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AN EXTENSION OF UNICITY THEOREM FOR MEROMORPHIC FUNCTIONS

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Abstract. Meromorphic functions on the complex plane which have the same inverse images counting multiplicities for four values are Möbius transforms of each other. The aim of this paper is to give an extension of this statement to moving targets.

1. Introduction. We say that two meromorphic functions f and g on $\boldsymbol{C}$ share the value a if the zeros of $f-a$ and $g-a$ ($1/f$ and $1/g$ if $a=\infty$) are the same. Nevanlinna [4] proved the following theorems:

THEOREM A. *If two distinct nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share four values $a_1, \dots, a_4$ by counting multiplicities, then g is a Möbius transformation of f, two shared values, say a_3 and a_4, are Picard values, and the cross ratio $(a_1, a_2, a_3, a_4)=-1$.*

THEOREM B. *If two nonconstant meromorphic functions f and g share five values, then $f \equiv g$.*

In this paper, we give an extension of Theorem A by using the results of moving targets in [6] and [10]. An extension of Theorem B is conjectured, but the second main theorem for moving targets corresponding to that playing the main role in the proof of Theorem B is not proved yet.

2. Preliminaries from the value distribution theory. In this section, we define the tools in the value distribution theory.

The complex projective space of dimension 1 is denoted by $\boldsymbol{P}^1(\boldsymbol{C})$ and its homogeneous coordinate system by $(w_0 : w_1)$. Let f be a holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{P}^1(\boldsymbol{C})$. A holomorphic mapping $\tilde{f}=(f_0, f_1)$ of $\boldsymbol{C}$ into $\boldsymbol{C}^2$ is called a representation of f if $\tilde{f} \not\equiv \boldsymbol{o}$, where $\boldsymbol{o}$ is the origin of $\boldsymbol{C}^2$ and $f(z)=(f_0(z) : f_1(z))$ for each $z \in \boldsymbol{C}-\tilde{f}^{-1}(\boldsymbol{o})$. Moreover, if $\tilde{f}(z) \neq \boldsymbol{o}$ for any $z \in \boldsymbol{C}$, it is said to be reduced. In the rest of this section, let $\tilde{f}=(f_0, f_1)$ be a reduced representation of f. Then, we identify f with the meromorphic function f_1/f_0 if $f_0 \not\equiv 0$. Otherwise, we identify it with the constant mapping taking the point at infinity as its value. Also, we denote by f^* the holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{P}^1(\boldsymbol{C})$ with the reduced representation $(-f_1, f_0)$.

Let r_0 be a fixed positive number. The characteristic function of f is defined by

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$$T_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\log\|\tilde{f}(re^{i\theta})\|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log\|\tilde{f}(r_0e^{i\theta})\|d\theta \tag{2.1}$$

for $r\geq r_0$, where $\|w\|=(|w_0|^2+|w_1|^2)^{1/2}$ for $w=(w_0, w_1)$ is the norm in C^2. The characteristic function is non-negative, and if f is nonconstant, then $T_f(r)\to\infty$ as $r\to\infty$.

If $f\not\equiv 0$, we define the counting function of f for 0 by

$$N_{f;0}(r)=\int_{r_0}^{r}\frac{n_f(t)}{t}dt \tag{2.2}$$

for $r\geq r_0$, where $n_f(t)$ is the sum of the multiplicities of the zeros of f in $\{z; |z|\leq t\}$. Obviously, $N_{f;0}(r)\geq 0$. For $a\in C$, we define the counting functions $N_{f;a}(r):=N_{f-a;0}(r)$ of f for a if $f\not\equiv a$, and $N_{f;\infty}(r):=N_{1/f;0}(r)$ of f for ∞ if $f\not\equiv\infty$. By the Poisson-Jensen formula, we have

$$N_{f;0}(r)-N_{f;\infty}(r)=\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log|f(r_0e^{i\theta})|d\theta .$$

If $h:=a_0f_0+a_1f_1\not\equiv 0$ for a holomorphic mapping a of C into $P^1(C)$ with a reduced representation (a_0, a_1), then the counting function $N_{f,a}(r)$ of f for a is defined as $N_{h;0}(r)$.

REMARK. We have defined two kinds of counting functions $N_{f;a}(r)$ and $N_{f,a}(r)$ for $a\in\bar{C}:=C\cup\{\infty\}$ which is a constant holomorphic mapping of C into $P^1(C)$. However, if f is entire, then $N_{f;a}(r)=N_{f,a^*}(r)$.

