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CONSTRUCTION OF HIGHER GENUS MINIMAL SURFACES WITH ONE END AND FINITE TOTAL CURVATURE

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Abstract. We prove that there exist complete minimal surfaces in the Euclidean 3-space with one Enneper-type end and finite total curvature which have two parameters j, k and are of genus jk, where j and k are positive integers. Our main problem is *the period problem*: each surface has j periods to be killed. We prove that these periods can be killed simultaneously.

Introduction. Recently, many minimal surfaces with higher genus and finite total curvature have been found. Costa [Co] found a complete minimal surface of genus one in $\boldsymbol{R}^3$. Hoffman and Meeks [HM2] proved its embeddedness, and they found higher genus embedded surfaces which are similar to Costa's surface, but with higher-order rotational symmetry. Wohlgemuth [W] proved rigorously the existence of several higher genus minimal surfaces which have embedded ends, including a surface which was found by Hoffman, Meeks and Callahan only numerically. In 1982, Chen-Gackstatter [CG] found surfaces which have one end and are of genera one and two. The genus one C-G surface was generalized by Karcher [K] and the genus two C-G surface was generalized by Thayer [T]. These generalizations are similar to C-G surfaces, but with higher winding order at the end.

Since minimal surfaces in $\boldsymbol{R}^3$ are given by Weierstrass data and path integrals, we always must check well-definedness of the surfaces, and this is called *the period problem*. Chen-Gackstatter [CG] also gave Weierstrass data for a genus three surface, but they did not solve its period problem. Thayer [T] conjectured that the period problem can be solved for arbitrary genus and gave numerical evidence to support this. Espírito-Santo [E] solved the genus three case with a numerical argument.

Rossman suggested to me *a homotopy argument* (which can be thought of as an intermediate value theorem of several variables) for solving the period problem. Wohlgemuth [W] used the homotopy argument to solve period problems for minimal surfaces with four embedded ends.

In this paper, we solve the period problem of the generalized C-G surfaces for arbitrary genus (2.1) by using the intermediate value theorem of several variables.

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MAIN THEOREM. *There exist one-ended complete minimal surfaces* $X(M_{j,k})$ *in* $\boldsymbol{R}^3$ *of genus* jk *with total curvature* $-4(j+1)k\pi$ *for all* $j, k=1, 2, \dots$.

In Section 1, we summarize some basic fact about minimal surface theory. In Section 2, we introduce the generalized C-G surfaces $X(M_{j,k})$ and discuss their symmetries. In Section 3, we show that each $X(M_{j,k})$ is complete and regular in $\boldsymbol{R}^3$, and we see that the symmetries in Section 2 can be used to reduce the number of periods from jk to j. Furthermore, we write down a concrete condition for solving the periods. In Section 4, we rewrite some of the results of Chen-Gackstatter [CG] and Thayer [T] in preparation for Section 5. In Section 5, we solve the period problem inductively, by assuming several inequalities. Finally in Section 6, we prove these inequalities.

REMARK. Once we prove the existence of certain value for the parameters $a_2, \dots, a_j, c\in\boldsymbol{R}$ so that the minimal immersion (2.1) is well-defined, we say that *the period problem is solved.* Though we could not show uniqueness for the value of $a_2, \dots, a_j, c$, our numerical computations suggest that the surfaces are unique. (Recently, in the case $j=2$ and $k=1$, Lopez, Martin and Rodriguez [LMR] have shown the uniqueness for the value a_2 and c.) More information about the numerical results and several conjectures are written in [T].

Thanks are also due to the referee and Dr. Rossman for valuable suggestions.

1. Basic properties. The following Theorems 1.1–1.3 were given by Osserman [Os1][Os2] (see also [HM1]).

THEOREM 1.1. *Let M be a Riemann surface,* $g: M\to\boldsymbol{C}\cup\{\infty\}$ *a meromorphic function and dh a holomorphic* 1*-form on M. We define a vector-valued* 1*-form* Φ *by*

$$\Phi=(\phi_1, \phi_2, \phi_3)=\left(\left(\frac{1}{g}-g\right)dh, i\left(\frac{1}{g}+g\right)dh, 2dh\right). \tag{1.1}$$

Then the real part

$$X(p)=\Re e\int_{p_0}^{p}\Phi \tag{1.2}$$

defines a minimal mapping into $\boldsymbol{R}^3$, *which is well-defined on M if and only if*

$$\Re e\oint_{\alpha}\Phi=0 \tag{1.3}$$

for all closed curves α *on M. Moreover, X is regular provided that the order of the poles and zeros of g coincide with the order of the zeros of dh.*

Conversely, every conformal minimal immersion $X: M\to\boldsymbol{R}^3$ *can be represented in the form* (1.1), (1.2) *for some meromorphic function g and holomorphic* 1*-form dh. Moreover, g is the stereographic projection of the Gauss map* $N: M\to S^2$ *of X.*

THEOREM 1.2. *Let $X: M \to \boldsymbol{R}^3$ be a complete isometric minimal immersion whose total Gaussian curvature is finite. Then there exists a compact Riemann surface $\bar{M}_k$ with genus k and a finite number of points $p_1, p_2, \dots, p_r$ on $\bar{M}_k$ so that there exists a conformal diffeomorphism between M and $\bar{M}_k \setminus \{p_1, p_2, \dots, p_r\}$.*

THEOREM 1.3. *Let $X: M \to \boldsymbol{R}^3$ be a complete minimal immersion. Then the total curvature of X is $-4\pi m$ where m is either a non-negative integer or $-\infty$.*

In the following we assume that X has finite total curvature.

2. Symmetries of the minimal surfaces. Here we set

$$F_j(z, a_2, \dots, a_j) = z \prod_{1 \le m \le j/2} (z^2 - a_{2m}^2) \prod_{1 \le n \le (j+1)/2} (z^2 - a_{2n-1}^2)^{-1},$$

where $j, k, m, n \in \boldsymbol{N}$, $a_1, \dots, a_j \in \boldsymbol{R}$, $1 = a_1 < a_2 < \cdots < a_j$. We define surfaces by the following

$$\begin{aligned}
&\bar{M}_{j,k} = \{(z, w) \in (\boldsymbol{C} \cup \{\infty\})^2 \mid w^{k+1} = F_j(z, a_2, \dots, a_j)\}, \\
&M_{j,k} = \begin{cases} \bar{M}_{j,k} \setminus \{(\infty, \infty)\} & \text{for } j \text{ even}, \\ \bar{M}_{j,k} \setminus \{(\infty, 0)\} & \text{for } j \text{ odd}, \end{cases} \\
&g = c_{j,k} w^k, \qquad c_{j,k} \in \boldsymbol{R}, \\
&dh = dz, \\
&\Phi = \left(\left(\frac{1}{g} - g \right) dh, \, i \left(\frac{1}{g} + g \right) dh, \, 2dh \right), \\
&X: M_{j,k} \to \boldsymbol{R}^3, \qquad p \mapsto \Re e \int_{p_0}^{p} \Phi,
\end{aligned} \tag{2.1}$$

where $p_0 = (0, 0)$, and each $c_{j,k}$ will be determined later.

