=m_{(\beta,\gamma)}=m_{(\beta+\gamma,\delta)}=m_{(\gamma,\delta)}=m_{(\beta,\gamma+\delta)}$. Applying the commutator relation (0.5) to $a=x_\beta(r)$, $b=x_\gamma(s)$, $c=x_\delta(t)$, $r,s,t\in R$ and (Gr3), we have

$$((x_\beta(r),x_\gamma(s)),x_\delta(t))=(x_\beta(r),(x_\gamma(s),x_\delta(t)));$$

$$(x_{\beta+\gamma}(c_{\beta,\gamma}m(r,s)),x_\delta(t))=(x_\beta(r),x_{\gamma+\delta}(c_{\gamma,\delta}m(s,t)));$$

$$x_{\beta+\gamma+\delta}(c_{\beta+\gamma,\delta}(c_{\beta,\gamma}m(r,s),t))=x_{\beta+\gamma+\delta}(c_{\beta,\gamma+\delta}m(r,c_{\gamma,\delta}m(s,t)))\,.$$

By calculating the identity $[[E_\beta,E_\gamma],E_\delta]=[E_\beta,[E_\gamma,E_\delta]]$ (see the definition of Chevalley bases), we get $c_{\beta,\gamma}c_{\beta+\gamma,\delta}=c_{\beta,\gamma+\delta}c_{\gamma,\delta}$. Also m is biadditive by Lemma (2.19). Thus $m(m(r,s),t)=m(r,m(s,t))$, that is, m is associative.

Identity. We show $1=1_\alpha\in \boldsymbol{K}$ is the unit element of R. Take an element $w\in W$ such that $w\beta=\alpha$ (the fixed root). Let $\delta=w\gamma$. Then $m=m_{(\beta,\gamma)}=m_{(\alpha,\delta)}$ and

$$\begin{aligned}
x_{\alpha+\delta}(c_{\alpha,\delta}r)&=n_\alpha(1)x_\delta(r)n_\alpha(-1)\quad\text{by Lemma (2.18)}\\
&=x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)x_\delta(r)x_\alpha(-1)x_{-\alpha}(1)x_\alpha(-1)\\
&=x_\alpha(1)x_{-\alpha}(-1)x_{\alpha+\delta}(c_{\alpha,\delta}m(1,r))x_\delta(r)x_{-\alpha}(1)x_\alpha(-1)\quad\text{by the definition of } m\\
&=x_\alpha(1)(x_{-\alpha}(-1),x_{\alpha+\delta}(c_{\alpha,\delta}m(1,r)))x_{\alpha+\delta}(c_{\alpha,\delta}m(1,r))x_\delta(r)x_\alpha(-1)\quad\text{by (Gr3)}\\
&=x_\alpha(1)x_\delta(-c_{-\alpha,\alpha+\delta}(c_{\alpha,\delta}m(1,m(1,r))))x_{\alpha+\delta}(c_{\alpha,\delta}m(1,r))x_\delta(r)x_\alpha(-1)\,.
\end{aligned}$$

Now by (Gr3), $(G^\alpha,G^{\alpha+\delta})=(G^\delta,G^{\alpha+\delta})=1$, thus bringing the conjugation with respect to $x_\alpha(1)$ to the left hand side the last equality, we get

$$x_\delta(r-c_{-\alpha,\alpha+\delta}(c_{\alpha,\delta}m(1,m(1,r))))=x_{\alpha+\delta}(c_{\alpha,\delta}r-c_{\alpha,\delta}m(1,r))\,.$$

So $r=m(1,r)$ by (Gr4). By considering $x_{\alpha+\delta}(c_{\delta,\alpha}r)=n_\delta(1)x_\alpha(r)n_\delta(-1)$ and using $m_{(\alpha,\delta)}(r,s)=m_{(\delta,\alpha)}(s,r)$, we get $r=m_{(\delta,\alpha)}(1,r)=m(r,1)$. This proves that 1 is the unit element of R with respect to the multiplication m. □

Now we can conclude Theorem (2.3). Here is its restatement.

(2.3) THEOREM. *Let G be a group graded by a finite root system Δ (of Type A_l, $l\geq 3$, D_l, $l\geq 4$ or E_6, E_7, E_8) relative to an element $(\dot{G},\dot{\phi})\in\mathfrak{S}=\mathfrak{S}(\Delta,\boldsymbol{K})$, where $\boldsymbol{K}$ is a commutative ring. Fix any root $\alpha\in\Delta$ and let $R=G^\alpha$ as an abelian group. Relative to a Chevalley basis $\{E_\beta\}_{\beta\in\Delta}\cup\{H_i\}_{i=1}^l$ define the maps $\lambda_{\beta,\alpha}\colon G^\alpha\to G^\beta$ of (2.9) and the elements*

$x_\beta(r)$, $\beta \in \Delta$, $r \in R$ *of* (2.15). *Any positive* A_2*-pair* (β, γ) *define the multiplication in* R *by* (2.17). *Then* R *is an associative ring with* 1, *containing* $\boldsymbol{K}$ *as a subring, and the generators* $x_\beta(r)$*'s satisfy the relations* (R1) *and* (R2) ((2.16) *and* (2.17)). *In particular,* G *is a homomorphic image of the Steinberg group* $\mathrm{St}(\Delta, R)$. *In addition,* R *is commutative if* Δ *is of Type* D_l *or* E_l.

