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COXETER ARRANGEMENTS ARE HEREDITARILY FREE

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Abstract. A Coxeter arrangement is the set of reflecting hyperplanes in the reflection representation of a finite Coxeter group. Arnold and Saito showed that every Coxeter arrangement is free. We prove that any restriction of a Coxeter arrangement is again a free arrangement. It explains why the characteristic polynomial of any restriction of a Coxeter arrangement has only positive integer roots, which was observed by P. Orlik and L. Solomon. We use the classification of Coxeter groups.

1. Introduction. Basic definitions below may be stated over an arbitrary field, but we restrict attention to the real numbers in this paper. We refer to [8] for details.

Let V be a real vector space of dimension l. A hyperplane H in V is a subspace of codimension one. An arrangement $\mathscr{A}$ is a finite collection of hyperplanes in V. Let $L(\mathscr{A})$ be the set of all intersections of elements of $\mathscr{A}$. We agree that $L(\mathscr{A})$ includes V as the intersection of the empty collection of hyperplanes. We should remember that if $X \in L(\mathscr{A})$, then $X \subseteq V$. Strictly speaking, these objects should have different names, but it is always clear from the context which one is in consideration. If $\mathscr{B} \subseteq \mathscr{A}$ is a subset, then $\mathscr{B}$ is called a *subarrangement*. For $X \in L(\mathscr{A})$, define a subarrangement $\mathscr{A}_X$ of $\mathscr{A}$ by $\mathscr{A}_X = \{H \in \mathscr{A} \mid X \subseteq H\}$. Define an arrangement $\mathscr{A}^X$ in X by

$$\mathscr{A}^X = \{X \cap H \mid H \in \mathscr{A} - \mathscr{A}_X\} .$$

We call $\mathscr{A}^X$ the *restriction* of $\mathscr{A}$ to X. Note that $\mathscr{A}^V = \mathscr{A}$.

Let V^* be the dual space of V, the space of linear forms on V. Let $S = S(V^*)$ be the symmetric algebra of V^*. Choose a basis $\{e_1, \dots, e_l\}$ in V and let $\{x_1, \dots, x_l\}$ be the dual basis in V^* so $x_i(e_j) = \delta_{i,j}$. We may identify $S(V^*)$ with the polynomial algebra $S = \boldsymbol{R}[x_1, \dots, x_l]$. Each hyperplane $H \in \mathscr{A}$ is the kernel of a linear form α_H, defined up to a constant. The product

$$Q(\mathscr{A}) = \prod_{H \in \mathscr{A}} \alpha_H$$

is called a *defining polynomial* of $\mathscr{A}$. An $\boldsymbol{R}$-linear map $\theta: S \to S$ is a *derivation* if $\theta(fg) = f\theta(g) + g\theta(f)$ for $f, g \in S$. Let $\mathrm{Der}_{\boldsymbol{R}}(S)$ be the S-module of derivations of S. Define an S-submodule of $\mathrm{Der}_{\boldsymbol{R}}(S)$, called the *module of $\mathscr{A}$-derivations*, by

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$$D(\mathscr{A})=\{\theta\in \mathrm{Der}_{\mathbf{R}}(S)\,|\,\theta(Q)\in QS\}\,.$$

The arrangement $\mathscr{A}$ is called *free* if $D(\mathscr{A})$ is a free S-module [12]. If $\mathscr{A}$ is a free arrangement, then the subarrangement $\mathscr{A}_X$ is free for all $X\in L(\mathscr{A})$. Recently, Edelman and Reiner [4] constructed a free arrangement $\mathscr{A}$ which contains a hyperplane $H\in\mathscr{A}$ so that the restriction, $\mathscr{A}^H$ is not free. We call $\mathscr{A}$ *hereditarily free* if $\mathscr{A}^X$ is free for all $X\in L(\mathscr{A})$. The Edelman-Reiner example shows that not all free arrangements are hereditarily free.

Let $GL(V)$ denote the general linear group of V. An element $s\in GL(V)$ is a *reflection* if it has order 2 and its fixed point set is a hyperplane H_s. We call H_s the *reflecting hyperplane* of s. A finite subgroup $G\subset GL(V)$ is called a *reflection group* if it is generated by reflections. These groups were classified by Coxeter and are often called finite Coxeter groups. Let $G\subset GL(V)$ be a finite Coxeter group. The set $\mathscr{A}=\mathscr{A}(G)$ of reflecting hyperplanes of G is called the *Coxeter arrangement* of G.

Arnold [1], [2] and Saito [9], [10] proved independently that Coxeter arrangements are free. If $\mathscr{A}$ is a Coxeter arrangement and $X\in L(\mathscr{A})$, then the restriction $\mathscr{A}^X$ is not always a Coxeter arrangement. Thus these restrictions are not automatically free. Restrictions of Coxeter arrangements were studied in [6]. There is a close and still unexplained connection between the numerical results in [6] and the Springer representations of the corresponding Weyl groups, see [11] and [5]. The results of [6] led to the conjecture that $\mathscr{A}^X$ is free for all $X\in L(\mathscr{A})$. In this paper we prove this conjecture.

THEOREM 1.1. *Coxeter arrangements are hereditarily free.*

The argument uses the classification of finite Coxeter groups [3]. It follows from [8, Proposition 4.28] that it suffices to prove the assertion for irreducible groups. We proved Theorem 1.1 for the infinite families in [8, Section 6.4]. For the exceptional groups we utilize additional facts. Every 2-arrangement is free [8, Example 4.20]. The results in [12] and [8, Appendix D] imply that every 3-dimensional restriction of a Coxeter arrangement is free. We proved in [7] that in a Coxeter arrangement the restriction $\mathscr{A}^H$ is free for every hyperplane $H\in\mathscr{A}$. Thus the assertion is true for exceptional groups of rank ≤ 5.

This leaves the arrangements for Coxeter groups of types E_r, $r=6$, 7, 8 and in the arrangement $\mathscr{A}(E_r)$, restrictions to subspaces X of dimensions $4,\dots,r-2$. The rest of this paper is devoted to proving that these 19 arrangements are free. In Section 2 we collect the basic tools needed in our constructions. Given $\mathscr{A}$, we apply the Addition-Deletion Theorem 2.3 to build $\mathscr{A}$ from a known free arrangement $\mathscr{B}$ adding hyperplanes one-by-one, each time satisfying the conditions of Theorem 2.3. We consider the arrangements in the order of Table 1. In 13 cases we can find $\mathscr{B}$ so that the hyperplanes may be added in any order. We call these pairs $(\mathscr{A},\mathscr{B})$ pure and consider them in Section 3. In the remaining six cases the order of the hyperplanes is essential.