$\bar{M}_{j,k}$ can be considered as a branched covering over the complex plane $\boldsymbol{C}$. Let $[\alpha, \beta]_+^l$ be the upper side of the interval $[\alpha, \beta]$ of the l-th sheet, and let $[\alpha, \beta]_-^l$ be the lower side. By cutting the Riemann sphere $\boldsymbol{C} \cup \{\infty\}$ along the alternating segments $[-a_j, -a_{j-1}]$, $[-a_{j-2}, -a_{j-3}], \dots, [a_{j-2}, a_{j-1}]$, $[a_j, \infty]$ (we denote these segments by $[\alpha, \beta]$), and then by pasting $[\alpha, \beta]_-^l$ with $[\alpha, \beta]_+^{l+1}$ for $1 \le l \le k$, and pasting $[\alpha, \beta]_-^{k+1}$ with $[\alpha, \beta]_+^1$, we observe that $\bar{M}_{j,k}$ is of genus jk. In the case $\bar{M}_{2,2}$, a rough sketch is given in Figure 2.1.

We introduce the following conformal mappings of $(\boldsymbol{C} \cup \{\infty\})^2$:

$$\gamma(z, w) = (\bar{z}, \bar{w}), \qquad \rho(z, w) = (-z, \zeta w),$$

where $\zeta = \exp(\pi i/(k+1))$.

The next lemma can be found in [HM1] and can be proven easily.

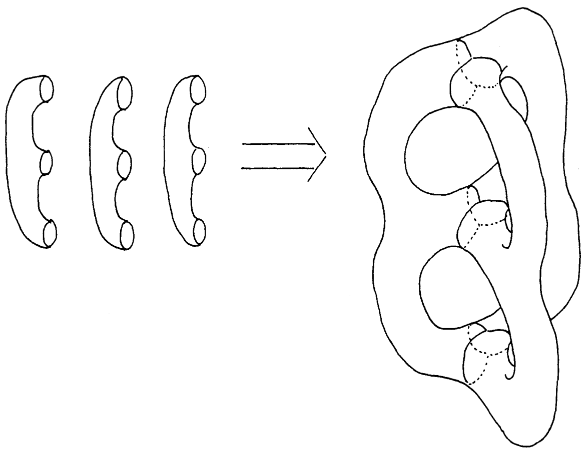


FIGURE 2.1.

LEMMA 2.1. *Let γ, ρ be as above. Then γ is of order 2 while ρ is of order $2k+2$. The group generated by γ and ρ is the dihedral group $D(2k+2)$ with $4k+4$ elements.*

We will need a homology basis of $\bar{M}_{j,k}$. Let $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_j$ be any fixed lift to $\bar{M}_{j,k}$ of the closed curves $\alpha_1, \ldots, \alpha_j$ about the intervals $[-a_j, -a_{j-1}]$, $[-a_{j-2}, -a_{j-3}], \ldots$, $[a_{j-2}, a_{j-1}]$ as shown in Figure 2.2. Then we define

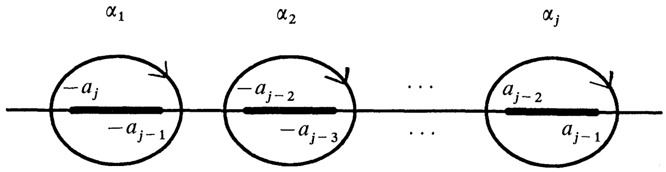


FIGURE 2.2.

$$H_{j,k} = \{s \circ \tilde{\alpha}_l \,|\, s \in D(2k+2),\ l = 1, \ldots, j\}\,.$$

It is easy to see that $H_{j,k}$ contains a homology basis of $\bar{M}_{j,k}$.

REMARK. γ can be omitted if we wish only to obtain a set containing a homology basis. But to see that $X(M_{j,k})$ has the same symmetries as the Costa-Hoffman-Meeks embedded minimal surfaces, it is valuable to consider γ.

LEMMA 2.2. *Consider the real orthogonal matrices*

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $\theta=\pi/(k+1)$. *Then*

$$\gamma^*\Phi=C\bar{\Phi}\,,\qquad \rho^*\Phi=R\Phi\,,$$

where Φ *is as in* (2.1).

PROPOSITION 2.3. *For any* $\tilde{\alpha}\in H_{j,k}$

$$\Re e\int_{\tilde{\alpha}}\Phi=0 \tag{2.2}$$

holds if and only if

$$\Re e\int_{\tilde{\alpha}_l}\Phi=0 \qquad \textit{for}\quad l=1,\dots,j\,. \tag{2.3}$$

PROOF. (2.2)⇒(2.3) is trivial. We assume that (2.3) holds. For any $\tilde{\alpha}\in H_{j,k}$, there exists an $s\in D(2k+2)$ and $\tilde{\alpha}_l$ $(1\le l\le j)$ such that $\tilde{\alpha}=s\circ\tilde{\alpha}_l$. By Lemma 2.2, there exists a real orthogonal matrix S such that

$$\Re e\int_{\tilde{\alpha}}\Phi=\Re e\int_{s\circ\tilde{\alpha}_l}\Phi=\Re e\int_{\tilde{\alpha}_l}s^*\Phi=S\Re e\int_{\tilde{\alpha}_l}\Phi\,.$$

Hence (2.2) holds. □

3. Completeness, regularity and period conditions of the surfaces. The information about the poles and zeros of g and dh is as follows: When j is even,

z	0	$\pm a_1$	$\pm a_2$	$\pm a_3$	$\cdots$	$\pm a_j$	∞
g	0^k	∞^k	0^k	∞^k	$\cdots$	0^k	∞^k
dh	0^k	0^k	0^k	0^k	$\cdots$	0^k	∞^{k+2}

and when j is odd,

z	0	$\pm a_1$	$\pm a_2$	$\pm a_3$	$\cdots$	$\pm a_j$	∞
g	0^k	∞^k	0^k	∞^k	$\cdots$	∞^k	0^k
dh	0^k	0^k	0^k	0^k	$\cdots$	0^k	∞^{k+2}.