(2.24) REMARK. If we start with $\boldsymbol{K}=\boldsymbol{Z}/n\boldsymbol{Z}$, then any ring could possibly appear here. The chosen R is independent of the choice of the root α up to isomorphism, since $\lambda_{\beta,\alpha}$'s are ring isomorphisms.

3. Central extensions of Δ-graded groups. In this section we study central extensions of Δ-graded groups. Let us recall some notion about central extensions of groups (cf. [8]). A surjective group homomorphism ϕ from U onto G is a *central extension* of G if the kernel is contained in the center of U. A central extension (ϕ, U) of a group G is called a *covering* of G if U is *perfect*, that is, $(U, U)=U$. A central extension (ϕ, U) of a group G is said to be *universal* if it covers all other central extensions of G, i.e. if (ϕ', G'') is any central extension of G, then there is a homomorphism $\bar{\phi}$ from U into G' such that $\phi=\phi'\bar{\phi}$. Any perfect group has a universal central extension which is unique up to isomorphism. Two perfect groups are said to be *centrally isogenous* if they have the same (isomorphic) universal central extension. We will use the previous notation unless otherwise specified.

Let G be a group graded by Δ (of Type A_l, $l \geq 3$, D_l, $l \geq 4$, or E_6, E_7, E_8) with $(\dot{G}, \dot{\phi}) \in \mathfrak{S}=\mathfrak{S}(\Delta, \boldsymbol{K})$. Then there is an associative ring R such that G is a homomorphic image of $\mathrm{St}(\Delta, R)$. R is chosen for a fixed root with its multiplication defined by any fixed positive A_2-pair. We will simply write rs instead of $m(r, s)$ for $r, s \in R$. Now we give the following definition and the main result Theorem (3.2) of this section.

(3.1) DEFINITION. Let $\boldsymbol{K}$ be a commutative ring. Suppose G_1 and G_2 are Δ-graded groups relative to $(\dot{G}_1, \dot{\phi}_1)$ and $(\dot{G}_2, \dot{\phi}_2) \in \mathfrak{S}(\Delta, \boldsymbol{K})$, respectively. A group homomorphism σ from G_1 to G_2 is a *Δ-homomorphism* if $\sigma\dot{\phi}_1=\dot{\phi}_2$.

(3.2) THEOREM. *Let* Δ *be of Type* A_l, $l \geq 4$, E_l, $l=6, 7, 8$ *or* D_l, $l \geq 5$. *Any covering* (U, ψ) *of a* Δ*-graded group* G *is also* Δ*-graded and* ψ *is a* Δ*-homomorphism. Moreover there is a surjective homomorphism* Ψ *from* $\mathrm{St}(\Delta, R)$ *onto* U *such that*

$$\begin{tikzcd} \mathrm{St}(\Delta, R) \arrow[r, "\Psi"] \arrow[dr] & U \arrow[d, "\psi"] \\ & G, \end{tikzcd}$$

where R *is chosen as above for* G.

Our proof will be constructive and the relations (R1) and (R2) play a major role in the proof. Note that G has a set of generators $\{x_\alpha(r) \mid \alpha \in \Delta, r \in R\}$ which satisfies (R1) and (R2). We will define a set of generators $\{\bar{x}_\alpha(r) \mid \alpha \in \Delta, r \in R\}$ in U which satisfies the relations (R1) and (R2). Then U is a homomorphic image of $\mathrm{St}(\Delta, R)$. The idea of this proof is based on showing that $\mathrm{St}(R) \to E(R)$ (the elementary group in the Chevalley group) is a universal central extension (cf. [5], [8], [7]). Technically, [7] has been very helpful. Before going to the proof, we state a consequence. Again we give the proof later.

(3.3) THEOREM. *Let Δ be of Type A_l, $l \geq 4$; D_l, $l \geq 5$; or E_l, $l = 6, 7, 8$. Let G and G' be perfect, and G Δ-graded. If there is a group which is a covering for both G and G', then G' is also Δ-graded in such a way that G and G' are Δ-homomorphic images of the same Steinberg group* $\mathrm{St}(\Delta, R)$. *In particular, if G and G' are centrally isogenous and G is Δ-graded, then G' is also Δ-graded.*

The proof of Theorem (3.2) will be given later as a consequence of a series of preliminary results.

Let (U, ψ) be a covering of a Δ-graded group G and C the kernel of the central extension $\psi: U \to G$. First note that G is perfect. Indeed, for any $\alpha \in \Delta$, there is an A_2-pair (β, γ) such that $\beta + \gamma = \alpha$. Since $x_\alpha(r) = (x_\beta(1), x_\gamma(c_{\beta,\gamma} r))$, we have $G^\alpha = (G^\beta, G^\gamma)$. So $G^\alpha \subseteq (G, G)$. By (Gr1), we have $G = (G, G)$. The perfectness makes sure the existence of a covering.

The following standard lemma, sometimes called the *central trick*, is technically important. It will be used repeatedly.

(3.4) LEMMA (the central trick). *Let $p: H_1 \to H_2$ be a central extension of a group H_2. If $x_1, x_2, y_1, y_2 \in H_1$ so that $px_1 = px_2$, $py_1 = py_2$, then $(x_1, y_1) = (x_2, y_2)$.*

For any $\alpha \in \Delta$, let

$$\tilde{U}^\alpha = \psi^{-1}(G^\alpha), \tag{3.5}$$

and

$$G(\alpha) := \langle G^\beta \mid \beta \in \Delta, (\beta \mid \alpha) \geqslant 0 \rangle < G. \tag{3.6}$$

Then $G(\alpha)$ is contained in the centralizer of G^α.