We use the notation $\mathcal{M}$ to represent the meromorphic function field on C. For a subfield $\mathcal{K}$ of $\mathcal{M}$, put $\bar{\mathcal{K}}=\mathcal{K}\cup\{\infty\}$. If f is nonconstant, we define $\Gamma_f=\{h\in\mathcal{M}; T_h(r)=o(T_f(r))\,(r\to\infty)\}$ which is a field. Also, if $f\not\equiv\infty$, we define the proximity function of f for ∞ by

$$m_{f;\infty}(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta ,$$

where $\log^+ x=\log(\max(1, x))$ for $x\geq 0$, and if $f\not\equiv a$ for $a\in\mathcal{M}$, the proximity function of f for a is defined by $m_{f;a}(r):=m_{1/(f-a);\infty}(r)$. It is easy to see that

$$T_f(r)=N_{f;a}(r)+m_f(r)+O(1) \tag{2.3}$$

if $f\not\equiv a$ for $a\in\bar{C}$.

If f is nonconstant and $a\in\bar{\Gamma}_f$, then the defect of f for a is defined by

$$\delta(f, a)=\liminf_{r\to\infty}\left(1-\frac{N_{f,a}(r)}{T_f(r)}\right).$$

It follows from (2.3) and the non-negativity of characteristic functions, counting functions and proximity functions that $0\leq\delta(f, a)\leq 1$.

We use the notation "$A(r) \le B(r) //$" to mean $A(r) \le B(r)$ for all $r \in (r_0, \infty) - E$, where E is a subset of (r_0, ∞) of finite Lebesgue measure. We complete this section with the following lemma (for the proof, see [3, Chapter 3] and [9]):

LEMMA 2.1. *For a nonconstant meromorphic function h on $\boldsymbol{C}$ and $j = 1, 2, \dots,$*

$$m_{h^{(j)}/h;\infty}(r) = o(T_h(r)) // \qquad \text{as} \quad r \to \infty .$$

3. Defect relation and Borel's lemma. Let f be a nonconstant holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{P}^1(\boldsymbol{C})$ with a reduced representation $\tilde{f} = (f_0, f_1)$.

THEOREM 3.1. *If $a_1, \dots, a_q \in \bar{\Gamma}_f$ are distinct, then for $\varepsilon > 0$*

$$(q - 2 - \varepsilon) T_f(r) \le \sum_{j=1}^{q} N_{f,a_j}(r) + o(T_f(r)) // .$$

COROLLARY 3.2. *If $a_1, \dots, a_q \in \bar{\Gamma}_f$ are distinct, then*

$$\sum_{j=1}^{q} \delta(f, a_j) \le 2 .$$

This is an extension of Nevanlinna's defect relation and was obtained by Steinmetz [10]. We do not give the proof for this theorem, but prove the following theorem called Borel's lemma:

THEOREM 3.3. *Let $N \ge 2$ be an integer, $F_1, \dots, F_N$ nonvanishing entire functions, and $a_1, \dots, a_N$ meromorphic functions such that $a_j \not\equiv 0$ and*

$$T_{a_j}(r) = o(T(r)) // \qquad \text{as} \quad r \to \infty \tag{3.1}$$

$(1 \le j \le N)$, where $T(r) = \sum_{j=1}^{N} T_{F_j}(r)$. Assume that

$$a_1 F_1 + \cdots + a_N F_N \equiv 1 . \tag{3.2}$$

Then, $a_1 F_1, \dots, a_N F_N$ are linearly dependent over $\boldsymbol{C}$.

PROOF. Put $G_j = a_j F_j$. Assume that $G_1, \dots, G_N$ are linearly independent over $\boldsymbol{C}$. Then some G_j are not constant. The Wronskian determinant W of $G_1, \dots, G_N$ is not identically zero, i.e.,

$$W = \begin{vmatrix} G_1 & \cdots & G_N \\ G_1' & \cdots & G_N' \\ & \cdots & \\ G_1^{(N-1)} & \cdots & G_N^{(N-1)} \end{vmatrix} \not\equiv 0 .$$