By the tables above, in the case of $z=0$, $\pm a_1,\dots,\pm a_j$, the regularity of X holds, and in the case of $z=\infty$, completeness holds. All that remains to solve is *the period problem*, that is, the problem whether (1.3) holds. Again by the tables above, ϕ_1, ϕ_2, ϕ_3 are holomorphic 1-forms on $M_{j,k}$ and have poles only at $z=\infty$, so they have no residues. Hence, all we have to do is to check whether

$$\Re e\int_{\tilde{\alpha}}\Phi=0$$

holds for every closed curve $\tilde{\alpha}\in H_{j,k}$. By Proposition 2.3, this reduces to the following:

$$\Re e \int_{\tilde{\alpha}_l} \Phi = 0\,, \qquad l = 1, \dots, j\,, \tag{3.1}$$

where $\tilde{\alpha}_1, \dots, \tilde{\alpha}_j$ are as given in Section 2. This means that the actual number of periods to be killed is at most j. Moreover, since $\phi_3 = dz$ is exact,

$$\Re e = \int_{\tilde{\alpha}_l} \phi_3 = 0 \qquad l = 1, \dots, j$$

is always true. Hence (3.1) is equivalent to

$$\Re e \int_{\tilde{\alpha}_l} \phi_1 = \Re e \int_{\tilde{\alpha}_l} \phi_2 = 0 \qquad l = 1, \dots, j\,. \tag{3.2}$$

Now we introduce the notation (recall $g = c_{j,k} w^k$, $dh = dz$)

$$A_l = \int_{\tilde{\alpha}_l} \frac{1}{w^k} dz\,, \qquad B_l = \int_{\tilde{\alpha}_l} w^k dz\,.$$

Then rewriting (3.2) we obtain

$$\Re e\left(\frac{1}{c_{j,k}} A_l - c_{j,k} B_l\right) = \Re e\, i\left(\frac{1}{c_{j,k}} A_l + c_{j,k} B_l\right) = 0 \qquad l = 1, \dots, j\,, \tag{3.3}$$

which is equivalent to

$$c_{j,k}^2 = A_l \bar{B}_l^{-1} \qquad l = 1, \dots, j\,. \tag{3.4}$$

We will calculate A_l and B_l, as an example, in the case where $l = j$ and is an even number. Set $a = a_{j-2}$ and $b = a_{j-1}$. We can assume that the branches are taken as follows:

$$w = w(x) = \begin{cases} h(x)^{1/(k+1)} \in \boldsymbol{R} & \text{for} \quad x \in [a, b]_+^1\,, \\ e^{\varphi i} h(x)^{1/(k+1)} & \text{for} \quad x \in [a, b]_-^1 = [a, b]_+^2\,, \end{cases}$$

where

$$\varphi = \frac{2\pi}{k+1}\,, \qquad h(x) = F_j(x, a_2, \dots, a_j)\,,$$

and we can assume that $\tilde{\alpha}_j$ is the lift of the closed curve α_j (as in Figure 2.2) to the first sheet. Then $\tilde{\alpha}_j$ can be deformed as in Figure 3.1.

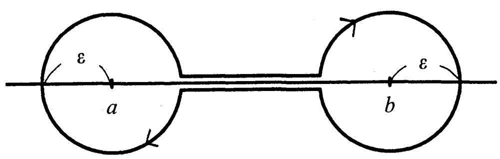


FIGURE 3.1.

w^{k+1} can be written as:

$$w^{k+1} = \frac{p(z)}{z - a_{j-1}} = \frac{p(z)}{z-b},$$

where $p(z)$ denotes a meromorphic function of z which is holomorphic in a neighborhood of b, $p(b) \neq 0$. Hence for small $\varepsilon > 0$, there exists M such that $|p(z)| \leq M$ for all $z \in \{z \mid |z-b| \leq \varepsilon\}$. We have the following inequality

$$\left| \int_{|z-b|=\varepsilon} w^k dz \right| = \int_{|z-b|=\varepsilon} |w^k| \, |dz| = \int_{|z-b|=\varepsilon} \frac{|p(z)|^{k/(k+1)}}{|z-b|^{k/(k+1)}} |dz|$$
$$\leq \frac{M^{k/(k+1)}}{\varepsilon^{k/(k+1)}} 2\pi\varepsilon = 2\pi M^{k/(k+1)} \varepsilon^{k/(k+1)}.$$

Therefore

$$\int_{|z-b|=\varepsilon} w^k dz \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Similarly,

$$\int_{|z-a|=\varepsilon} \frac{1}{w^k} dz \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

On the other hand, since $p(b) \neq 0$, obviously the following holds:

$$\int_{|z-b|=\varepsilon} \frac{1}{w^k} dz \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

and similarly,

$$\int_{|z-a|=\varepsilon} w^k dz \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Hence the integral path of A_j and B_j can be deformed as $\overrightarrow{[a, b]}^1_+ \cup \overleftarrow{[a, b]}^1_-$, that is,

$$A_j = \int_{\tilde{\alpha}_j} \frac{1}{w^k} dz = \int_a^b \frac{dx}{h(x)^{k/(k+1)}} + \int_b^a \frac{dx}{(e^{\varphi i} h(x)^{1/(k+1)})^k}$$
$$= (1 - e^{-k\varphi i}) \int_a^b \frac{dx}{h(x)^{k/(k+1)}},$$

$$B_j = \int_{\tilde{\alpha}_j} w^k dz = \int_a^b h(x)^{k/(k+1)} dx + \int_b^a (e^{\varphi i} h(x)^{1/(k+1)})^k dx$$
$$= (1 - e^{k\varphi i}) \int_a^b h(x)^{k/(k+1)} dx.$$