(3.7) LEMMA. (i) *When Δ is of Type A_l, $l \geq 4$, or E_l, $l \geq 6, 7, 8$ for any two roots α, β with $(\alpha \mid \beta) = 0$, there are $\gamma, \delta \in \Delta$ such that $(\alpha, \gamma, \beta, \delta)$ is an A_4-quadruple.*

(ii) *In D_l, $l \geq 5$, for any two A_2-pairs (β, γ) and (β', γ'), there exists a third A_2-pair (β'', γ'') such that $\{\beta, \gamma, \beta', \gamma'\}$ and $\{\beta', \gamma', \beta'', \gamma''\}$ are contained in some (possibly different) A_{l-1}-subroot systems of D_l.*

PROOF. Examine the explicit constructions of these root systems in [2]. □

(3.8) LEMMA. *Assume that Δ is of Type A_l, $l \geq 4$, or E_l, $l = 6, 7, 8$. Let $\alpha, \beta \in \Delta$*

with $(\alpha \mid \beta) \geq 0$, $\alpha \neq \beta$. *Then* $G^{\beta} \subseteq (G(\alpha), G(\alpha))$.

PROOF. We discuss two cases $(\alpha \mid \beta)=1$ and $(\alpha \mid \beta)=0$. In the first case, take a root γ such that $(\alpha, -\beta, \gamma)$ is an A_3-triple. Without loss of generality, we may assume $\alpha=\varepsilon_1-\varepsilon_2$, $-\beta=\varepsilon_2-\varepsilon_3$, $\gamma=\varepsilon_3-\varepsilon_4$ (cf. Example (1.1)). Then for some $c \in \{\pm 1\}$,

$$x_{\beta}(r)=(x_{\varepsilon_3-\varepsilon_4}(cr), x_{\varepsilon_4-\varepsilon_2}(1)) \in (G(\alpha), G(\alpha)) ,$$

since $(\varepsilon_1-\varepsilon_2 \mid \varepsilon_3-\varepsilon_4)=0$ and $(\varepsilon_1-\varepsilon_2 \mid \varepsilon_4-\varepsilon_2)=1$ are nonnegative. In the second case when $(\alpha \mid \beta)=0$, we may take two roots γ, δ such that $(\alpha, \gamma, \beta, \delta)$ is an A_2-quadruple by Lemma (3.7). Without loss of generality again, we may assume $\alpha=\varepsilon_1-\varepsilon_2$, $\gamma=\varepsilon_2-\varepsilon_3$, $\beta=\varepsilon_3-\varepsilon_4$ and $\delta=\varepsilon_4-\varepsilon_5$. Since $(\varepsilon_1-\varepsilon_2 \mid \varepsilon_3-\varepsilon_5)=0$ and $(\varepsilon_1-\varepsilon_2 \mid \varepsilon_5-\varepsilon_4)=0$ are nonnegative, then for some $c \in \{\pm 1\}$,

$$x_{\beta}(r)=(x_{\varepsilon_3-\varepsilon_5}(cr), x_{\varepsilon_5-\varepsilon_4}(1)) \in (G(\alpha), G(\alpha)) .$$

So the result follows. □

(3.9) LEMMA. *Assume that* Δ *is of Type* A_l, $l \geq 4$, *or* E_l, $l=6, 7, 8$. *Let* $\alpha, \beta \in \Delta$ *and* $\alpha \neq \beta$. *If* $(G^{\alpha}, G^{\beta})=1$, *then* $(\tilde{U}^{\alpha}, \tilde{U}^{\beta})=1$ *(cf.* (3.5)).

PROOF. $(G^{\alpha}, G^{\beta})=1$ implies $(\alpha \mid \beta) \geq 0$. Then $G^{\beta} \subseteq G(\alpha)$. Let $\tilde{x}$, $\tilde{y}$ be arbitrary preimages of $x \in G^{\alpha}$ and $y \in G(\alpha)$, respectively. Then by Lemma (3.4), $(\tilde{x}, \tilde{y})$ depends only on x, y. Furthermore, $(\tilde{x}, \tilde{y})$ is in C since $(\psi(\tilde{x}, \tilde{y}))=(x, y)=1$. Define a map λ_x from $G(\alpha)$ to C by $\lambda_x(y):=(\tilde{x}, \tilde{y})$, where $\tilde{y}$ is any preimage of y. Since C is central, we see from (0.2) that λ_x is a group homomorphism and hence $\lambda_x\{(G(\alpha), G(\alpha))\}=1$. But $G^{\beta} \subseteq (G(\alpha), G(\alpha))$, so $\lambda_x(G^{\beta})=1$. Thus $(\tilde{x}, \tilde{U}^{\beta})=1$. Since $\tilde{x}$ is arbitrary, we have $(\tilde{U}^{\alpha}, \tilde{U}^{\beta})=1$. □

For the generators $x_{\alpha}(r)$, $r \in R$, $\alpha \in \Delta$, let $y_{\alpha}(r) \in U$ be *any* preimage of $x_{\alpha}(r)$. For $\alpha \in \Delta$, choose any two roots β, γ with $\alpha=\beta+\gamma$. Define

$$\bar{x}_{\alpha}(r):=(y_{\beta}(c_{\beta,\gamma}r), y_{\gamma}(1)) . \tag{3.10}$$

So by the central trick

$$\bar{x}_{\alpha}(r)=(\bar{x}_{\beta}(c_{\beta,\gamma}r), \bar{x}_{\gamma}(1)) . \tag{3.11}$$

Let $U^{\alpha}=\{\bar{x}_{\alpha}(r) \mid r \in R\}$.