Denote the $(1, j)$-minors of W by W_j. Then

$$G_j = \frac{(-1)^{j+1} W_j}{G_1 \cdots G_{j-1} G_{j+1} \cdots G_N} \cdot \left(\frac{W}{G_1 \cdots G_N} \right)^{-1} = (-1)^{j+1} \Delta_j / \Delta \tag{3.3}$$

by (3.2), where

$$(3.4)\qquad \Delta=\begin{vmatrix} 1 & \cdots & 1 \\ G_1'/G_1 & \cdots & G_N'/G_N \\ & \cdots & \\ G_1^{(N-1)}/G_1 & \cdots & G_N^{(N-1)}/G_N \end{vmatrix}$$

and Δ_j are its $(1,j)$-minors. By (3.3) and (2.3), we get

$$m_{G_j;\infty}(r)\leq m_{\Delta_j;\infty}(r)+T_\Delta(r)+O(1)=m_{\Delta_j;\infty}(r)+m_{\Delta;\infty}(r)+o(T(r))\,//\,.$$

By applying Lemma 2.1 to each of Δ, Δ_j, we obtain

$$m_{G_j;\infty}(r)\leq o(T(r))\,//\,.$$

Also, we can easily check that $\sum_{j=1}^{N} T_{G_j}(r)=T(r)(1+o(1))\,//$. Hence, $T(r)\leq o(T(r))\,//$, which is a contradiction. q.e.d.

4. Unicity Theorem. We extend Theorem A by dividing it into two parts.

Let f and g be distinct nonconstant meromorphic functions with reduced representations (f_0, f_1) and (g_0, g_1), respectively. Let a_j be distinct elements of $\bar{\Gamma}_f$ with reduced representations (a_{j0}, a_{j1}) $(1\leq j\leq 4)$. We define entire functions by $F_j=a_{j0}f_0+a_{j1}f_1$ and $G_j=a_{j0}g_0+a_{j1}g_1$. Then $F_j\not\equiv 0$. Also, we define meromorphic functions ψ_j by

$$(4.1)\qquad G_j=\psi_j F_j\,.$$

THEOREM 4.1. *If all ψ_j are nonvanishing entire functions, then there exist A, B, C, $D\in\Gamma_f$ such that $AD-BC\not\equiv 0$ and*

$$(4.2)\qquad g=\frac{Af+B}{Cf+D}\,.$$

PROOF. By (4.1), we get

$$\begin{pmatrix} a_{10} & a_{11} & -a_{10}\psi_1 & -a_{11}\psi_1 \\ a_{20} & a_{21} & -a_{20}\psi_2 & -a_{21}\psi_2 \\ a_{30} & a_{31} & -a_{30}\psi_3 & -a_{31}\psi_3 \\ a_{40} & a_{41} & -a_{40}\psi_4 & -a_{41}\psi_4 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \\ f_0 \\ f_1 \end{pmatrix}\equiv\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $(g_0, g_1, f_0, f_1)\not\equiv(0,0,0,0)$, the determinant of the 4×4 matrix above is identically equal to zero. By expanding it, we have

$$(4.3)\qquad b_{12}\psi_1\psi_2+b_{34}\psi_3\psi_4+b_{13}\psi_1\psi_3+b_{24}\psi_2\psi_4+b_{14}\psi_1\psi_4+b_{23}\psi_2\psi_3\equiv 0\,,$$

where

$$b_{12}=b_{34}=(a_{10}a_{21}-a_{11}a_{20})(a_{30}a_{41}-a_{31}a_{40})$$

$$b_{13}=b_{24}=-(a_{10}a_{31}-a_{11}a_{30})(a_{20}a_{41}-a_{21}a_{40})$$

$$b_{14}=b_{23}=(a_{10}a_{41}-a_{11}a_{40})(a_{20}a_{31}-a_{21}a_{30})\,.$$

For distinct j and k, we have

$$\frac{\psi_j}{\psi_k}-1=\frac{(a_{j1}a_{k0}-a_{j0}a_{k1})(f_0g_1-f_1g_0)}{F_jG_k}\,. \tag{4.4}$$