TABLE 1. Structure formulas.

type $\mathscr{A}$	type $\mathscr{B}$	type $\mathscr{A}$ − type $\mathscr{B}$
$E_6^{A_2}$	$D_5^{A_1}$	$2E_6^{A_1\times A_2}$
$E_6^{A_1^2}$	$D_5^{A_1}$	$3E_6^{A_1^3}+E_6^{A_1\times A_2}$
$E_7^{A_3}$	$E_6^{A_1^2}$	$E_7^{(A_1\times A_3)'}$
$E_7^{A_1\times A_2}$	$E_6^{A_1^2}$	$3E_7^{A_1^2\times A_2}+E_7^{A_2^2}$
$E_7^{(A_1^3)''}$	$E_6^{A_1^2}$	$E_7^{A_1^4}+4E_7^{A_1^2\times A_2}$
$E_7^{(A_1^3)'}$	$E_6^{A_1^2}$	$6E_7^{A_1^4}+E_7^{(A_1\times A_3)'}$
$E_7^{A_2}$	$E_6^{A_1}$	$5E_7^{A_1\times A_2}$
$E_7^{A_1^2}$	$E_6^{A_1}$	$6E_7^{(A_1^3)''}+2E_7^{A_1\times A_2}-3E_7^{A_1^4}$
$E_8^{D_4}$	$E_7^{(A_1^3)'}$	0
$E_8^{A_4}$	$E_7^{A_1\times A_2}$	$4E_8^{A_1\times A_4}$
$E_8^{A_1\times A_3}$	$E_7^{(A_1^3)''}$	$4E_8^{A_1^2\times A_3}+2E_8^{A_2\times A_3}$
$E_8^{A_2^2}$	$E_7^{A_1\times A_2}$	$4E_8^{A_1\times A_2^2}+5E_8^{A_2\times A_3}-4E_8^{A_1\times A_2\times A_3}-4E_8^{A_1^2\times A_2^2}$
$E_8^{A_1^2\times A_2}$	$E_7^{(A_1^3)'}$	$8E_8^{A_1\times A_2^2}$
$E_8^{A_1^4}$	$E_7^{(A_1^3)'}$	$8E_8^{A_1^3\times A_2}$
$E_8^{A_3}$	$E_7^{A_1^2}$	$8E_8^{A_1\times A_3}$
$E_8^{A_1\times A_2}$	$E_7^{A_1^2}$	$8E_8^{A_1^2\times A_2}+4E_8^{A_2^2}+E_8^{A_1\times A_3}-12E_8^{A_1^3\times A_2}-8E_8^{A_1\times A_2^2}$
$E_8^{A_1^3}$	$E_7^{A_1^2}$	$6E_8^{A_1^4}+9E_8^{A_1^2\times A_2}+E_8^{A_1\times A_3}-24E_8^{A_1^3\times A_2}-4E_8^{A_1\times A_2^2}$
$E_8^{A_2}$	$E_7^{A_1}$	$16E_8^{A_1\times A_2}+E_8^{A_3}-40E_8^{A_1^2\times A_2}$
$E_8^{A_1^2}$	$E_7^{A_1}$	$15E_8^{A_1^3}+7E_8^{A_1\times A_2}-45E_8^{A_1^4}-30E_8^{A_1^2\times A_2}$

These require more detailed study and we treat them in Section 4. The largest is a 6-arrangement with 68 hyperplanes. The corresponding addition tables are collected in Section 5. The results may be summarized as follows.

THEOREM 1.2. *The* 19 *restrictions to subspaces of dimensions* $4, \ldots, r-2$ *in the Coxeter arrangements of* E_r, $r=6, 7, 8$ *are free and satisfy the structure formulas in Table* 1.

2. Basic tools. The set $L(\mathscr{A})$ is partially ordered by reverse inclusion. The Möbius function $\mu\colon L(\mathscr{A})\to \boldsymbol{Z}$ is defined by $\mu(V)=1$ and for $Y>V$ we obtain $\mu(Y)$ recursively from $\sum_{Z\le Y}\mu(Z)=0$. Let t be an indeterminate. Define the *Poincaré polynomial* of $\mathscr{A}$ by

$$\pi(\mathscr{A}, t)=\sum_{X\in L(\mathscr{A})}\mu(X)(-t)^{r(X)}.$$

It follows from [8, Theorem 2.47] that $\pi(\mathscr{A}, t)$ has nonnegative coefficients. Let $|\mathscr{A}|$ denote the number of hyperplanes in $\mathscr{A}$. Then $|\mathscr{A}|$ is the coefficient of t in $\pi(\mathscr{A}, t)$.

Next we define deletion and restriction. Let $\mathscr{A}$ be a nonempty arrangement and let $H_0 \in \mathscr{A}$. Let $\mathscr{A}' = \mathscr{A} \setminus \{H_0\}$ and let $\mathscr{A}'' = \mathscr{A}^{H_0}$. We call $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ a *triple* of arrangements and H_0 the distinguished hyperplane. The next result was proved in [8, Theorem 2.56].

THEOREM 2.1. *If $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ is a triple of arrangements, then*

$$\pi(\mathscr{A}, t) = \pi(\mathscr{A}', t) + t\pi(\mathscr{A}'', t) .$$

Let $G \subset GL(V)$ be a finite Coxeter group with reflection arrangement $\mathscr{A} = \mathscr{A}(G)$. The group G acts by permutation on the poset $L(\mathscr{A})$. A complete set of orbit types was determined in [6]; see also [8, Appendix C]. The type T of $X \in L(\mathscr{A})$ is the Coxeter group which fixes X pointwise. If two orbits have type T, we label them T', T''. This determines the structure of $L(\mathscr{A})$ and the structure of the restriction $L(\mathscr{A}^X)$ for all $X \in L(\mathscr{A})$. We identify a restriction $\mathscr{A}^X$ by a pair (G, T) and write type $\mathscr{A}^X = (G, T)$ or type $\mathscr{A}^X = G^T$. Here G is the Coxeter group of the reflection arrangement $\mathscr{A} = \mathscr{A}(G)$ in which we are restricting and T is the orbit type of the subspace $X \in L(\mathscr{A})$ whose restriction $\mathscr{A}^X$ is in consideration. The results of [6] give the following.

PROPOSITION 2.2. *Let $G \subset GL(V)$ be a finite Coxeter group with reflection arrangement $\mathscr{A} = \mathscr{A}(G)$. For each $X \in L(\mathscr{A})$ with* $\dim(X) = p$, *there exist positive integers $b_1^X, \dots, b_p^X$ such that*

$$\pi(\mathscr{A}^X, t) = \prod_{i=1}^{p} (1 + b_i^X t) .$$

The values of b_i^X are tabulated in [6] and [8, Appendix C] for all exceptional groups and all orbit types. We make frequent use of these tables in this paper.