(3.12) LEMMA. *Let* Δ *be of Type* A_l, $l \geq 4$, E_l, $l=6, 7, 8$ *or* D_l, $l \geq 5$. *Then* $\bar{x}_{\alpha}(r)$ *is independent of the choice and the order of* β, γ.

PROOF FOR A_l OR E_l. Independence of the choice. Suppose $\alpha=\beta'+\gamma'$ is another such representation of α with $\{\beta, \gamma\} \neq \{\beta', \gamma'\}$ (set-theoretically). $1=(\alpha \mid \beta)=(\beta'+\gamma' \mid \beta)=(\beta' \mid \beta)+(\gamma' \mid \beta)$. So, either $(\beta' \mid \beta)=1$, $(\gamma' \mid \beta)=0$, or $(\beta' \mid \beta)=0$, $(\gamma' \mid \beta)=1$. We study these cases separately.

Case 1: $(\beta' \mid \beta)=1$, $(\gamma' \mid \beta)=0$. We may and will apply the commutator formula

(0.5) with $a=y_\beta(\varepsilon r)$, $b=y_{\beta'-\beta}(\eta\varepsilon\varepsilon')$, $c=y_{\gamma'}(1)$ where $\varepsilon=c_{\beta,\gamma}$, $\varepsilon'=c_{\beta',\gamma'}$, $\eta=c_{\beta,\beta'-\beta}$, because we have $(\beta\,|\,\gamma')=0$, $(\beta'\,|\,\beta'-\beta+\gamma)=0$, $(\gamma'\,|\,\beta'-\beta+\gamma)=0$ and Lemma (3.9). So

$$\begin{aligned}(y_{\beta'}(\varepsilon' r), y_{\gamma'}(1)) &= ((y_\beta(\varepsilon r), y_{\beta'-\beta}(\eta\varepsilon\varepsilon')), y_{\gamma'}(1)) \\ &= (y_\beta(\varepsilon r), (y_{\beta'-\beta}(\eta\varepsilon\varepsilon'), y_{\gamma'}(1))) = (y_\beta(\varepsilon r), y_\gamma(\eta\eta'\varepsilon\varepsilon'))\,,\end{aligned}$$

where $\eta'=c_{\gamma-\gamma',\gamma'}$, and the central trick has been applied. Now the following calculation yields that $\eta\eta'\varepsilon\varepsilon'=1$: $c_{\beta',\gamma'}c_{\beta,\beta'-\beta}E_{\beta'+\gamma'}=[[E_\beta, E_{\beta'-\beta}], E_{\gamma'}]=[E_\beta, [E_{\beta'-\beta}, E_{\gamma'}]]=c_{\beta,\gamma}c_{\gamma-\gamma',\gamma'}E_{\beta+\gamma}$. So, $\bar{x}_\alpha(r)$ is independent of the choice of β, γ in this case.

Case 2: $(\beta'\,|\,\beta)=0, (\gamma'\,|\,\beta)=1$. By using Lemma (3.9), and then (0.5) and the central trick, we have for any $r\in R$,

$$\begin{aligned}(y_{\beta'}(\varepsilon' r), y_{\gamma'}(1)) &= ((y_\gamma(-1), y_{\beta'-\gamma}(-\eta\varepsilon' r)), y_{\gamma'}(1)) = (y_\gamma(-1), (y_{\beta'-\gamma}(-\eta\varepsilon' r), y_{\gamma'}(1))) \\ &= (y_\gamma(-1), y_\beta(-\eta\eta'\varepsilon' r)) = (y_\gamma(1)^{-1}, y_\beta(\eta\eta'\varepsilon' r)^{-1}) = (y_\beta(\eta\eta'\varepsilon' r), y_\gamma(1))\,,\end{aligned}$$

where $\varepsilon=c_{\beta,\gamma}$, $\varepsilon'=c_{\beta',\gamma'}$, $\eta=c_{\gamma,\beta'-\gamma}$, $\eta'=c_{\beta,-\gamma',\gamma'}$. Also the Jacobi identity,

$$[[E_\gamma, [E_{\beta'-\gamma}], E_{\gamma'}]=[E_\gamma, [E_{\beta'-\gamma}, E_{\gamma'}]]\,,$$

and (1.1) imply $\varepsilon=-\varepsilon'\eta\eta'$. Then it follows that $\bar{x}_\alpha(r)$ is independent of the choice of β, γ.

We still have to show that $\bar{x}_\alpha(r)$ is independent of the order of β, γ. By examining Example (1.1), we see that in an A_l, $l\geq 3$, there are at least two distinct representations $\alpha=\beta+\gamma=\beta'+\gamma'$. We chose such a pair $\{\beta', \gamma'\}$ not equal to $\{\beta, \gamma\}$ as sets. Then by the independence of choice,

$$\bar{x}_\alpha(r)=(y_\beta(c_{\beta,\gamma}r), y_\gamma(1))=(y_{\beta'}(c_{\beta',\gamma'}r), y_{\gamma'}(1))=(y_\gamma(c_{\gamma,\beta}r), y_\beta(1))\,.$$

This shows that $\bar{x}_\alpha(r)$ is independent of the order of β, γ.

Proof for D_l. For $l\geq 5$, $\Delta=D_l$ contains two subroot systems of Type A_{l-1} $(l-1\geq 4)$, whose union contains a base for Δ and whose intersection is an A_{l-2}-subroot system of Δ.