Since both $A_j' := \int_a^b h(x)^{-k/(k+1)}dx$ and $B_j' := \int_a^b h(x)^{k/(k+1)}dx$ are real positive numbers, if $c_{j,k}$ exists then it satisfies:

$$c_{j,k}^2 = \frac{(1-e^{-k\varphi i})A_j'}{\overline{(1-e^{k\varphi i})}B_j'} = \frac{A_j'}{B_j'} .$$

When $l=1,\dots,j-1$ and j is any positive integer, we get the following similarly to the above case:

$$c_{j,k} = \frac{A_l'}{B_l'} \qquad (l=1,\dots,j-1) .$$

Thus we can rewrite (3.4) as follows:

LEMMA 3.1. *Let X be the minimal immersion as in* (2.1). *Then X is well-defined if and only if*

$$c_{j,k}^2 := \frac{A_1'}{B_1'} = \frac{A_2'}{B_2'} = \cdots = \frac{A_j'}{B_j'} . \tag{3.5}$$

Furthermore, we introduce

$$f_{j,k,l}(a_2,\dots,a_j) = \frac{\displaystyle\int_{a_{l-1}}^{a_l} (s_{j,l}F_j(x,a_2,\dots,a_j))^{-k/(k+1)}dx}{\displaystyle\int_{a_{l-1}}^{a_l} (s_{j,l}F_j(x,a_2,\dots,a_j))^{k/(k+1)}dx}$$

for $l=1,\dots,j$ where the constant $s_{j,l}$ takes the value $+1$ or -1 to keep $s_{j,l}F_j$ positive on $[a_{l-1}, a_l]$. Note that the set $\{f_{j,k,l}(a_2,\dots,a_j) \mid l=1,\dots,j\}$ coincides the set $\{A_m' B_m'^{-1} \mid m=1,\dots,j\}$.

Define

$$\varphi_{j,k,l}(a_2,\dots,a_j) = f_{j,k,l+1}(a_2,\dots,a_j) - f_{j,k,l}(a_2,\dots,a_j)$$

for $l=1,\dots,j-1$.

Then the period condition of $X(M_{j,k})$, $j\geq 2$ can be written as follows:

LEMMA 3.1′. *X is well-defined if and only if there exist $a_{j,k,2},\dots,a_{j,k,j}$ such that*

$$\varphi_{j,k,1}(a_{j,k,2},\dots,a_{j,k,j}) = \cdots = \varphi_{j,k,j-1}(a_{j,k,2},\dots,a_{j,k,j}) = 0 , \qquad j\geq 2 . \tag{3.6}$$

REMARK. (3.5) always holds when $j=1$. Hence the minimal surface $X(M_{1,k})$ do exist. These surfaces were described by Karcher [Ka]. When $j=2$, Lemma 3.1 was first solved by Thayer [T].

In Section 5, we will see that this condition actually holds, which establishes the main theorem.

4. The genus $2k$ case. In this section, rewriting parts of the results of Chen-Gackstatter [CG] and Thayer [T], we prove (3.6) when $j=2$. By definition,

$$f_{2,k,1}(a_2)=\frac{\displaystyle\int_0^1\left(\frac{x(a_2^2-x^2)}{1-x^2}\right)^{-k/(k+1)}dx}{\displaystyle\int_0^1\left(\frac{x(a_2^2-x^2)}{1-x^2}\right)^{k/(k+1)}dx},$$

$$f_{2,k,2}(a_2)=\frac{\displaystyle\int_1^{a_2}\left(\frac{x(a_2^2-x^2)}{x^2-1}\right)^{-k/(k+1)}dx}{\displaystyle\int_1^{a_2}\left(\frac{x(a_2^2-x^2)}{x^2-1}\right)^{k/(k+1)}dx},$$

$$\varphi_{2,k,1}(a_2)=f_{2,k,2}(a_2)-f_{2,k,1}(a_2)\,.$$

We will extend the domain $(1,\infty)$ of $f_{2,k,1}$ and $f_{2,k,2}$ to its boundary $a_2=1$. Since the mean value theorem can be applied to generalized integrals, there exist α, β such that $1<\alpha$, $\beta<a_2$ and

$$f_{2,k,2}(a_2)=\frac{\displaystyle\int_1^{a_2}\left(\frac{x(a_2+x)}{1+x}\right)^{-k/(k+1)}\left(\frac{a_2-x}{x-1}\right)^{-k/(k+1)}dx}{\displaystyle\int_1^{a_2}\left(\frac{x(a_2+x)}{1+x}\right)^{k/(k+1)}\left(\frac{a_2-x}{x-1}\right)^{k/(k+1)}dx},$$

$$=\frac{\displaystyle\left(\frac{\alpha(a_2+\alpha)}{1+\alpha}\right)^{-k/(k+1)}\int_1^{a_2}\left(\frac{a_2-x}{x-1}\right)^{-k/(k+1)}dx}{\displaystyle\left(\frac{\beta(a_2+\beta)}{1+\beta}\right)^{k/(k+1)}\int_1^{a_2}\left(\frac{a_2-x}{x-1}\right)^{k/(k+1)}dx}.$$

Since

$$\int_p^q\left(\frac{q-x}{x-p}\right)^{k/(k+1)}dx=\int_p^q\left(\frac{q-x}{x-p}\right)^{-k/(k+1)}dx\,,$$

we obtain

$$f_{2,k,2}(a_2)=\left(\frac{\alpha\beta(a_2+\alpha)(a_2+\beta)}{(1+\alpha)(1+\beta)}\right)^{-k/(k+1)}\to 1$$

as $a_2\to 1$.

Hence we can define $f_{2,k,2}(1)$ as 1. Furthermore, since the integrand of the denominator of $f_{2,k,1}(a_2)$ uniformly converges to $x^{k/(k+1)}$ as $a_2\to 1$ on an arbitrary closed interval $I\subseteq[0,1)$,

$$f_{2,k,1}(1)=\lim_{a_2\to 1} f_{2,k,1}(a_2)=\frac{\displaystyle\int_0^1 x^{-k/(k+1)}dx}{\displaystyle\int_0^1 x^{k/(k+1)}dx}=2k+1>1\ .$$

Therefore $\varphi_{2,k,1}(1)<0$. Moreover, by an order argument with a_2, we see that $\varphi_{2,k,1}(a_2)>0$ for sufficiently large a_2. Consequently, by the intermediate value theorem, there exists $a_2>1$ such that $\varphi_{2,k,1}(1)=0$.