The S-module $\mathrm{Der}_{\boldsymbol{R}}(S)$ is free with basis $D_i = \partial/\partial x_i$. Thus any derivation $\theta \in \mathrm{Der}_{\boldsymbol{R}}(S)$ is expressed uniquely as

$$\theta = f_1 D_1 + \cdots + f_l D_l , \qquad f_1, \dots, f_l \in S .$$

We grade $\mathrm{Der}_{\boldsymbol{R}}(S)$ by polynomial degree. Thus $\theta \in \mathrm{Der}_{\boldsymbol{R}}(S)$ is homogeneous of pdegree q if the polynomials $f_1, \dots, f_l$ are homogeneous of degree q. In this case we write $\operatorname{pdeg} \theta = q$.

If $\mathscr{A}$ is free, then by [8, Proposition 4.18] there exists a homogeneous basis $\{\theta_1, \dots, \theta_l\}$ for $D(\mathscr{A})$. The polynomial degrees of a basis (with multiplicity but neglecting the order) depend only on $\mathscr{A}$. We call $\operatorname{pdeg} \theta_1, \dots, \operatorname{pdeg} \theta_l$ the *exponents* of $\mathscr{A}$ and write

$$\exp \mathscr{A} = \{\operatorname{pdeg} \theta_1, \dots, \operatorname{pdeg} \theta_l\} .$$

The following fundamental result is needed in our calculations [12], [8, Theorem 4.51].

THEOREM 2.3. *Suppose $\mathscr{A}$ is nonempty. Let $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ be a triple. Any two of the following statements imply the third:*

$$\mathscr{A} \text{ is free with } \exp \mathscr{A} = \{b_1, \ldots, b_{l-1}, b_l\},$$

$$\mathscr{A}' \text{ is free with } \exp \mathscr{A}' = \{b_1, \ldots, b_{l-1}, b_l - 1\},$$

$$\mathscr{A}'' \text{ is free with } \exp \mathscr{A}'' = \{b_1, \ldots, b_{l-1}\}.$$

If we assume that $\mathscr{A}'$ and $\mathscr{A}''$ are free, then the result is called the Addition Theorem. If we assume that $\mathscr{A}$ and $\mathscr{A}''$ are free, then the result is called the Deletion Theorem. The following Factorization Theorem was proved in [13], [8, Theorem 4.137].

THEOREM 2.4. *If $\mathscr{A}$ is a free arrangement with $\exp \mathscr{A} = \{b_1, \ldots, b_l\}$, then*

$$\pi(\mathscr{A}, t) = \prod_{i=1}^{l} (1 + b_i t).$$

Comparison of Proposition 2.2 and Theorem 2.4 shows that if $\mathscr{A}^X$ is free, then its exponents must equal $b_1^X, \ldots, b_p^X$.

In the explicit calculations we need coordinates for the remaining arrangements. Define

$$P_m = \prod_{1 \le i < j \le m} (x_i - x_j) \prod_{1 \le i < j < k \le m} (x_i + x_j + x_k).$$

Let $s_m = \sum_{i=1}^{m} x_i$. Cartan's coordinates are as follows.

$$Q(D_5) = P_5,$$

$$Q(E_6) = P_6 s_6,$$

$$Q(E_7) = P_7 \prod_{i=1}^{7} (s_7 - x_i),$$

$$Q(E_8) = P_8 \prod_{i=1}^{8} (s_8 + x_i) \prod_{1 \le i < j \le 8} (s_8 - x_i - x_j).$$

The inner product on V^* is given by:

$$\langle x_i, x_j \rangle = \begin{cases} 8/9 & \text{if } i = j, \\ -1/9 & \text{otherwise}. \end{cases}$$

3. Pure pairs of arrangements. Suppose $\mathscr{A}$ is one of the 19 arrangements in consideration. It follows from Proposition 2.2 that $\pi(\mathscr{A}, t) = \prod_{i=1}^{l} (1 + a_i t)$ for positive integers $a_1, \ldots, a_l$. We want to prove that $\mathscr{A}$ is free with $\exp \mathscr{A} = \{a_1, \ldots, a_l\}$. We choose a subarrangement $\mathscr{B} \subseteq \mathscr{A}$ with the following properties:

(1) $\mathscr{B}$ is a restriction of a Coxeter arrangement which is already known to be free with $\exp \mathscr{B} = \{b_1, \ldots, b_l\}$,

(2) after a suitable permutation of the subscripts we have $b_i = a_i$ for as many

indices as possible.

DEFINITION 3.1. Given an ordering of the hyperplanes of $\mathscr{A}-\mathscr{B}$, say $H_1, \ldots, H_m$, we set $\mathscr{A}_0=\mathscr{B}$ and $\mathscr{A}_i=\mathscr{B}\cup\{H_1, \ldots, H_i\}$ for $i=1, \ldots, m$. We call an ordering of the hyperplanes in $\mathscr{A}-\mathscr{B}$ *admissible* if we can apply the Addition Theorem 2.3 adding one hyperplane at a time in the given order to $\mathscr{B}$ to show that $\mathscr{A}$ is free.

We must consider the restrictions of the successive arrangements $\mathscr{A}_i^{H_i}$. The most favorable situation is if these restrictions are already known and do not depend on the hyperplanes to be added. This notion is stated precisely below. The next two definitions and three propositions are valid for all arrangements.

DEFINITION 3.2. Suppose H_1, H_2 are two distinct hyperplanes not in $\mathscr{B}$. We call the hyperplanes H_1, H_2 *compatible* with respect to $\mathscr{B}$ if there exists $K\in\mathscr{B}$ so that $H_1\cap H_2=H_1\cap K=H_2\cap K$.

Equivalently, the hyperplanes H_1, H_2, K are dependent. The next result is immediate from this definition.

PROPOSITION 3.3. *If two distinct hyperplanes H_1, H_2 are compatible with respect to $\mathscr{B}$, then $\mathscr{C}^{H_2}=(\mathscr{C}\setminus\{H_1\})^{H_2}$ for any arrangement $\mathscr{C}$ with $\mathscr{B}\cup\{H_1, H_2\}\subseteq\mathscr{C}$.*

DEFINITION 3.4. Call the pair $(\mathscr{A}, \mathscr{B})$ *pure* if any two hyperplanes in $\mathscr{A}-\mathscr{B}$ are compatible with respect to $\mathscr{B}$. We call the pair $(\mathscr{A}, \mathscr{B})$ *mixed* if it is not pure.