With this observation, we see from Lemma (3.7) that given two representations $\alpha=\beta'+\gamma'=\beta+\gamma$, we may always find a third distinct representation of $\alpha=\beta''+\gamma''$ such that $\{\beta, \gamma, \beta'', \gamma''\}$ and $\{\beta', \gamma', \beta'', \gamma''\}$ each lie in an A_{l-1}-subroot system $(l-1)\geq 4)$ of Δ. Then we can apply the result for A_{l-1} $(l-1\leq 4)$, and get

$$\bar{x}_\alpha(r)=(y_\beta(c_{\beta,\gamma}r), y_\gamma(1))=(y_{\beta''}(c_{\beta'',\gamma''}r), y_{\gamma''}(1))=(y_{\beta'}(c_{\beta',\gamma'}r), y_{\gamma'}(1))\,.$$

□

(3.13) Lemma. *Let Δ be of Type A_l, $l\geq 4$, E_l, $l=6, 7, 8$ or D_l, $l\geq 5$. The generators $\bar{x}_\alpha(r)$, $r\in R$, $\alpha\in\Delta$ satisfy the relations* (R1) *and* (R2).

Proof for A_l or E_l. Let $\alpha=\beta+\gamma$, α, β, $\gamma\in\Delta$. We use the notation defined in (3.10). Then for $\varepsilon=c_{\beta,\gamma}$,

$$\begin{aligned}\bar{x}_\alpha(r+s) &= (y_\beta(c_{\beta,\gamma}(r+s)),\, y_\gamma(1)) = (y_\beta(c_{\beta,\gamma}r)y_\beta(c_{\beta,\gamma}s),\, y_\gamma(1)) \qquad \text{by the central trick}\\ &= y_\beta(c_{\beta,\gamma}r)(y_\beta(c_{\beta,\gamma}s),\, y_\gamma(1))y_\beta(c_{\beta,\gamma}r)^{-1}(y_\beta(c_{\beta,\gamma}r),\, y_\gamma(1)) \qquad \text{by (0.2)}\\ &= (y_\beta(c_{\beta,\gamma}s),\, y_\gamma(1))(y_\beta(c_{\beta,\gamma}r),\, y_\gamma(1)) = \bar{x}_\alpha(s)\bar{x}_\alpha(r) = \bar{x}_\alpha(r)\bar{x}_\alpha(s)\,.\end{aligned}$$

So (R1) holds for $\bar{x}_\alpha(r)$'s.

If $(\alpha\,|\,\delta)\geq 0$, then by Lemmas (3.8) and (3.9) and the central trick we have $(\bar{x}_\alpha(r), \bar{x}_\delta(s))=1$.

Now we assume $(\alpha\,|\,\delta)=-1$ and (α, δ) is a positive A_2-pair. It remains to show

$$(\bar{x}_\alpha(r),\, \bar{x}_\delta(s)) = \bar{x}_{\alpha+\delta}(c_{\alpha,\delta}rs)\,.$$

Case 1: $\Delta=A_l$. We need to show $(\bar{x}_{ij}(r), \bar{x}_{jk}(s))=\bar{x}_{ik}(rs)$ for i, j, k distinct, where $\bar{x}_{ij}(r):=\bar{x}_{\varepsilon_i-\varepsilon_j}(r)$, etc. (cf. Lemma (1.5)).

Take m not equal to i, j, k. Then applications of the central trick and (0.5) (see (3.11) as well) yield

$$\begin{aligned}(\bar{x}_{ij}(r),\, \bar{x}_{jk}(s)) &= (\bar{x}_{ij}(r),\, (\bar{x}_{jm}(s),\, \bar{x}_{mk}(1))) = ((\bar{x}_{ij}(r),\, \bar{x}_{jm}(s)),\, \bar{x}_{mk}(1)) = (\bar{x}_{im}(rs),\, \bar{x}_{mk}(1))\\ &= (y_{im}(rs),\, y_{mk}(1)) = \bar{x}_{ik}(rs)\,.\end{aligned}$$

Case 2: $\Delta=E_l$. There is only one class of positive A_2-pairs. Choose $\beta'\in\Delta$ so that (α, δ, β') is an A_3-triple. Then $\beta:=\delta+\beta'$ satisfies $(\beta\,|\,\delta)=1$ and $(\beta\,|\,\alpha)=-1$. Hence $\gamma:=\delta-\beta,\ \alpha+\beta\in\Delta$. Thus

$$\begin{aligned}(\bar{x}_\alpha(r),\, \bar{x}_\delta(s)) &= (\bar{x}_\alpha(r),\, (\bar{x}_\beta(\varepsilon s),\, \bar{x}_\gamma(1))) = ((\bar{x}_\alpha(r),\, \bar{x}_\beta(\varepsilon s)),\, \bar{x}_\gamma(1))\\ &= (\bar{x}_{\alpha+\beta}(\varepsilon'\varepsilon rs),\, \bar{x}_\gamma(1)) = \bar{x}_{\alpha+\beta+\gamma}(\eta\varepsilon\varepsilon' rs)\,,\end{aligned}$$

where $\varepsilon=c_{\beta,\gamma}$, $\varepsilon'=c_{\alpha,\beta}$, $\eta=c_{\alpha+\beta,\gamma}$. As before, we have $\eta\varepsilon\varepsilon'=c_{\alpha,\delta}$. So (R2) is satisfied by $\bar{x}_\alpha(r)$'s.

Proof for D_l. We will need an A_4-quadruple (cf. Lemma (3.7)). That is why we assume $l\geq 5$. The relation (R1) follows from the same proof as that for A_l and E_l. For (R2), suppose $\alpha, \delta\in\Delta$, $r, s\in R$.