REMARK. We do not argue here the uniqueness of a_2, so we could just take the smallest value $a_{2,k,2}$ amongst the numbers a_2 which have the property $\varphi_{2,k,1}(a_2)=0$. But our numerical computation implies uniqueness, as is mentioned in the Introduction. For instance, $a_{2,1,2}\sim 1.713$ by the numerical computation.

5. Higher genus type. We begin with computations of boundary values of $f_{j,k,l}$. The following equalities are obtained (we can prove them in the same way as in Section 4).

$$\lim_{a_j\to a_{j-1}} f_{j,k,l}(a_2,\dots,a_{j-2},a_{j-1},a_j)=f_{j-2,k,l}(a_2,\dots,a_{j-2}) \tag{5.1}$$

for $j\geq 3$, $k\geq 1$, $l=1,\dots,j-2$.

$$\lim_{a_j\to a_{j-1}} f_{j,k,j-1}(a_2,\dots,a_j)=\frac{\displaystyle\int_{a_{j-2}}^{a_{j-1}} F_{j-2}(x,a_2,\dots,a_{j-2})^{-k/(k+1)}dx}{\displaystyle\int_{a_{j-2}}^{a_{j-1}} F_{j-2}(x,a_2,\dots,a_{j-2})^{k/(k+1)}dx}, \tag{5.2}$$

$$\lim_{a_j\to a_{j-1}} f_{j,k,j}(a_2,\dots,a_j)=F_{j-2}(a_{j-1},a_2,\dots,a_{j-2})^{-2k/(k+1)} \tag{5.3}$$

for $j\geq 3$, $k\geq 1$.

Because of the form of F_j, we must argue the two cases where j is odd or even separately. Here we introduce a notation for convenience:

$$S_j=\{(a_2,\dots,a_j)\in \boldsymbol{R}^{j-1}\,|\,1<a_2<\cdots<a_j\}\ .$$

The following lemma will be proved in Section 6.

LEMMA 5.1. *We assume that each* $(a_2,\dots,a_j)$ *denotes an element of* S_j. *The following inequalities* (i) *to* (v) *hold for each odd number* $j\geq 3$.

(i) *If* $\varphi_{j-1,k,l}(a_2,\dots,a_{j-1})\leq 0$, *then* $\varphi_{j,k,l}(a_2,\dots,a_j)<0$ *for arbitrary* $a_j>a_{j-1}$ *and* $l=1,\dots,j-2$.

(ii) *If* $\varphi_{j,k,l}(a_2,\dots,a_j)\geq 0$, *then* $\varphi_{j,k,l}(a_2,\dots,a_j+\delta)>0$ *for arbitrary* $\delta>0$ *and* $l=1,\dots,j-2$.

(iii) $\varphi_{j,k,j-1}(a_2, \ldots, a_j)<0$ *for sufficiently large* a_j.
(iv) $\lim_{a_j \to a_{j-1}} \varphi_{j,k,j-1}(a_2, \ldots, a_j)>0$.
(v) $\lim_{a_j \to a_{j-1}} \varphi_{j,k,j-2}(a_2, \ldots, a_j)>0$ *for sufficiently large* a_{j-1}.

The following inequalities (i)′ *to* (v)′ *hold for each even number* $j \geq 4$.

(i)′ *If* $\varphi_{j-1,k,l}(a_2, \ldots, a_{j-1}) \geq 0$, *then* $\varphi_{j,k,l}(a_2, \ldots, a_j)>0$ *for arbitrary* $a_j > a_{j-1}$ *and* $l=1, \ldots, j-2$.

(ii)′ *If* $\varphi_{j,k,l}(a_2, \ldots, a_j) \leq 0$, *then* $\varphi_{j,k,l}(a_2, \ldots, a_j+\delta)<0$ *for arbitrary* $\delta>0$ *and* $l=1, \ldots, j-2$.

(iii)′ $\varphi_{j,k,j-1}(a_2, \ldots, a_j)>0$ *for sufficiently large* a_j.
(iv)′ $\lim_{a_j \to a_{j-1}} \varphi_{j,k,j-1}(a_2, \ldots, a_j)<0$.
(v)′ $\lim_{a_j \to a_{j-1}} \varphi_{j,k,j-2}(a_2, \ldots, a_j)<0$ *for sufficiently large* a_{j-1}.

We will prove the case of genus *three* (with $j=3$, $k=1$) before the arbitrary genus case (by Rossman's advice). By (v) and (ii), $\varphi_{3,k,1}(a_2, a_3)>0$ for sufficiently large a_2 and any $a_3>a_2$. By (i) and the argument of $\varphi_{2,k,1}$ in Section 2, $\varphi_{3,k,1}(a_2, a_3)<0$ for a_2 near to 1 and any $a_3>a_2$. These situations are illustrated in Figure 5.1. Moreover, by (iii) and (iv), the signs of $\varphi_{3,k,2}$ are as in Figure 5.2. Then the line segments as in Figure 5.3 are mapped by $\varphi_{3,k}(a_2, a_3)=(\varphi_{3,k,1}(a_2, a_3), \varphi_{3,k,2}(a_2, a_3))$ to a closed curve whose image surrounds the origin as in Figure 5.4. Since $\varphi_{3,k}$ is continuous in the region D (as in Figure 5.3), there exist $a_{3,k,2}$, $a_{3,k,3}$ such that $\varphi_{3,k}=(0, 0)$. Thus Lemma 3.1′ and the main theorem is proved for $j=3$.

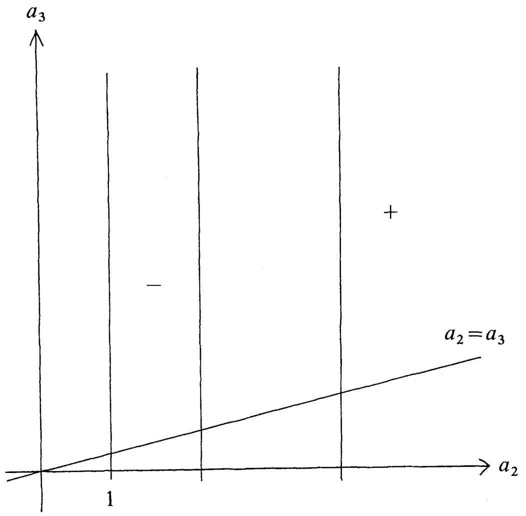


FIGURE 5.1. Signs of $\varphi_{3,1}$.