PROPOSITION 3.5. *Let $\mathscr{B}\subseteq\mathscr{A}$. The following five conditions are equivalent*:

(1) *$(\mathscr{A}, \mathscr{B})$ is pure*,
(2) *for an arbitrary linear order of $\mathscr{A}-\mathscr{B}$ we have $\mathscr{A}^{H_i}=(\mathscr{A}_i)^{H_i}$*,
(3) *there exists a linear order of $\mathscr{A}-\mathscr{B}$ so that $\mathscr{A}^{H_i}=(\mathscr{A}_i)^{H_i}$*,
(4) *$\pi(\mathscr{A}, t)=\pi(\mathscr{B}, t)+t\sum_{H\in\mathscr{A}-\mathscr{B}}\pi(\mathscr{A}^H, t)$*,
(5) *$\mathscr{A}^H=(\mathscr{B}\cup\{H\})^H$ for any $H\in\mathscr{A}-\mathscr{B}$.*

PROOF. (1) $\Rightarrow$ (2): Since H_i and H_{i+1} are compatible with respect to $\mathscr{B}$, we have

$$(\mathscr{A}_i)^{H_i}=(\mathscr{A}_{i+1}\setminus\{H_{i+1}\})^{H_i}=(\mathscr{A}_{i+1})^{H_i}$$

by Proposition 3.3. Next, since H_i and H_{i+2} are compatible with respect to $\mathscr{B}$, we similarly have $(\mathscr{A}_{i+1})^{H_i}=(\mathscr{A}_{i+2})^{H_i}$. Repeating this process, we finally have (2) because $\mathscr{A}_m=\mathscr{A}$.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (4): Apply Theorem 2.1 to obtain

$$\pi(\mathscr{A}_i, t)-\pi(\mathscr{A}_{i-1}, t)=t\pi((\mathscr{A}_i)^{H_i}, t)$$

for $i=1, \ldots, m$. Combining these equalities, we have

$$\pi(\mathscr{A}, t)-\pi(\mathscr{B}, t)=t\sum_{i=1}^{m}\pi((\mathscr{A}_i)^{H_i}, t)\,.$$

(4) $\Rightarrow$ (5): Give a linear order to $\mathscr{A}-\mathscr{B}=\{H_1, \ldots, H_m\}$ such that $H=H_1$. Since

$$t\sum_{i=1}^{m} \pi((\mathscr{A}_i)^{H_i}, t)=\pi(\mathscr{A}, t)-\pi(\mathscr{B}, t)=t\sum_{i=1}^{m} \pi(\mathscr{A}^{H_i}, t)\,,$$

we have

$$\sum_{i=1}^{m} |(\mathscr{A}_i)^{H_i}|=\sum_{i=1}^{m} |\mathscr{A}^{H_i}|$$

by comparing the coefficients of t^2. In general, each $(\mathscr{A}_i)^{H_i}$ is a subarrangement of $\mathscr{A}^{H_i}$. Therefore we have $(\mathscr{A}_i)^{H_i}=\mathscr{A}^{H_i}$ for $i=1, \ldots, m$. When $i=1$ this is the assertion in (5).

(5) $\Rightarrow$ (1): Note that $H_1\cap H_2\in\mathscr{A}^{H_1}=(\mathscr{B}\cup\{H_1\})^{H_1}$. Thus there exists $K\in\mathscr{B}$ such that $H_1\cap H_2=H_1\cap K$. □

PROPOSITION 3.6. *Suppose* $\mathscr{B}\subseteq\mathscr{A}$. *If*

(1) $\pi(\mathscr{A}, t)=\prod_{i=1}^{l}(1+a_i t)$,

(2) $\mathscr{B}$ *is free with* $\exp(\mathscr{B})=\{a_1, \ldots, a_{l-1}, a_l-|\mathscr{A}|+|\mathscr{B}|\}$, *and*

(3) $\mathscr{A}^H$ *is free with* $\exp(\mathscr{A}^H)=\{a_1, \ldots, a_{l-1}\}$ *for all* $H\in\mathscr{A}-\mathscr{B}$,

then any arrangement $\mathscr{C}$ *with* $\mathscr{B}\subseteq\mathscr{C}\subseteq\mathscr{A}$ *is free with* $\exp(\mathscr{C})=\{a_1, \ldots, a_{l-1}, a_l-|\mathscr{C}|+|\mathscr{B}|\}$. *In particular,* $\mathscr{A}$ *is free and any ordering of the hyperplanes in* $\mathscr{A}-\mathscr{B}$ *is admissible. We express this with the symbolic structure formula*

$$\operatorname{type}\mathscr{A}-\operatorname{type}\mathscr{B}=\sum_{H\in\mathscr{A}-\mathscr{B}}\operatorname{type}\mathscr{A}^H\,.$$

PROOF. It is easy to check that

$$\pi(\mathscr{A}, t)-\pi(\mathscr{B}, t)=t\sum_{H\in\mathscr{A}-\mathscr{B}}\pi(\mathscr{A}^H, t)\,.$$

By Proposition 3.5, $(\mathscr{A}, \mathscr{B})$ is pure. Let $\mathscr{C}=\mathscr{B}\cup\{H_1, \ldots, H_p\}$ and $\mathscr{A}_i=\mathscr{B}\cup\{H_1, \ldots, H_i\}$ $(i=1, \ldots, p)$. Since $(\mathscr{C}, \mathscr{A})$ is also pure, we have $\mathscr{A}_i=\mathscr{A}_{i-1}\cup\{H_i\}$ and $(\mathscr{A}_i)^{H_i}=\mathscr{A}^{H_i}$ for $i=1, \ldots, p$ by Proposition 3.5. Apply the Addition Theorem 2.3 repeatedly. □

EXAMPLE 3.7. Consider the arrangement $\mathscr{A}=(E_6, A_1^2)$. We see from the tables in [6] and [8, Appendix C] that $\pi(\mathscr{A}, t)=(1+t)(1+4t)(1+5t)(1+7t)$. We show that $\mathscr{A}$ is free with $\exp\mathscr{A}=\{1, 4, 5, 7\}$.

Recall $Q(E_6)$ and let $\beta_1=x_3-x_4$, $\beta_2=x_5-x_6$. Let $K_i=\ker\beta_i$ for $i=1, 2$. With respect to the inner product, the two hyperplanes are orthogonal. Thus the reflections in K_1 and K_2 commute. If $X=K_1\cap K_2$, then $\mathscr{A}=\mathscr{A}(E_6)^X$. Choose coordinates x_1, x_2, x_3, x_5 on X. Then $Q(\mathscr{A})=Q(E_6)|_{x_3=x_4, x_5=x_6}$ where subscripts indicate restrictions. Thus

$$Q(\mathscr{A})=P_5|_{x_3=x_4}(\alpha_1\alpha_2\alpha_3\alpha_4)|_{x_3=x_4, x_5=x_6}\,,$$

$$\alpha_1=x_1+x_5+x_6\,,$$

$$\alpha_2 = x_2 + x_5 + x_6 \,,$$
$$\alpha_3 = x_3 + x_5 + x_6 \,,$$
$$\alpha_4 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \,.$$

Let $\mathscr{B} = (D_5, A_1)$ and note that $P_5|_{x_3 = x_4} = Q(\mathscr{B})$. We get from [8, Proposition 6.85] that $\mathscr{B}$ is free with $\exp \mathscr{B} = \{1, 3, 4, 5\}$. Thus $\mathscr{B}$ satisfies conditions (1) and (2) of Proposition 3.6. We check (3). Let $H_i = \ker \alpha_i$ for $i = 1, 2, 3, 4$. The type of $\mathscr{A}^{H_i}$ is the subgroup generated by the reflections in the hyperplanes $\{K_1, K_2, H_i\}$. For $i = 1, 2, 4$, the three hyperplanes K_1, K_2, H_i are orthogonal to each other, so that $\mathscr{A}^{H_1} \simeq \mathscr{A}^{H_2} \simeq \mathscr{A}^{H_4} \simeq (E_6, A_1^3)$. For H_3, the angle θ between H_3 and K_1 satisfies

$$\cos\theta = \frac{\langle \alpha_3, \beta_1 \rangle}{|\alpha_3||\beta_1|} = \frac{1}{2} \,.$$