If $(\alpha\,|\,\delta)=2$, i.e. $\alpha=\delta$, then $(\bar{x}_\alpha(r), \bar{x}_\alpha(s))=1$ from (R1).

If $(\alpha\,|\,\delta)=1$. Lemmas (3.8) and (3.9) hold for α, δ by replacing α, β there. Using $(G^\alpha, G^\delta)=1$, we have $(\bar{x}_\alpha(r), \bar{x}_\delta(s))=1$.

If $(\alpha\,|\,\delta)=-1$, we know that $\{\alpha, \delta\}$ can be imbedded into a subroot system of Type A_{l-1}. Then $(\bar{x}_\alpha(r), \bar{x}_\alpha(s))=\bar{x}_{\alpha+\delta}(c_{\alpha,\delta}rs)$.

Finally, assume $(\alpha\,|\,\delta)=0$. We will use the explicit construction for the root system $\Delta=D_l$, that is,

$$\Delta=\{\pm\varepsilon_i\pm\varepsilon_j\,|\,1\leq i\neq j\leq l\}\,,$$

where $\{\varepsilon_i\}$ is the standard basis of $\boldsymbol{R}^l$ (cf. [2]).

Recall that Δ has only one W-orbit of roots. Without loss of generality, assume $\alpha=\varepsilon_1-\varepsilon_2$. Then δ is one of the following roots.

$$\{\pm(\varepsilon_1+\varepsilon_2),\ \pm\varepsilon_i\pm\varepsilon_j,\ 3\le i\ne j\le l\}\ .$$

We claim that if $\delta\in\{\pm\varepsilon_i\pm\varepsilon_j, 3\le i\ne j\le l\}$, there exist two roots $\beta, \gamma\in\Delta$ such that $(\alpha, \beta, \delta, \gamma)$ is an A_4-quadruple. Indeed, if $\delta=\varepsilon_i+\varepsilon_j$, take $\beta=\varepsilon_2-\varepsilon_i$, $\gamma=\varepsilon_k-\varepsilon_j$; if $\delta=\varepsilon_i-\varepsilon_j$, take $\beta=\varepsilon_2-\varepsilon_i$, $\gamma=\varepsilon_j-\varepsilon_k$; if $\delta=-\varepsilon_i+\varepsilon_j$, take $\beta=\varepsilon_2+\varepsilon_i$, $\gamma=-\varepsilon_j+\varepsilon_k$; if $\delta=-\varepsilon_i-\varepsilon_j$, take $\beta=\varepsilon_2+\varepsilon_i$, $\gamma=\varepsilon_j+\varepsilon_k$ where $3\le i, j, k\le l$ and i, j, k are distinct. Also it is clear from the point of view of the Weyl group, since $W=S_l\rtimes 2^{l-1}$ where S_l is the symmetric group on l letters and 2^{l-1} consists of an even number of sign changes. Then applying the result on A_l $(l\ge 4)$, we get $(\bar{x}_\alpha(r), \bar{x}_\delta(s))=1$ for $\delta\in\{\pm\varepsilon_i\pm\varepsilon_j, 3\le i\ne j\le l\}$ $(\alpha=\varepsilon_1-\varepsilon_2)$.

It remains to show $(\bar{x}_\alpha(r), \bar{x}_\delta(s))=1$ for $\delta=\pm(\varepsilon_1+\varepsilon_2)$ $(\alpha=\varepsilon_1-\varepsilon_2)$. Applying the conjugation with respect to $\bar{n}_\delta(1)$ and the central trick, we need only to prove $(\bar{x}_\alpha(r), \bar{x}_\delta(s))=1$ for $\delta=\varepsilon_1+\varepsilon_2$.

Let $\beta=-\varepsilon_1+\varepsilon_3$. Then (α, β, δ) is an A_3-triple. Note that $(\bar{x}_\alpha(r), \bar{x}_\delta(s))$ is central since $\psi(U^\alpha, U^\delta)=(G^\alpha, G^\delta)=1$. Then

$$\begin{aligned}
(\bar{x}_\alpha(r), \bar{x}_\delta(s))&=(\bar{x}_\alpha(r), \bar{x}_\delta(s))^{\bar{x}_\beta(1)}=((\bar{x}_\beta(1), \bar{x}_\alpha(r))\bar{x}_\alpha(r), (\bar{x}_\beta(1), \bar{x}_\delta(r))\bar{x}_\delta(s))\\
&=(\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r)\bar{x}_\alpha(r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s)\bar{x}_\delta(s))\\
&=(\bar{x}_\alpha(r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s))^{\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r)}\cdot(\bar{x}_\alpha(r), \bar{x}_\delta(s))^{\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r)\bar{x}_{\beta+\delta}(c_{\beta,\delta}s)}\cdot(\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s))\\
&\quad\cdot(\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_\delta(s))^{\bar{x}_{\beta+\delta}(c_{\beta,\delta}s)} \qquad \text{by (0.3)}\\
&=\bar{x}_{\alpha+\beta+\delta}(c_{\alpha,\beta+\delta}c_{\beta,\delta}rs)\cdot(\bar{x}_\alpha(r), \bar{x}_\delta(s))\cdot(\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s))\cdot\bar{x}_{\alpha+\beta+\delta}(c_{\alpha,\beta+\delta}c_{\beta,\delta}rs)\ .
\end{aligned}$$