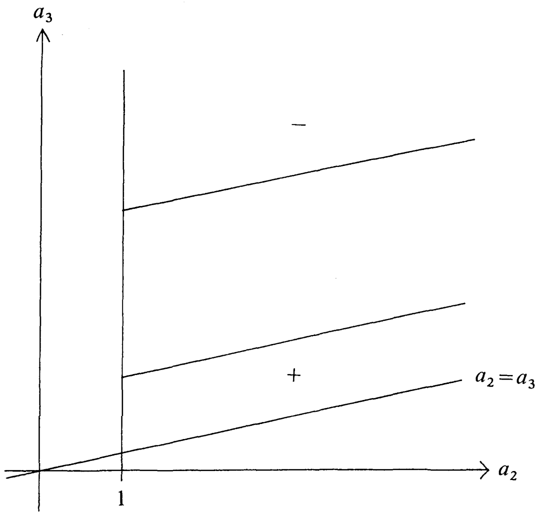


FIGURE 5.2. Signs of $\varphi_{3,2}$.

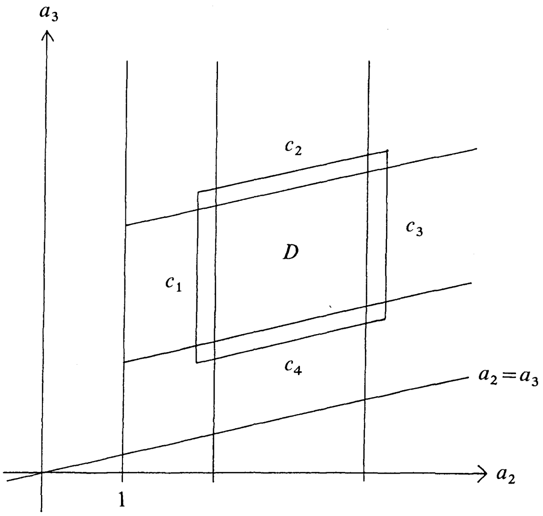


FIGURE 5.3.

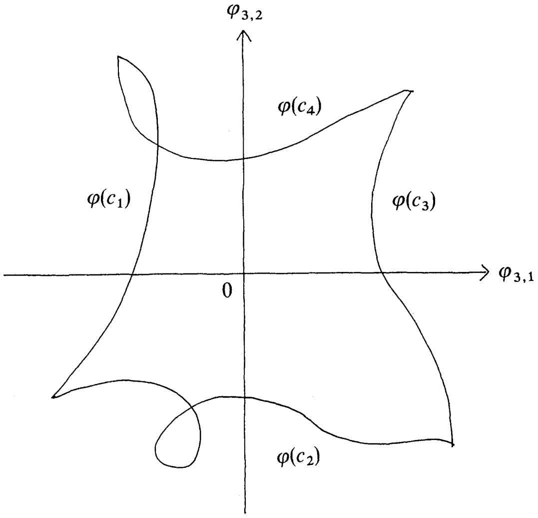


FIGURE 5.4. Image of the line segments in Figure 5.3 by $\varphi=(\varphi_{3,1}, \varphi_{3,2})$.

When j is more than three, similarly as above, we get a $j-2$ dimensional $2(j-1)$-hedron whose image by $\varphi_{j,k}=(\varphi_{j,k,1}, \ldots, \varphi_{j,k,j-1})$ has the origin in its interior.

Now let us prove that the condition in Lemma 3.1′ can be satisfied.

We prove that there exists connected region $D^{\pm}_{j,k,l}$ $(l=1, \ldots, j-1)$ such that

$$\varphi_{j,k,l}(a_2, \ldots, a_j)>0 \qquad \text{for any } (a_2, \ldots, a_j)\in D^{+}_{j,k,l}\,,$$

$$\varphi_{j,k,l}(a_2, \ldots, a_j)<0 \qquad \text{for any } (a_2, \ldots, a_j)\in D^{-}_{j,k,l}$$

and we take a $(j-2)$-dimensional plane in each $D^{\pm}_{j,k,l}$ $(l=1, \ldots, j-1)$, with which we make a polyhedron. The origin is within the interior of the image of the polyhedron under the map $\varphi_{j,k}=(\varphi_{j,k,1}, \ldots, \varphi_{i,k,j-1})$. Since $\varphi_{j,k}$ is continuous in $a_2, \ldots, a_j$ in the interior of the polyhedron, there exist $a_{j,k,2}, \ldots, a_{j,k,j}$ such that $(\varphi_{j,k,1}, \ldots, \varphi_{j,k,j-1})=(0, \ldots, 0)$. Now we consider how to construct $D^{\pm}_{j,k,l}$:

(a) $j=2$: By (iii)′ and (iv)′, there exist p_2, q_2, r_2, s_2 such that $\varphi_{2,k,1}(x)<0$ for any $x\in(p_2, q_2)$ and $\varphi_{2,k,1}(x)>0$ for any $x\in(r_2, s_2)$. We set

$$D^{-}_{2,k,1}=(p_2, q_2)\,, \qquad D^{+}_{2,k,1}=(r_2, s_2)\,.$$

(b) $j=3$: By (v) and (ii), for sufficiently large t_3, u_3, we can set

$$D^{+}_{3,k,1}=((t_3, u_3)\times \boldsymbol{R})\cap S_3\,.$$

And by (i),

$$D^{-}_{3,k,1}=(D^{-}_{2,1}\times \boldsymbol{R})\cap S_3\,.$$

Furthermore, by (iii) and (iv), there exist p_3, q_3, r_3, s_3, M_3 such that

$$\varphi_{3,k,2}(a_2, a_2+x)>0 \qquad \text{for any } x\in(p_3, q_3)\,,$$

$$\varphi_{3,k,2}(a_2, a_2+x)<0 \qquad \text{for any } x\in(r_3, s_3)\,,$$

where $1<a_2<M_3$ for sufficiently large M_3. Put

$$D^+_{3,k,2}=\{(a_2, a_2+x)\,|\,1<a_2<M_3,\, x\in(p_3, q_3)\}\,,$$

$$D^-_{3,k,2}=\{(a_2, a_2+x)\,|\,1<a_2<M_3,\, x\in(r_3, s_3)\}$$

for sufficiently large M_3.