Also $\langle \alpha_3, \beta_2 \rangle = 0$. This implies $\mathscr{A}^{H_3} \simeq (E_6, A_1 \times A_2)$. Thus $\mathscr{A}^{H_i}$ is free with $\exp \mathscr{A}^{H_i} = \{1, 4, 5\}$ for $1 \le i \le 4$ from [8, Appendix D]. Therefore the conditions of Proposition 3.6 are satisfied. The corresponding structure formula is

$$E_6^{A_1^2} - D_5^{A_1} = 3E_6^{A_1^3} + E_6^{A_1 \times A_2} \,.$$

4. Mixed pairs of arrangements. Suppose $\mathscr{A}$ is one of the arrangements (E_7, A_1^2), (E_8, A_2^2), $(E_8, A_1 \times A_2)$, (E_8, A_1^3), (E_8, A_2), (E_8, A_1^2). Choose $\mathscr{B}$ as indicated in Table 1. We want to prove that $\mathscr{A}$ is free. Since the pair $(\mathscr{A}, \mathscr{B})$ is mixed, we must find an admissible order of the hyperplanes in $\mathscr{A} - \mathscr{B}$. In order to apply the Addition Theorem, we must know the restrictions of the successive arrangements to the hyperplanes to be added.

We encode the information necessary to find an admissible order in the $p \times p$ *compatibility matrix* C associated to $\mathscr{A}$ and $\mathscr{B}$. The rows and columns are labeled by the hyperplanes $H_i \in \mathscr{A} - \mathscr{B}$. There is no entry $C_{i,j}$ if $i = j$ or if H_i, H_j are compatible with respect to $\mathscr{B}$. If H_i, H_j are not compatible with respect to $\mathscr{B}$, then $C_{i,j} = \text{type}\, \mathscr{A}^{H_i \cap H_j}$. Note that C is a symmetric matrix which depends on the order of the hyperplanes in $\mathscr{A} - \mathscr{B}$.

PROPOSITION 4.1. *The structure formulas for the six mixed pairs are independent of the admissible order of the hyperplanes in* $\mathscr{A} - \mathscr{B}$.

PROOF. Choose the admissible orders given in the addition tables. If H_i and H_j $(j = i+1, \ldots, m)$ are compatible with respect to $\mathscr{B}$, then $\mathscr{A}_i^{H_i} = \mathscr{A}^{H_i}$. Otherwise, $\mathscr{A}_i^{H_i}$ is a subarrangement of $\mathscr{A}^{H_i}$. We want to apply the Deletion Theorem to $\mathscr{A}^{H_i}$ the necessary number of times to show that $\mathscr{A}_i^{H_i}$ is free before we can proceed to $\mathscr{A}_i$. Direct computation shows that the pair $(\mathscr{A}_i^{H_i}, \mathscr{A}^{H_i})$ is always pure so we have

$$\operatorname{type} \mathscr{A}^{H_i} - \operatorname{type} \mathscr{A}_i^{H_i} = \sum_{X \in \mathscr{A}^{H_i} - \mathscr{A}_i^{H_i}} \operatorname{type} \mathscr{A}^X .$$

Thus we can apply the Deletion Theorem to show that $\mathscr{A}_i^{H_i}$ is free. Next we can apply the Addition Theorem to show that $\mathscr{A}$ itself is free. We have

$$\operatorname{type} \mathscr{A} - \operatorname{type} \mathscr{B} = \sum_{i=1}^{p} \operatorname{type} \mathscr{A}_i^{H_i} .$$

We substitute and collect terms to obtain

$$\operatorname{type} \mathscr{A} - \operatorname{type} \mathscr{B} = \sum_{H \in \mathscr{A} - \mathscr{B}} \operatorname{type} \mathscr{A}^H - \frac{1}{2} \sum_{i,j} C_{i,j} .$$

The sum of the entries of C is independent of the admissible order. □

EXAMPLE 4.2. Consider the arrangement $\mathscr{A} = (E_8, A_2^2)$. We see from the tables in [6] and [8, Appendix C] that $\pi(\mathscr{A}, t) = (1+t)(1+7t)(1+11t)(1+11t)$. We show that $\mathscr{A}$ is free with $\exp \mathscr{A} = \{1, 7, 11, 11\}$.

Recall $Q(E_8)$ and let $\beta_1 = x_3 - x_4$, $\beta_2 = x_4 - x_5$, $\beta_3 = x_6 - x_7$, $\beta_4 = x_7 - x_8$. Let $K_i = \ker \beta_i$ for $i = 1, 2, 3, 4$. If $X = \bigcap_{i=1}^4 K_i$, then $\mathscr{A} = \mathscr{A}(E_8)^X$. Then $Q(\mathscr{A}) = Q(E_8)|_{x_3 = x_4 = x_5, x_6 = x_7 = x_8}$ where subscripts indicate restrictions. Thus

$$Q(\mathscr{A}) = Q(E_7)|_{x_3 = x_4 = x_5, x_6 = x_7} \left(\prod_{i=1}^{9} \gamma_i \right) \Bigg|_{x_3 = x_4 = x_5, x_6 = x_7 = x_8} ,$$

$$\gamma_1 = x_6 + x_7 + x_8 ,$$

$$\gamma_2 = s_8 + x_1 ,$$

$$\gamma_3 = s_8 + x_2 ,$$

$$\gamma_4 = s_8 + x_3 ,$$

$$\gamma_5 = s_8 + x_6 ,$$

$$\gamma_6 = s_8 - x_1 - x_2 ,$$

$$\gamma_7 = s_8 - x_1 - x_5 ,$$

$$\gamma_8 = s_8 - x_2 - x_5 ,$$

$$\gamma_9 = s_8 - x_4 - x_5 .$$

Let $\mathscr{B} = (E_7, A_1 \times A_2)$. Then $\mathscr{B} \subseteq \mathscr{A}$ and it follows from our earlier work that $\mathscr{B}$ is free with $\exp \mathscr{B} = \{1, 5, 7, 8\}$. Let $G_i = \ker \gamma_i$. The compatibility matrix C is presented in Table 2. Here we use the symbols $T_1 = (E_8, A_1 \times A_2 \times A_3)$ and $T_2 = (E_8, A_1^2 \times A_2^2)$. Table 2 also includes a column for the type of the restriction $\mathscr{A}^{G_i}$. Here we use the symbols $S_1 = (E_8, A_1 \times A_2^2)$ and $S_2 = (E_8, A_2 \times A_3)$. We see from [6] and [8, Appendix C] that

TABLE 2. Compatibility matrix for $(E_8^{A_2^2}, E_7^{A_1\times A_2})$.