Note that the middle two terms of the last expression are central. Again by calculating the Jacobi identity, $[[E_\beta, E_\alpha], E_\delta]=-[E_\alpha, [E_\beta, E_\delta]]$, we have $c_{\alpha,\beta+\delta}c_{\beta,\delta}=-c_{\alpha,\beta+\delta}c_{\beta,\delta}$. Then $(\bar{x}_\alpha(r), \bar{x}_\delta(s))=(\bar{x}_\alpha(r), \bar{x}_\delta(s))(\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s))$, and $(\bar{x}_{\beta+\alpha}(c_{\beta,\alpha}r), \bar{x}_{\beta+\delta}(c_{\beta,\delta}s))=1$. Since r, s are arbitrary, we have $(\bar{x}_{\beta+\alpha}(r), \bar{x}_{\beta+\delta}(s))=1$ for all $r, s\in R$. Now applying the conjugation by $\bar{n}_\beta(1)$ and the central trick, we have $(\bar{x}_\alpha(r), \bar{x}_\delta(s))=1$ for all $r, s\in R$. □

(3.14) LEMMA. *$U=\langle U^\alpha \mid \alpha\in\Delta\rangle$. Hence* (Gr1) *holds for U.*

PROOF. Let $U':=\langle U^\alpha \mid \alpha\in\Delta\rangle$ and C be the kernel of ψ from U onto G. Since $\psi(U')=G$, then $U=U'C$, then $U=(U, U)=(U'C, U'C)=(U', U')=U'$, where the last equality follows from the relations (R1) and (R2) for $\bar{x}_\alpha(r)$'s. □

PROOF OF THEOREM (3.2). Up to now, we have constructed a surjective homomorphism Ψ from $\mathrm{St}(\Delta, R)$ onto U by sending $\hat{x}_\alpha(r)$ to $\bar{x}_\alpha(r)$. Let $\dot{U}$ be the subgroup generated by $\{\bar{x}_\alpha(r) \mid \alpha\in\Delta, r\in\boldsymbol{K}\}$ and $\dot{U}^\alpha=\{\bar{x}_\alpha(r) \mid r\in\boldsymbol{K}\}$ for each $\alpha\in\Delta$. We show $\dot{U}\in\mathfrak{S}$.

Clearly, $\hat{x}_\alpha(r)\to\bar{x}_\alpha(r)$ defines a surjective homomorphism, denoted by ϕ_u, from $\mathrm{St}(\Delta, \boldsymbol{K})$ onto $\dot{U}$. So the diagram commutes.

$$\begin{tikzcd}
\mathrm{St}(\Delta, \boldsymbol{K}) \arrow[r, "\dot{\phi}_u"] \arrow[d, equal] & \dot{U} \arrow[d, "\psi"] \\
\mathrm{St}(\Delta, \boldsymbol{K}) \arrow[r, "\dot{\phi}"] & \dot{G}\,.
\end{tikzcd}$$

Since $\dot{G} \in \mathfrak{S}$, so $\dot{\phi}$, restricted to $\chi^+ = \chi^+(\boldsymbol{K})$ relative to a positive system Δ_+ of Δ, is an isomorphism. The commutative diagram implies $\dot{\phi}_u$ restricted to χ^+, is an isomorphism as well. This implies $(\dot{U}, \dot{\phi}_u) \in \mathfrak{S}$.

It remains to verify the axioms (Gr1) through (Gr5). (Gr1) follows from Lemma (3.14). (Gr2) is clear by the definition of $\dot{U}^\alpha$. (Gr3) follows from Lemma (3.13). (Gr4) holds for U, since it holds for $\mathrm{St}(\Delta, R)$ and G. (Gr5) is from the relation (R3). It is clear from the construction that ψ is a Δ-homomorphism. □

Proof of Theorem (3.3). Let R be the associative ring relative to G, Let ψ (resp. ψ'): $U \to G$ (resp. G') be the universal central extension of G (resp. G'). By Theorem (3.2), U is graded by Δ. Moreover, the set of the generators $[x_\alpha(r) \mid \alpha \in \Delta, r \in R\}$ in G can be lifted to a set of generators $\{\bar{x}_\alpha(r) \mid \alpha \in \Delta, r \in R\}$ in U which satisfies the relations (R1) and (R2). Denote the element in $\mathfrak{S} = \mathfrak{S}(\Delta, \boldsymbol{K})$ relative to G (resp. U) by $(\dot{G}, \dot{\phi})$ (resp. $(\dot{U}, \dot{\phi}_u)$). The meanings of U^α, $U^\pm$ (relative to a positive system of Δ), $\bar{n}_\alpha(u)$, $\bar{h}_\alpha(u)$, $\dot{U}^\alpha$, $\dot{U}^\pm$, etc. are defined as before in an obvious manner. Pass these objects to G' by the central extension homomorphism ψ', for example, $G'^\alpha := \psi'(U^\alpha)$, $x'_\alpha(r) := \psi'(\bar{x}_\alpha(r))$, $\dot{G}' := \psi'(\dot{U})$, $\dot{G}'^\pm = \psi'(\dot{U}^\pm)$, etc. Then $\dot{\phi}' := \psi'\dot{\phi}_u$ is a homomorphism from $\mathrm{St}(\Delta, \boldsymbol{K})$ onto $\dot{G}'$. We will show that G' is Δ-graded relative to $(\dot{G}', \dot{\phi}')$. It suffices to show that $(\dot{G}', \dot{\phi}') \in \mathfrak{S} = \mathfrak{S}(\Delta, \boldsymbol{K})$ and that the axiom (Gr4) holds, since the other axioms are direct consequences of the relations (R1) and (R2) and the fact that ψ' is a homomorphism.