(c) $j\geq 4$, when j is an even number: By (5.1) and (ii)$'$, we set

$$D^-_{j,k,l}=(D^-_{j-2,k,l}\times \boldsymbol{R}^2)\cap S_j$$

for $l=1,\dots,j-3$. By (i)$'$,

$$D^+_{j,k,l}=(D^+_{j-1,k,l}\times \boldsymbol{R})\cap S_j$$

for $l=1,\dots,j-2$. By (v)$'$ and (ii)$'$, with sufficiently large t_j, u_j, we set

$$D^-_{j,k,j-2}=(\boldsymbol{R}^{j-3}\times(t_j, u_j)\times \boldsymbol{R})\cap S_j\,.$$

Furthermore, by (iii)$'$ and (iv)$'$, there exist p_j, q_j, r_j, s_j, M_j such that

$$\varphi_{j,k,j-1}(a_2,\dots, a_{j-1}, a_{j-1}+x)<0 \qquad \text{for any } x\in(p_j, q_j)\,,$$

$$\varphi_{j,k,j-1}(a_2,\dots, a_{j-1}, a_{j-1}+x)>0 \qquad \text{for any } x\in(r_j, s_j)\,,$$

where $1<a_2<\cdots<a_{j-1}<M_j$ for sufficiently large M_j. We set

$$D^-_{j,k,j-1}=\{(a_2,\dots, a_{j-1}, a_{j-1}+x)\,|\,1<a_2<\cdots<a_{j-1}<M_j,\, x\in(p_j, q_j)\}\,,$$

$$D^+_{j,k,j-1}=\{(a_2,\dots, a_{j-1}, a_{j-1}+x)\,|\,1<a_2<\cdots<a_{j-1}<M_j,\, x\in(r_j, s_j)\}$$

for sufficiently large M_j.

(d) $j\geq 5$, when j is an odd number: By (5.1) and (ii), we set

$$D^+_{j,k,l}=(D^+_{j-2,k,l}\times \boldsymbol{R}^2)\cap S_j$$

for $l=1,\dots,j-3$. By (i),

$$D^-_{j,k,l}=(D^-_{j-1,k,l}\times \boldsymbol{R})\cap S_j$$

for $l=1,\dots,j-2$. By (v) and (ii), for sufficiently large t_j, u_j, we set

$$D^+_{j,k,j-2}=(\boldsymbol{R}^{j-3}\times(t_j, u_j)\times \boldsymbol{R})\cap S_j\,.$$

Furthermore, by (iii) and (iv), there exist p_j, q_j, r_j, s_j, M_j such that

$$\varphi_{j,k,j-1}(a_2,\dots, a_{j-1}, a_{j-1}+x)>0 \qquad \text{for any } x\in(p_j, q_j)\,,$$

$$\varphi_{j,k,j-1}(a_2,\dots, a_{j-1}, a_{j-1}+x)<0 \qquad \text{for any } x\in(r_j, s_j)\,,$$

where $1<a_2<\cdots<a_{j-1}<M_j$ for sufficiently large M_j. We set

$$D^+_{j,k,j-1}=\{(a_2,\dots,a_{j-1},a_{j-1}+x)\,|\,1<a_2<\cdots<a_{j-1}<M_j, x\in(p_j,q_j)\},$$
$$D^-_{j,k,j-1}=\{(a_2,\dots,a_{j-1},a_{j-1}+x)\,|\,1<a_2<\cdots<a_{j-1}<M_j, x\in(r_j,s_j)\}$$

for sufficiently large M_j. □

6. Proof of the lemma. In this section we prove (i)–(v) of Lemma 5.1. (i)′–(v)′ can be proved similarly.

6.1. Proof of (i). When j is an odd number, F_j can be expressed in the form

$$F_j(x,a_2,\dots,a_j)=F_{j-1}(x,a_2,\dots,a_{j-1})\cdot(x^2-a_j^2)^{-1}.$$

Hence

$$\begin{aligned}
f_{j,k,l+1}(a_2,\dots,a_j)&=\frac{\displaystyle\int_{a_l}^{a_{l+1}}(-s_{j,l+1}F_{j-1}(x,a_2,\dots,a_{j-1})\cdot(a_j^2-x^2)^{-1})^{-k/(k+1)}dx}{\displaystyle\int_{a_l}^{a_{l+1}}(-s_{j,l+1}F_{j-1}(x,a_2,\dots,a_{j-1})\cdot(a_j^2-x^2)^{-1})^{k/(k+1)}dx}\\
&<(a_j^2-a_l^2)^{2k/(k+1)}f_{j-1,k,l+1}(a_2,\dots,a_j)\\
&\le(a_j^2-a_l^2)^{2k/(k+1)}f_{j-1,k,l}(a_2,\dots,a_j)\qquad(\because\quad \varphi_{j-1,k,l}\le 0)\\
&<\frac{\displaystyle\int_{a_{l-1}}^{a_l}(-s_{j,l}F_{j-1}(x,a_2,\dots,a_{j-1})\cdot(a_j^2-x^2)^{-1})^{-k/(k+1)}dx}{\displaystyle\int_{a_{l-1}}^{a_l}(-s_{j,l}F_{j-1}(x,a_2,\dots,a_{j-1})\cdot(a_j^2-x^2)^{-1})^{k/(k+1)}dx}\\
&=f_{j,k,l}(a_2,\dots,a_j).
\end{aligned}$$

□

6.2. Proof of (ii).

$$\begin{aligned}
&f_{j,k,l+1}(a_2,\dots,a_j+\delta)\\
&\quad=\frac{\displaystyle\int_{a_l}^{a_{l+1}}(-s_{j,l+1}F_{j-1}(\cdots)(a_j^2-x^2)^{-1}(a_j^2-x^2)((a_j+\delta)^2-x^2)^{-1})^{-k/(k+1)}dx}{\displaystyle\int_{a_l}^{a_{l+1}}(-s_{j,l+1}F_{j-1}(\cdots)(a_j^2-x^2)^{-1}(a_j^2-x^2)((a_j+\delta)^2-x^2)^{-1})^{k/(k+1)}dx}\\
&\quad>\left(\frac{a_j^2-a_l^2}{(a_j+\delta)^2-a_l^2}\right)^{-2k/(k+1)}f_{j,k,l+1}(a_2,\dots,a_j)
\end{aligned}$$

$$\geq\left(\frac{a_j^2-a_l^2}{(a_j+\delta)^2-a_l^2}\right)^{-2k/(k+1)} f_{j,k,l}(a_2, \ldots, a_j) \qquad (\because \quad \varphi_{j,k,l}\geq 0)$$