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	$\mathscr{A}^{G_i}$
G_1		T_2	T_2	T_1						S_1
G_2	T_2					T_2	T_1			S_1
G_3	T_2					T_2		T_1		S_1
G_4	T_1									S_2
G_5										S_2
G_6		T_2	T_2						T_1	S_1
G_7		T_1								S_2
G_8			T_1							S_2
G_9						T_1				S_2

$\exp T_1 = \exp T_2 = \{1, 7\}$, $\exp S_1 = \{1, 7, 11\}$ and $\exp S_2 = \{1, 7, 9\}$.

The only hyperplane compatible with all others is G_5 with restriction of type S_2 and exponents $\{1, 7, 9\}$. Since $\exp \mathscr{B} = \{1, 5, 7, 8\}$, we cannot add G_5 to $\mathscr{B}$ using the Addition Theorem. In this example we can start with any other hyperplane, but in each case the restriction is a proper subarrangement of S_1 or S_2.

We add G_9 first. Let $\mathscr{A}_1 = \mathscr{B} \cup \{G_9\}$. Since G_9 is compatible with all hyperplanes except G_6, the difference between $\mathscr{A}^{G_9} \simeq S_2$ and $\mathscr{A}_1^{G_9}$ is the hyperplane $G_9 \cap G_6 \in \mathscr{A}^{G_9}$. The restriction to this hyperplane has type T_1. Since $\exp T_1 = \{1, 7\}$ and $\exp S_2 = \{1, 7, 9\}$, the Deletion Theorem 2.3 shows that $\mathscr{A}_1^{G_9}$ is free with $\exp \mathscr{A}_1^{G_9} = \{1, 7, 8\}$. We write this symbolically as type $\mathscr{A}_1^{G_9} = S_2 - T_1$. Now we use the Addition Theorem to show that $\mathscr{A}_1$ is free with $\exp \mathscr{A}_1 = \{1, 6, 7, 8\}$, and we write type $\mathscr{A}_1 - \text{type}\,\mathscr{B} = S_2 - T_1$.

The hyperplanes may be added in several different orders. We choose the admissible order G_9, G_8, G_7, G_4, G_5, G_6, G_3, G_2, G_1. This order is labeled $H_1, \dots, H_9$ in the corresponding *addition table*, Table 5. At each step the restriction must be checked as above. The symbolic formula becomes $\text{type}\,\mathscr{A} - \text{type}\,\mathscr{B} = (S_2 - T_1) + (S_2 - T_1) + (S_2 - T_1) + (S_2 - T_1) + S_2 + (S_1 - 2T_2) + (S_1 - T_2) + (S_1 - T_2) + S_1$. We collect terms symbolically to get the structure formula

$$E_8^{A_2^2} - E_7^{A_1\times A_2} = 4E_8^{A_1\times A_2^2} + 5E_8^{A_2\times A_3} - 4E_8^{A_1\times A_2\times A_3} - 4E_8^{A_1^2\times A_2^2}.$$

It follows from Proposition 4.1 that this formula is independent of the admissible order.

5. Addition tables. We present addition tables for the six mixed pairs. Table 3 lists our choice of coordinates.

TABLE 3. Coordinates.

type $\mathscr{A}$		type $\mathscr{B}$	
$E_7^{A_1^2}$	x_4-x_5, x_6-x_7	$E_6^{A_1}$	x_4-x_5
$E_8^{A_2^2}$	$x_3-x_4, x_4-x_5, x_6-x_7, x_7-x_8$	$E_7^{A_1\times A_2}$	$x_3-x_4, x_4-x_5, x_6-x_7$
$E_8^{A_1\times A_2}$	$x_4-x_5, x_6-x_7, x_7-x_8$	$E_7^{A_1^2}$	x_4-x_5, x_6-x_7
$E_8^{A_1^3}$	$x_3-x_4, x_5-x_6, x_7-x_8$	$E_7^{A_1^2}$	x_3-x_4, x_5-x_6
$E_8^{A_2}$	x_6-x_7, x_7-x_8	$E_7^{A_1}$	x_6-x_7
$E_8^{A_1^2}$	x_5-x_6, x_7-x_8	$E_7^{A_1}$	x_5-x_6

TABLE 4. Addition from $E_6^{A_1}$ to $E_7^{A_1^2}$.

$\exp\mathscr{A}_{i-1}$	α_i	$\mathscr{A}_i^{H_i}$	$\exp\mathscr{A}_i^{H_i}$
1, 5, 7, 4, 8	$x_4+x_6+x_7$	$E_7^{A_1\times A_2}$	1, 5, 7, 8
1, 5, 7, 5, 8	s_7-x_5	$E_7^{A_1\times A_2}$	1, 5, 7, 8
1, 5, 7, 6, 8	s_7-x_3	$E_7^{(A_1^3)''}-E_7^{A_1^4}$	1, 5, 7, 8
1, 5, 7, 7, 8	s_7-x_2	$E_7^{(A_1^3)''}-E_7^{A_1^4}$	1, 5, 7, 8
1, 5, 7, 8, 8	s_7-x_1	$E_7^{(A_1^3)''}-E_7^{A_1^4}$	1, 5, 7, 8
1, 5, 7, 8, 9	$x_1+x_6+x_7$	$E_7^{(A_1^3)''}$	1, 5, 7, 9
1, 5, 7, 9, 9	$x_2+x_6+x_7$	$E_7^{(A_1^3)''}$	1, 5, 7, 9
1, 5, 7, 9, 10	$x_3+x_6+x_7$	$E_7^{(A_1^3)''}$	1, 5, 7, 9
1, 5, 7, 9, 11			

TABLE 5. Addition from $E_7^{A_1\times A_2}$ to $E_8^{A_2^2}$.

$\exp\mathscr{A}_{i-1}$	α_i	$\mathscr{A}_i^{H_i}$	$\exp\mathscr{A}_i^{H_i}$
1, 7, 5, 8	$s_8-x_4-x_5$	$E_8^{A_2\times A_3}-E_8^{A_1\times A_2\times A_3}$	1, 7, 8
1, 7, 6, 8	$s_8-x_2-x_5$	$E_8^{A_2\times A_3}-E_8^{A_1\times A_2\times A_3}$	1, 7, 8
1, 7, 7, 8	$s_8-x_1-x_5$	$E_8^{A_2\times A_3}-E_8^{A_1\times A_2\times A_3}$	1, 7, 8
1, 7, 8, 8	s_8+x_3	$E_8^{A_2\times A_3}-E_8^{A_1\times A_2\times A_3}$	1, 7, 8
1, 7, 8, 9	s_8+x_6	$E_8^{A_2\times A_3}$	1, 7, 9
1, 7, 9, 9	$s_8-x_1-x_2$	$E_8^{A_1\times A_2^2}-2E_8^{A_1^2\times A_2^2}$	1, 7, 9
1, 7, 9, 10	s_8+x_2	$E_8^{A_1\times A_2^2}-E_8^{A_1^2\times A_2^2}$	1, 7, 10
1, 7, 10, 10	s_8+x_1	$E_8^{A_1\times A_2^2}-E_8^{A_1^2\times A_2^2}$	1, 7, 10
1, 7, 10, 11	$x_6+x_7+x_8$	$E_8^{A_1\times A_2^2}$	1, 7, 11
1, 7, 11, 11			

TABLE 6. Addition from $E_7^{A_1^2}$ to $E_8^{A_1\times A_2}$.

