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FREE RESOLUTIONS FOR SEMI-DIRECT PRODUCTS

THOMAS BRADY

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Abstract. We define a notion of one group acting on a free resolution for another and show how this can lead to a free resolution for the semi-direct product. We apply this result to obtain a free resolution for dihedral groups.

In §1 below, we consider an action of a group H on a group K and we define the notion of an action of H on a free resolution for K. We show how this can be used to give a free resolution for the semi-direct product. In §2 we look at the case where $H=\boldsymbol{Z}_2$, $K=\boldsymbol{Z}_n$ and the semi-direct product is the dihedral group D_{2n}. The author wishes to thank Ken Brown for several helpful conversations and M.S.R.I., where most of this work was completed.

1. Actions of resolutions. Let H and K be groups, $\phi: H\to\mathrm{Aut}(K)$ be a group homomorphism and $G=K\rtimes_{\phi}H$. We write k^h for $\phi(h)k$.

DEFINITION 1.1. Given a free resolution $\varepsilon: F\to\boldsymbol{Z}$ for K we say that it admits an action of H compatible with ϕ if for all $h\in H$ there is an augmentation-preserving chain map $\tau(h): F\to F$ satisfying

(1) $\tau(h)[k\cdot f]=k^h\cdot[\tau(h)f]$ for all $k\in K$ and $f\in F$, and
(2) $\tau(h)\tau(h')=\tau(hh')$ for all $h, h'\in H$.

If such an action exists we can give F a G-module structure as follows. If $g\in G$ then g can be expressed uniquely as $g=kh$, with $k\in K$ and $h\in H$. We set $(kh)\cdot f=k\cdot\tau(h)f$.

PROPOSITION 1.2. *If $\varepsilon: F\to\boldsymbol{Z}$ is a free resolution for K, which admits an action of H compatible with ϕ and $\varepsilon': P\to\boldsymbol{Z}$ is any free resolution for H, then $\varepsilon\otimes\varepsilon': F\otimes P\to\boldsymbol{Z}$ is a free resolution for $G=K\rtimes_{\phi}H$. Moreover, if F is K-free on $\{f_i\}_{i\in I}$ and P is H-free on $\{p_j\}_{j\in J}$ then $F\otimes P$ is G-free on $\{f_i\otimes p_j\}_{i\in I, j\in J}$.*

PROOF. $F\otimes P$ is an acyclic complex of free abelian groups. Thus it suffices to show that it has a G-module structure and that it is G-free. P has a G-module structure given by $(kh)\cdot p=h\cdot p$ and we give $F\otimes P$ the diagonal action. Thus $kh\cdot(f\otimes p)=[k\cdot(\tau(h)f)]\otimes[h\cdot p]$. To show that $F\otimes P$ is a free G-module we first observe that $P=\bigoplus_{j\in J}P_j$ where each P_j is a copy of the free H-module $\boldsymbol{Z}H$. Then $F\otimes P=\bigoplus_{j\in J}(F\otimes P_j)$, so it suffices to show that $F\otimes\boldsymbol{Z}H$ is G-free. We now use the follow-

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ing two facts about induced modules:

(1) If F is a G-module and $K<G$, then [B, p. 69] there is a G-module isomorphism

$$F\otimes \mathbf{Z}[G/K]\approx \mathrm{Ind}_K^G \mathrm{Res}_K^G F,$$

where $F\otimes \mathbf{Z}[G/K]$ is given the diagonal G-action.

(2) If M is a free K-module then $\mathrm{Ind}_K^G M$ is a free G-module.

In our case $\mathrm{Res}_K^G F$ is the original K-module F, which was assumed to be K-free. Thus $F\otimes \mathbf{Z}H=F\otimes \mathbf{Z}[G/K]$ is G-free. The second assertion of the Proposition follows from the proof of the above facts.

NOTE 1.3. Furthermore, if such an action exists we get an explicit expression for the differential map, d, in terms of the basis $\{f_i\otimes p_j\}$, i.e., we can write $d(f\otimes p)$ as a sum of terms of the form $kh(f'\otimes p')$. To see this, first note that

$$d(f\otimes p)=d(f)\otimes p+(-1)^{\deg(f)}f\otimes d(p)=d_0(f\otimes p)+d_1(f\otimes p)$$

using the notation of Wall in [W]. The first term, $d_0(f\otimes p)$, consists of a sum of terms of the form $k(f')\otimes p$. However, since K acts trivially on P, $k(f')\otimes p=k(f'\otimes p)$. As for the term $d_1(f\otimes p)$, it consists of a sum of terms of the form $f\otimes h(p')$. By definition of the action $f\otimes h(p')=h(\tau(h^{-1})(f)\otimes p')$ which in turn consists of terms of the form $h(k(f')\otimes p')$. Finally,