$$>\frac{\displaystyle\int_{a_{l-1}}^{a_l} (-s_{j,l}F_{j-1}(\cdots)(a_j^2-x^2)^{-1}(a_j^2-x^2)((a_j+\delta)^2-x^2)^{-1})^{-k/(k+1)}dx}{\displaystyle\int_{a_{l-1}}^{a_l} (-s_{j,l}F_{j-1}(\cdots)(a_j^2-x^2)^{-1}(a_j^2-x^2)((a_j+\delta)^2-x^2)^{-1})^{k/(k+1)}dx}$$

$$=f_{j,k,l}(a_2, \ldots, a_j+\delta)\,,$$

where $F_{j-1}(\cdots)$ denotes $F_{j-1}(x, a_2, \ldots, a_{j-1})$. □

6.3. Proof of (iii). By an other argument of $f_{j,j-1}$, $f_{j,j}$ in terms of a_j, (iii) can be proved easily.

6.4. Proof of (iv). Noting that j is an odd number, we see that $F_{j-2}(x, a_2, \ldots, a_{j-2})$ is monotone decreasing in x on (a_{j-2}, ∞).

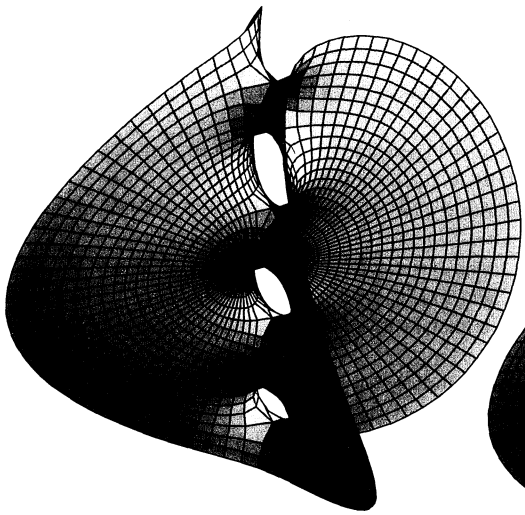
$$\lim_{a_j\to a_{j-1}} f_{j,k,j-1}(a_2, \ldots, a_j)=\frac{\displaystyle\int_{a_{j-2}}^{a_{j-1}} F_{j-2}(x, a_2, \ldots, a_{j-2})^{-k/(k+1)}dx}{\displaystyle\int_{a_{j-2}}^{a_{j-1}} F_{j-2}(x, a_2, \ldots, a_{j-2})^{k/(k+1)}dx} \qquad \text{(by (5.2))}$$

$$<\frac{(a_{j-1}-a_{j-2})F_{j-2}(a_{j-1}, a_2, \ldots, a_{j-2})^{-k/(k+1)}}{(a_{j-1}-a_{j-2})F_{j-2}(a_{j-1}, a_2, \ldots, a_{j-2})^{k/(k+1)}}$$

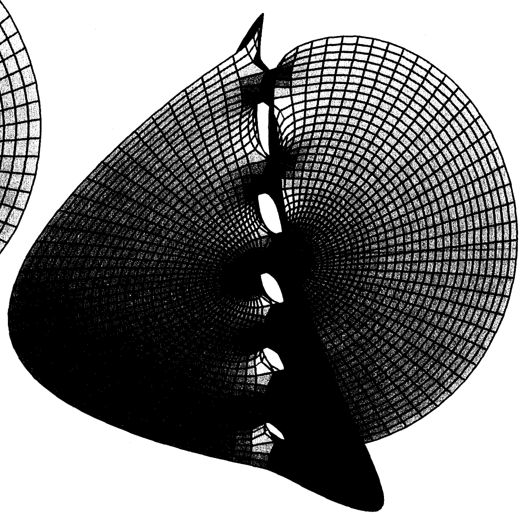
$$=\lim_{a_j\to a_{j-1}} f_{j,k,j}(a_2, \ldots, a_j) \qquad \text{(by (5.3))}\,.$$

□

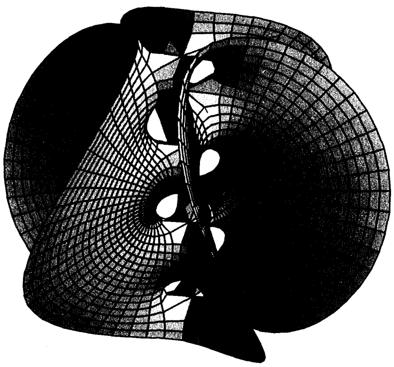
6.5. Proof of (v). By (5.1), $\lim_{a_j\to a_{j-1}} f_{j,k,j-2}$ does not depend on a_{j-1}. On the other hand, since the order of the numerator of $\lim_{a_j\to a_{j-1}} f_{j,k,j-1}$ in terms of a_{j-1} is more than that of the denominator, $\lim_{a_j\to a_{j-1}} f_{j,k,j-1} > \lim_{a_j\to a_{j-1}} f_{j,k,j-2}$ for sufficiently large a_{j-1}. □



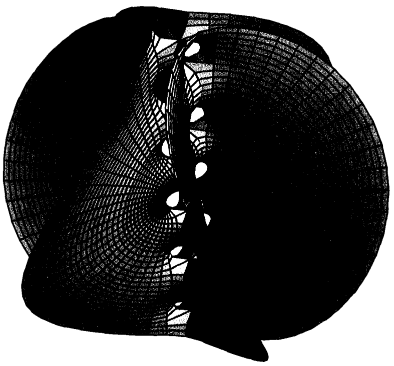
$M_{3,1}$



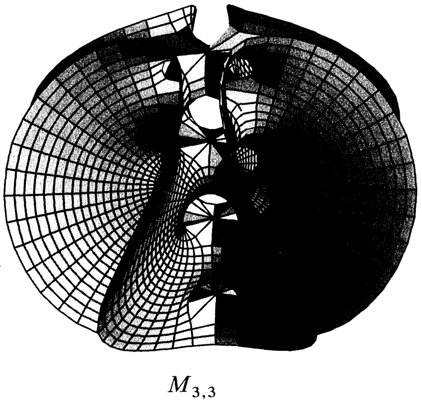
$M_{5,1}$



$M_{3,2}$



$M_{5,2}$



$M_{3,3}$

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