$\exp\mathscr{A}_{i-1}$	α_i	$\mathscr{A}_i^{H_i}$	$\exp\mathscr{A}_i^{H_i}$
1, 7, 11, 5, 9	$s_8-x_4-x_5$	$E_8^{A_1^2\times A_2}-3E_8^{A_1^3\times A_2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 9
1, 7, 11, 6, 9	s_8+x_1	$E_8^{A_1^2\times A_2}-3E_8^{A_1^3\times A_2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 9
1, 7, 11, 7, 9	s_8+x_2	$E_8^{A_1^2\times A_2}-3E_8^{A_1^3\times A_2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 9
1, 7, 11, 8, 9	s_8+x_3	$E_8^{A_1^2\times A_2}-3E_8^{A_1^3\times A_2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 9
1, 7, 11, 9, 9	s_8+x_6	$E_8^{A_1\times A_3}$	1, 7, 11, 9
1, 7, 11, 9, 10	$s_8-x_3-x_5$	$E_8^{A_2^2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 10
1, 7, 11, 10, 10	$s_8-x_2-x_5$	$E_8^{A_2^2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 10
1, 7, 11, 10, 11	$s_8-x_1-x_5$	$E_8^{A_2^2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 10
1, 7, 11, 10, 12	s_8+x_4	$E_8^{A_2^2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 10
1, 7, 11, 10, 13	$x_6+x_7+x_8$	$E_8^{A_1^2\times A_2}$	1, 7, 11, 13
1, 7, 11, 11, 13	$s_8-x_2-x_3$	$E_8^{A_1^2\times A_2}$	1, 7, 11, 13
1, 7, 11, 12, 13	$s_8-x_1-x_3$	$E_8^{A_1^2\times A_2}$	1, 7, 11, 13
1, 7, 11, 13, 13	$s_8-x_1-x_2$	$E_8^{A_1^2\times A_2}$	1, 7, 11, 13
1, 7, 11, 13, 14			

TABLE 7. Addition from $E_7^{A_1^2}$ to $E_8^{A_1^3}$.

$\exp\mathscr{A}_{i-1}$	α_i	$\mathscr{A}_i^{H_i}$	$\exp\mathscr{A}_i^{H_i}$
1, 7, 11, 5, 9	$s_8-x_4-x_6$	$E_8^{A_1\times A_3}$	1, 7, 11, 9
1, 7, 11, 6, 9	s_8+x_3	$E_8^{A_1^2\times A_2}-3E_8^{A_1^3\times A_2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 9
1, 7, 11, 7, 9	s_8+x_5	$E_8^{A_1^2\times A_2}-3E_8^{A_1^3\times A_2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 9
1, 7, 11, 8, 9	s_8+x_1	$E_8^{A_1^4}-4E_8^{A_1^3\times A_2}$	1, 7, 11, 9
1, 7, 11, 9, 9	s_8+x_2	$E_8^{A_1^4}-4E_8^{A_1^3\times A_2}$	1, 7, 11, 9
1, 7, 11, 9, 10	$s_8-x_1-x_6$	$E_8^{A_1^2\times A_2}-2E_8^{A_1^3\times A_2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 10
1, 7, 11, 10, 10	$s_8-x_1-x_4$	$E_8^{A_1^2\times A_2}-2E_8^{A_1^3\times A_2}-E_8^{A_1\times A_2^2}$	1, 7, 11, 10
1, 7, 11, 10, 11	$s_8-x_2-x_4$	$E_8^{A_1^2\times A_2}-2E_8^{A_1^3\times A_2}$	1, 7, 11, 11
1, 7, 11, 11, 11	$s_8-x_3-x_4$	$E_8^{A_1^4}-2E_8^{A_1^3\times A_2}$	1, 7, 11, 11
1, 7, 11, 11, 12	$s_8-x_2-x_6$	$E_8^{A_1^2\times A_2}-E_8^{A_1^3\times A_2}$	1, 7, 11, 12
1, 7, 11, 12, 12	$s_8-x_5-x_6$	$E_8^{A_1^4}-E_8^{A_1^3\times A_2}$	1, 7, 11, 12
1, 7, 11, 12, 13	s_8+x_7	$E_8^{A_1^2\times A_2}$	1, 7, 11, 13
1, 7, 11, 13, 13	$x_3+x_7+x_8$	$E_8^{A_1^2\times A_2}$	1, 7, 11, 13
1, 7, 11, 13, 14	$x_5+x_7+x_8$	$E_8^{A_1^2\times A_2}$	1, 7, 11, 13
1, 7, 11, 13, 15	$x_1+x_7+x_8$	$E_8^{A_1^4}$	1, 7, 11, 13
1, 7, 11, 13, 16	$x_2+x_7+x_8$	$E_8^{A_1^4}$	1, 7, 11, 13
1, 7, 11, 13, 17			

TABLE 8. Addition from $E_7^{A_1}$ to $E_8^{A_2}$.

$\exp \mathscr{A}_{i-1}$	α_i	$\mathscr{A}_i^{H_i}$	$\exp \mathscr{A}_i^{H_i}$
1, 7, 11, 13, 5, 9	s_8+x_6	$E_8^{A_3}$	1, 7, 11, 13, 9
1, 7, 11, 13, 6, 9	s_8+x_5	$E_8^{A_1\times A_2}-5E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 7, 9	s_8+x_4	$E_8^{A_1\times A_2}-5E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 8, 9	s_8+x_3	$E_8^{A_1\times A_2}-5E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 9, 9	s_8+x_2	$E_8^{A_1\times A_2}-5E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 9, 10	s_8+x_1	$E_8^{A_1\times A_2}-5E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 9, 11	$s_8-x_1-x_2$	$E_8^{A_1\times A_2}-3E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 11
1, 7, 11, 13, 10, 11	$s_8-x_1-x_3$	$E_8^{A_1\times A_2}-3E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 11
1, 7, 11, 13, 11, 11	$s_8-x_1-x_4$	$E_8^{A_1\times A_2}-3E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 11
1, 7, 11, 13, 11, 12	$s_8-x_1-x_5$	$E_8^{A_1\times A_2}-3E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 11
1, 7, 11, 13, 11, 13	$s_8-x_2-x_3$	$E_8^{A_1\times A_2}-E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 13
1, 7, 11, 13, 12, 13	$s_8-x_2-x_4$	$E_8^{A_1\times A_2}-E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 13
1, 7, 11, 13, 13, 13	$s_8-x_2-x_5$	$E_8^{A_1\times A_2}-E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 13
1, 7, 11, 13, 13, 14	$s_8-x_3-x_4$	$E_8^{A_1\times A_2}$	1, 7, 11, 13, 14
1, 7, 11, 13, 14, 14	$s_8-x_3-x_5$	$E_8^{A_1\times A_2}$	1, 7, 11, 13, 14
1, 7, 11, 13, 14, 15	$s_8-x_4-x_5$	$E_8^{A_1\times A_2}$	1, 7, 11, 13, 14
1, 7, 11, 13, 14, 16	$x_6+x_7+x_8$	$E_8^{A_1\times A_2}$	1, 7, 11, 13, 14
1, 7, 11, 13, 14, 17			