Since $F_l(z)=G_l(z)=0$ implies $f_0(z)g_1(z)-f_1(z)g_0(z)=0$,

$$N_{\psi_j/\psi_k;\,1}(r)\geq\sum_{l\neq j,k}N_{f,a_l}(r)+o(T_f(r))\,.$$

Hence, if $\#\{j,k,\mu,\nu\}\geq 3$, by (2.3) and Theorem 3.1

$$\begin{aligned}T_{\psi_j/\psi_k}(r)+T_{\psi_\mu/\psi_\nu}(r)&\geq N_{\psi_j/\psi_k;1}(r)+N_{\psi_\mu/\psi_\nu;1}(r)+O(1)\\&\geq\sum_{l\neq j,k}N_{f,a_l}(r)+\sum_{l\neq\mu,\nu}N_{f,a_l}(r)+o(T_f(r))\\&\geq\frac{1}{2}\,T_f(r)+o(T_f(r))\,//\,.\end{aligned}$$

Applying Theorem 3.3 to the identity obtained from (4.3)

$$\frac{b_{12}\psi_1}{b_{23}\psi_3}+\frac{b_{34}\psi_4}{b_{23}\psi_2}+\frac{b_{13}\psi_1}{b_{23}\psi_2}+\frac{b_{24}\psi_4}{b_{23}\psi_3}+\frac{b_{14}\psi_1\psi_4}{b_{23}\psi_2\psi_3}\equiv-1\,,$$

we have a shorter identity

$$\alpha_{12}b_{12}\psi_1\psi_2+\alpha_{34}b_{34}\psi_3\psi_4+\alpha_{13}b_{13}\psi_1\psi_3+\alpha_{24}b_{24}\psi_2\psi_4+\alpha_{14}b_{14}\psi_1\psi_4\equiv 0\,,$$

where α_{jk} are constants not all zero. By applying Theorem 3.3 successively, we deduce that some $(b_{jk}\psi_k)/(b_{jl}\psi_l)$ are nonzero constants, where $b_{jk}=b_{kj}$ if $j>k$. The conclusion of the theorem follows from this. q.e.d.

We state the second part of our extension of Theorem A. Let $A, B, C, D\in\mathscr{M}$ such that $AD-BC\not\equiv 0$. We define the mapping $S\colon\bar{\mathscr{M}}\to\bar{\mathscr{M}}$ by

$$S(F)=\begin{cases}(AF+B)/(CF+D) & (F\in\mathscr{M})\\ A/C & (F\equiv\infty)\,.\end{cases}$$

For a nonconstant meromorphic function f, we define the condition $\mathrm{P}(f)$ by

$$\mathrm{P}(f)\qquad N_{h;0}(r)+N_{h;\infty}(r)=o(T_f(r))\qquad(r\to\infty)$$

for $h\in\mathscr{M}$.

Remark. The conclusion of Theorem 4.1 is true under the weaker assumption that all ψ_j satisfy the condition $\mathrm{P}(f)$.

THEOREM 4.2. *Assume that* A, B, C, $D \in \Gamma_f$ *and that*

$$g = S(f) . \tag{4.5}$$

Moreover, assume that all ψ_j *satisfy the condition* $\mathrm{P}(f)$. *Then, for two* j, *say* $j=3, 4$, F_j *satisfy the condition* $\mathrm{P}(f)$, *and the meromorphic function of cross ratio* $(a_1^*, a_2^*, a_3^*, a_4^*)$ *is identically equal to* -1.

REMARK. Under the assumption above, the two conditions $\mathrm{P}(f)$ and $\mathrm{P}(g)$ are equivalent.

PROOF. It follows from (4.5) that

$$\frac{\psi_j}{\psi_k} = \frac{(Ba_{j1}+Da_{j0})f_0+(Aa_{j1}+Ca_{j0})f_1}{F_j} \frac{F_k}{(Ba_{k1}+Da_{k0})f_0+(Aa_{k1}+Ca_{k0})f_1} . \tag{4.6}$$

For distinct j and k, the common zeros of F_j and F_k are the zeros of $a_{j0}a_{k1}-a_{j1}a_{k0}$ ($\not\equiv 0$) which satisfies $\mathrm{P}(f)$, and also, the common zeros of F_j and $(Ba_{j1}+Da_{j0})f_0+(Aa_{j1}+Ca_{j0})f_1$ are the zeros of $(Ba_{j1}+Da_{j0})a_{j1}-(Aa_{j1}+Ca_{j0})a_{j0}$. Unless

$$(Ba_{j1}+Da_{j0})a_{j1}-(Aa_{j1}+Ca_{j0})a_{j0} \equiv 0 , \tag{4.7}$$

it satisfies $\mathrm{P}(f)$. Therefore, in this case, since ψ_j/ψ_k satisfies $\mathrm{P}(f)$,