To be clear, we describe the relations of above maps by the following commutative diagrams with the generators:

$$\begin{tikzcd}
\mathrm{St}(\Delta, R);\, \hat{x}_\alpha(r) \arrow[r] \arrow[d] \arrow[dr] & G;\, x_\alpha(r) \\
G';\, x'_\alpha(r) & U;\, \bar{x}_\alpha(r)\,, \arrow[l, "\psi'"] \arrow[u, "\psi"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{St}(\Delta, \boldsymbol{K});\, \hat{x}_\alpha(t) \arrow[r, "\dot{\phi}"] \arrow[d, "\dot{\phi}"'] \arrow[dr, "\dot{\phi}_u"] & \dot{G};\, x_\alpha(t) \\
\dot{G}';\, x'_\alpha(t) & \dot{U};\, \bar{x}_\alpha(t)\,. \arrow[l, "\psi'"] \arrow[u, "\psi"']
\end{tikzcd}$$

Now arbitrarily fix a positive system Δ_+ of Δ. Then $\dot{\phi}_u|_{\chi^+(\boldsymbol{K})}$ is injective since $(\dot{U}, \dot{\phi}_u) \in \mathfrak{S}$ by Theorem (3.2). Recall that the center of $\chi^+(\boldsymbol{K})$ is trivial ([4], [8]). Then the center of $\dot{U}^+$ is trivial. Suppose $\hat{x} \in \chi^+(\boldsymbol{K}) \cap \mathrm{Ker}(\dot{\phi}')$. Then $\dot{\phi}_u(\hat{x}) \in \dot{U}^+ \cap \mathrm{Ker}\,\psi'$. Since $\mathrm{Ker}\,\psi'$ is central by hypothesis. So $\dot{\phi}_u(\hat{x}) = 1$, and $\hat{x} = 1$. So $\dot{\phi}'|_{\chi^+(\boldsymbol{K})}$ is injective. This proves $(\dot{G}', \dot{\phi}') \in \mathfrak{S}$.

We show that $G'^\alpha \cap G'^\beta = 1$, if $\alpha \neq \beta$. Let $x'_\alpha(r) = x'_\beta(s)$. Then $\bar{x}_\alpha(r) = \bar{x}_\beta(s)z$ for some

$z \in \operatorname{Ker} \psi' \subseteq \operatorname{Center}(U)$. It suffices to show $r=s=0$. We need to consider four cases: $\alpha = -\beta$; $(\alpha \mid \beta) = -1$; $(\alpha \mid \beta) = -1$; and $(\alpha \mid \beta) = 0$.

If $\alpha = -\beta$, take $\gamma \in \Delta$ so that $(\alpha \mid \gamma) = -1$. Then by (R2), $1 = (\bar{x}_{-\alpha}(s)z, \bar{x}_\gamma(1)) = (\bar{x}_\alpha(r), x_\gamma(1)) = x_{\alpha+\gamma}(c_{\alpha,\gamma} r)$. Thus $r=0$ and $s=0$.

If $(\alpha \mid \beta) = -1$, i.e. $\alpha + \beta \in \Delta$, then $1 = (\bar{x}_\alpha(r), \bar{x}_\alpha(1)) = (\bar{x}_\beta(s)z, \bar{x}_\alpha(1)) = \bar{x}_{\beta+\alpha}(c_{\beta,\alpha} s)$. Hence $s=r=0$.

If $(\alpha \mid \beta) = 1$, we take $\gamma \in \Delta$ so that $(\alpha, -\beta, \gamma)$ is an A_3-triple. Then $1 = (\bar{x}_\alpha(r), \bar{x}_{-\gamma}(1)) = (\bar{x}_\beta(s)z, \bar{x}_{-\gamma}(1)) = \bar{x}_{\beta-\gamma}(c_{\beta,-\gamma} s)$. So $s=0$ and $r=0$.

Finally if $(\alpha \mid \beta) = 0$, then there exists a third root γ so that (α, γ, β) is an A_3-triple. Then

$$\bar{x}_{\alpha+\gamma}(c_{\alpha,\gamma} r) = (\bar{x}_\alpha(r), \bar{x}_\gamma(1)) = (\bar{x}_\beta(s)z, \bar{x}_\gamma(1)) = \bar{x}_{\beta+\gamma}(c_{\beta,\gamma} s),$$

but $(\alpha+\gamma \mid \beta+\gamma) = 1$. Thus $r=s=0$ follows from the third case. So (Gr4) holds. $\square$

References

[1] S. Berman and R. V. Moody, Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy, Invent. Math. (to appear).

[2] N. Bourbaki, Groupes et algèbres de Lie, Ch. IV, V, VI, Hermann, Paris, 1968.

[3] R. W. Carter, Simple Groups of Lie Type, Wiley, London, 1972.

[4] A. J. Hahn and O. T. O'Meara, The Classical Groups and K-theory, Springer-Verlag, Berlin Heidelberg, 1989.

[5] J. Milnor, Introduction to Algebraic K-theory, Ann. Math. Stud. 72, Princeton University Press, 1971.

[6] R. V. Moody and A. Pianzola, Lie Algebras with Triangular Decomposition, J. Wiley, 1992.

[7] M. R. Stein, Generators, relations and coverings of Chevalley groups over commutative rings, Amer. J. Math. 93 (1971), 965–1004.

[8] R. Steinberg, Lectures on Chevalley Groups (notes by J. Faulkner and R. Wilson) Yale Univ. Lecture Notes, 1967.

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