TABLE 9. Addition from $E_7^{A_1}$ to $E_8^{A_1^2}$.

$\exp \mathscr{A}_{i-1}$	α_i	$\mathscr{A}_i^{H_i}$	$\exp \mathscr{A}_i^{H_i}$
1, 7, 11, 13, 5, 9	$x_1+x_7+x_8$	$E_8^{A_1^3}-6E_8^{A_1^4}-2E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 6, 9	$x_2+x_7+x_8$	$E_8^{A_1^3}-6E_8^{A_1^4}-2E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 7, 9	$x_3+x_7+x_8$	$E_8^{A_1^3}-6E_8^{A_1^4}-2E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 8, 9	$x_4+x_7+x_8$	$E_8^{A_1^3}-6E_8^{A_1^4}-2E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 9, 9	$x_5+x_7+x_8$	$E_8^{A_1\times A_2}-5E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 9
1, 7, 11, 13, 9, 10	$s_8-x_1-x_5$	$E_8^{A_1\times A_2}-4E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 10
1, 7, 11, 13, 10, 10	$s_8-x_2-x_5$	$E_8^{A_1\times A_2}-4E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 10
1, 7, 11, 13, 10, 11	$s_8-x_3-x_5$	$E_8^{A_1\times A_2}-4E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 10
1, 7, 11, 13, 10, 12	$s_8-x_4-x_5$	$E_8^{A_1\times A_2}-4E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 10
1, 7, 11, 13, 10, 13	$s_8-x_5-x_6$	$E_8^{A_1^3}-6E_8^{A_1^4}-E_8^{A_1^2\times A_2}$	1, 7, 11, 13, 10
1, 7, 11, 13, 10, 14	s_8+x_1	$E_8^{A_1^3}-3E_8^{A_1^4}$	1, 7, 11, 13, 14
1, 7, 11, 13, 11, 14	s_8+x_2	$E_8^{A_1^3}-3E_8^{A_1^4}$	1, 7, 11, 13, 14
1, 7, 11, 13, 12, 14	s_8+x_3	$E_8^{A_1^3}-3E_8^{A_1^4}$	1, 7, 11, 13, 14
1, 7, 11, 13, 13, 14	s_8+x_4	$E_8^{A_1^3}-3E_8^{A_1^4}$	1, 7, 11, 13, 14
1, 7, 11, 13, 14, 14	s_8+x_5	$E_8^{A_1\times A_2}$	1, 7, 11, 13, 14
1, 7, 11, 13, 14, 15	s_8+x_7	$E_8^{A_1\times A_2}$	1, 7, 11, 13, 14
1, 7, 11, 13, 14, 16	$s_8-x_1-x_2$	$E_8^{A_1^3}-E_8^{A_1^4}$	1, 7, 11, 13, 16
1, 7, 11, 13, 15, 16	$s_8-x_1-x_3$	$E_8^{A_1^3}-E_8^{A_1^4}$	1, 7, 11, 13, 16
1, 7, 11, 13, 16, 16	$s_8-x_1-x_4$	$E_8^{A_1^3}-E_8^{A_1^4}$	1, 7, 11, 13, 16
1, 7, 11, 13, 16, 17	$s_8-x_2-x_3$	$E_8^{A_1^3}$	1, 7, 11, 13, 17
1, 7, 11, 13, 17, 17	$s_8-x_2-x_4$	$E_8^{A_1^3}$	1, 7, 11, 13, 17
1, 7, 11, 13, 17, 18	$s_8-x_3-x_4$	$E_8^{A_1^3}$	1, 7, 11, 13, 17
1, 7, 11, 13, 17, 19			

REFERENCES

[1] V. I. ARNOLD, Wave front evolution and equivariant Morse lemma, Comm. Pure Appl. Math. 29 (1976), 557–582.

[2] V. I. ARNOLD, Indices of singular points of 1-forms on a manifold with boundary, convolution of invariants of reflection groups, and singular projections of smooth surfaces, Russian Math. Surveys 34 (1979), 1–42.

[3] N. BOURBAKI, Groupes et Algèbres de Lie, Chapitres 4, 5 et 6, Hermann, Paris, 1968.

[4] P. EDELMAN AND V. REINER, A counterexample to Orlik's conjecture, Proc. Amer. Math. Soc., to appear.

[5] G. I. LEHRER AND T. SHOJI, On flag varieties, hyperplane complements and Springer representations of Weyl groups, J. Austral. Math. Soc. (A) 49 (1990), 449–485.

[6] P. ORLIK AND L. SOLOMON, Coxeter arrangements, in: Singularities (P. Orlik, ed.), Proc. Symp. Pure Math. 40 Part 2., Amer. Math. Soc., 1983, pp. 269–292.

[7] P. ORLIK, L. SOLOMON, AND H. TERAO, On Coxeter arrangements and the Coxeter number, in: Complex Analytic Singularities (T. Suwa and Ph. Wagreich, eds.), Adv. Studies in Pure Math. 8, Kinokuniya, Tokyo and North Holland, Amsterdam, 1987, pp. 461–477.

[8] P. ORLIK AND H. TERAO, Arrangements of hyperplanes, Grundleheren der mathermatischen Wissenschaften 300, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[9] K. SAITO, On the uniformization of complements of discriminant loci, in: Conference Notes, Amer. Math. Soc. Summer Institute, Williamstown, 1975.

[10] K. SAITO, Theory of logarithmic differential forms and logarithmic vector fields, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1981), 265–291.

[11] N. SPALTENSTEIN, Open Problem, in: Open Problems in Algebraic Groups, Proceedings of the Twelfth International Symposium, Division of Mathematics, The Taniguchi Foundation, R. Hotta and N. Kawanaka Organizing Committee, 1983.

[12] H. TERAO, Arrangements of hyperplanes and their freeness I, II, J. Fac. Sci. Univ. Tokyo 27 (1980), 293–320.

[13] H. TERAO, Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula, Invent. Math. 63 (1981), 159–179.

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