$$h(k(f')\otimes p')=hk(f'\otimes p')=k^h h(f'\otimes p').$$

2. Dihedral groups. Now let $K=\mathbf{Z}_n=\langle x\,|\,x^n=1\rangle$, $H=\mathbf{Z}_2=\langle x\,|\,x^2=1\rangle$ and $\phi\colon \mathbf{Z}_2\to\mathrm{Aut}(\mathbf{Z}_n)$ be given by $\phi(y)x=x^{-1}$. Set

$$G=D_{2n}=\mathbf{Z}_n\rtimes_\phi \mathbf{Z}_2=\langle x,y\,|\,x^n=y^2=1,\ yxy^{-1}=x^{-1}\rangle.$$

For $i\geq 0$, let F_i be a copy of $\mathbf{Z}[\mathbf{Z}_n]$ with generator denoted by a_i and set $F=\bigoplus_i F_i$. Define $d_K\colon F\to F$ by $d_K(a_i)=\sigma a_{i-1}$ where

$$\begin{array}{ll} \sigma=N_x & i=2m \\ \sigma=x-1 & i=2m+1\,. \end{array}$$

Here $N_x=1+x+x^2+\cdots+x^{n-1}$. The resolution (F,d_K) is the usual free resolution for $\mathbf{Z}_n$ [B, p. 21].

We define $\tau(y)\colon F\to F$ by $\tau(y)a_i=\gamma a_i$ where

$$\begin{array}{ll} \gamma=1 & i=4m \\ \gamma=-x^{-1} & i=4m+1 \\ \gamma=-1 & i=4m+2 \\ \gamma=x^{-1} & i=4m+3\,. \end{array}$$

It is easy to check that $\tau(y)d_K = d_K\tau(y)$ and $[\tau(y)]^2 = \mathrm{id}_F$. Thus we have an action of $\boldsymbol{Z}_2$ on F, which is compatible with ϕ. So, if (P, d_H) is the usual resolution for the cyclic group H of order $n=2$, then $F\otimes P$ is a free resolution for D_{2n} by Proposition 1.2.

In order to write out the resolution explicitly we observe that

$$(F\otimes P)_n = \bigoplus_{p+q=n} F_p \otimes P_q$$

and $F_p\otimes P_q$ is a copy of $\boldsymbol{Z}[D_{2n}]$ with generator $a_p\otimes b_q$. The differential is given, as usual, by

$$d(a_p\otimes b_q) = (d_K a_p)\otimes b_q + (-1)^p a_p \otimes (d_H b_q)\,.$$

Thus in the notation of [W], $d=d_0+d_1$, where $d_0(a_p\otimes b_q)=\alpha(a_{p-1}\otimes b_q)$ and $d_1(a_p\otimes b_q)=\beta(a_p\otimes b_{q-1})$.

Here

$$\begin{array}{ll} \alpha = x-1 & p=2r+1 \\ \alpha = N_x & p=2r \end{array}$$

and

$$\begin{array}{lll} \beta = y-1 & p=4r\,, & q=2s+1 \\ \beta = y+1 & p=4r\,, & q=2s \\ \beta = xy+1 & p=4r+1\,, & q=2s+1 \\ \beta = xy-1 & p=4r+1\,, & q=2s \\ \beta = -y-1 & p=4r+2\,, & q=2s+1 \\ \beta = -y+1 & p=4r+2\,, & q=2s \\ \beta = -xy+1 & p=4r+3\,, & q=2s+1 \\ \beta = -xy-1 & p=4r+3\,, & q=2s\,. \end{array}$$

NOTE 2.1. This resolution for D_{2n} is closely related to the one given by Hamada in [H]. In fact, if the latter resolution is expressed as a double complex by giving the module generator a_i^j the bidegree $(i-j+1, j-1)$, then the corresponding maps d_0 and d_1 can be shown to agree with those above, up to a factor of ± 1.

REFERENCES

[B] K. S. BROWN, Cohomology of groups, Springer-Verlag GTM 87, New York, 1982.

[H] S. HAMADA, On a free resolution of a dihedral group, Tôhoku Math. J. 15 (1963), 212–219.

[W] C. T. C. WALL, Resolutions for extensions of groups, Proc. Camb. Phil. Soc. 57 (1961), 251–255.

DEPARTMENT OF MATHEMATICS
BRIGHAM YOUNG UNIVERSITY
PROVO, UT 84602
U.S.A.