$$N_{F_j;0}(r) = o(T_f(r)) \qquad \text{as} \quad r \to \infty . \tag{4.8}$$

We conclude that at least one condition among (4.7) and (4.8) holds for each $j=1, \dots, 4$. However, the number of j's which satisfy (4.8) and (4.7), respectively, is at most two. Therefore, we may assume that for $j=1, 2$, (4.7) holds, but (4.8) does not, and that for $j=3, 4$, (4.8) holds, but (4.7) does not. In (4.6), we consider the case $j=3$, $k=1$. Then, we deduce that $(Ba_{31}+Da_{30})f_0+(Aa_{31}+Ca_{30})f_1$ satisfies $\mathrm{P}(f)$. However, (4.7) does not holds for $j=3$. It follows from these and Theorem 3.1 that

$$(Ba_{31}+Da_{30})a_{41}-(Aa_{31}+Ca_{30})a_{40} \equiv 0 .$$

Similarly, we have

$$(Ba_{41}+Da_{40})a_{31}-(Aa_{41}+Ca_{40})a_{30} \equiv 0 .$$

We obtain from these two identities

$$S(a_4^*) = a_3 , \qquad S(a_3^*) = a_4 . \tag{4.9}$$

Also, we have

$$S(a_j^*) = a_j^* \qquad (j=1, 2) \tag{4.10}$$

by (4.8). From (4.9) and (4.10), the identity $(a_1^*, a_2^*, a_3^*, a_4^*) \equiv -1$ is deduced. q.e.d.

We give an analogue of Theorem B.

COROLLARY 4.3. *Let f and g be nonconstant meromorphic functions with reduced representations (f_0, f_1) and (g_0, g_1), respectively, and $a_j \in \bar{\Gamma}_f$ distinct with reduced representations (a_{j0}, a_{j1}) $(1 \leq j \leq 5)$. Assume that all ψ_j defined by* (4.1) *are entire functions without zeros. Then, $f \equiv g$.*

PROOF. Assume that $f \not\equiv g$. Then, it follows from Theorems 4.1 and 4.2 that for two j in $\{1, 2, 3, 4\}$, say $j=3, 4$, F_j satisfy the condition P(f). In the same way, F_j satisfy the condition P(f) for two j in $\{1, 2, 3, 5\}$. Hence, the number of j in $\{1, 2, 3, 4, 5\}$ such that F_j satisfy the condition P(f) is three or four, a contradiction to Theorem 3.1. q.e.d.

In Corollary 4.3, F_j and G_j are required to have the same zeros counting multiplicities. However, Theorem B does not count the multiplicities. The following should be a complete extension of Theorem B:

CONJECTURE. We have $f \equiv g$, if F_j and G_j have the same zeros for each $j=1, \dots, 5$ (not counting multiplicities).

REFERENCES

[1] H. CARTAN, Sur les zéros des combinaisons linéaires de p fonctions holomorphes données, Mathematica (1933), 5–31.

[2] G. HIROMI AND M. OZAWA, On the existence of analytic mappings between two ultrahyperelliptic surfaces, Kodai Math. Sem. Rep. 17 (1965), 281–306.

[3] W. K. HAYMAN, Meromorphic functions, Clarendon Press, Oxford, 1964.

[4] R. NEVANLINNA, Einige Eindeutigkeitssätze in der Theorie der meromorphen Funktionen, Acta Math. 48 (1926), 367–391.

[5] J. NOGUCHI AND T. OCHIAI, Geometric function theory in several complex variables, Transl. Math. Monographs 80, Amer. Math. Soc., 1990.

[6] M. RU AND W. STOLL, The second main theorem for moving targets, J. Geometric Analysis 1 (1991), 99–138.

[7] B. SHIFFMAN, Introduction to the Carlson-Griffiths equidistribution theory, Lecture Notes in Math. 981, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983, 44–89.

[8] M. SHIROSAKI, On defect relations of moving hyperplanes, Nagoya Math. J. 120 (1990), 103–112.

[9] M. SHIROSAKI, Another proof of the defect relation for moving targets, Tôhoku Math. J. 43 (1991), 355–360.

[10] N. STEINMETZ, Eine Verallgemeinerung des zweiten Nevanlinnaschen Hauptsazes, J. Reine Angew. Math. 368 (1986), 134–141.

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