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PERIODIC SOLUTIONS OF CONVEX NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we consider the existence of periodic solutions of neutral functional differential equations. It has been proved that for convex neutral functional differential equations of D-operator type with finite (or infinite) delay and hyperneutral functional differential equations with finite delay, there is a periodic solution if and only if there is a bounded solution. The results proved by Massera, Chow and Makay are generalized.

1. Introduction. In this section we give a brief description of the background of and solutions to the problem considered. As we all know, the existence of periodic solutions of functional differential equations (i.e. FDE), which has been studied extensively, has great theoretical and practical significance. A classical theorem due to Yoshizawa [1] shows that if the solutions of a retarded functional differential equation (i.e. RFDE) with finite delay are uniformly bounded and uniformly ultimately bounded, then it has an ω-periodic solution provided that $r \leq \omega$, where r is the time delay, and ω is the period of the equation. In recent years, many authors devoted themselves to the generalization of Yoshizawa's theorem. Some authors removed the restriction $r \leq \omega$ successfully (Li [2], indexed by SCI). Some authors generalized Yoshizawa's theorem to different equations: Arino, Burton and Haddock [3], Wang and Huang [4, 5] to RFDE with infinite delay; Gao [6] to neutral functional differential equations of D-operator type (i.e. NFDE(D, f)) with finite delay; Shi [7] to NFDE(D, f) with infinite delay.

A remarkable fact is that the uniformly ultimate boundedness (i.e. UUB) does not imply the uniform boundedness (i.e. UB) for FDE as Kato [8] has shown, so a very interesting and meaningful question is brought out: is it possible that UUB without UB still guarantee the existence of periodic solutions? Burton and Zhang [9] succeeded in this way for RFDE with infinite delay under the assumption that the considered equation has a weak fading memory. Ma, Yu [10] and Zhang [11] considered the same problem for RFDE with finite delay and removed the UB assumption. A counterpart result was obtained by Fan and Wang for NFDE(D, f) with finite delay [12] and for NFDE(D, f) with infinite delay [13]. Though UB is dropped, UUB is still a very strong condition. However, for linear ordinary differential equations, Massera [14] proved that the existence of a bounded solution implies the existence

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of a periodic solution. Chow [15] considered a one dimensional FDE with finite delay in a special form: the right-hand side of the equation was a functional linear in ϕ plus a perturbing term depending only on t. Makay [16] generalized Chow's result to convex RFDE with finite and infinite delay and also to integral equations.

In this paper, we generalize the results proved by Massera for linear (inhomogeneous) ordinary differential equations, Chow for linear retarded functional differential equations with finite delay, and Makay for linear retarded functional differential equations with infinite delay to convex NFDE(D, f) with finite delay in Section 2, to convex NFDE(D, f) with infinite delay in Section 3, and to hyperneutral functional differential equations with finite delay in Section 4. It has been proved that for convex NFDE(D, f) with finite (or infinite) delay and hyperneutral functional differential equations with finite delay, there is a periodic solution if and only if there is a bounded solution. We define a stable operator in Section 2 and a B-uniformly stable operator in Section 3 to overcome the difficulty caused by x_t in the derivation of the left-hand side of the considered equation. In Section 3 we also introduce a new phase space with a strong fading memory to overcome the difficulty induced by the infinite delay. Our theorems give a necessary and sufficient condition for having an ω-periodic solution and generalize the known results.

2. Periodic solutions of convex neutral functional differential equations of D-operator type with finite delay. Let $(C, \|\cdot\|)$ be the Banach space of continuous functions $\phi : [-r, 0] \to R^n$ with the supremum norm. Consider the NFDE(D, f) with finite delay of the form

$$\frac{dDx_t}{dt} = f(t, x_t)\,, \tag{2.1}$$

where $x_t(s) = x(t+s)$ for $-r \le s \le 0$; $f : R \times C \to R^n$ be continuous and locally Lipschitz in ϕ and $f(t+\omega, \phi) = f(t, \phi)$ for some $\omega > 0$; $D\phi = \phi(0) - g(\phi)$ be a linear operator, and $g : C \to R^n$ is linear and continuous in ϕ, namely

$$g(\phi) = \int_{-r}^{0} d\mu(\theta)\phi(\theta)\,, \quad \text{for any } \phi \in C\,,$$

where $\mu(\theta)$ is a bounded variation matrix function. We also assume that there exists a nondecreasing continuous function $l(s)$, $s \in [0, r]$ with $l(0) = 0$ such that

$$\left| \int_{-s}^{0} d\mu(\theta)\phi(\theta) \right| \le l(s) \sup_{-s \le \theta \le 0} |\phi(\theta)|\,, \quad \text{for any } \phi \in C\,.$$

Under the above assumptions, for each $(t_0, \phi) \in R \times C$ there is a unique solution $x_t(t_0, \phi)$ of (2.1) that depends continuously on the initial data [17, 18], and if the solution is bounded, then it can be continued for all future time [19].

DEFINITION 2.1. A functional differential equation is said to be convex if the solution set of the considered equation is convex, i.e., x_t and y_t are solutions and $\alpha \in [0, 1]$, then $\alpha x_t + (1-\alpha)y_t$ is also a solution.

If an FDE is convex, we also say that the functional in the right-hand side of the equation is convex. There are many convex NFDE, for example, one can easily prove that if the right-hand side of (2.1) is linear in ϕ, then (2.1) is a convex NFDE.

Throughout the proof of the theorems in this paper we insist on not using the convexity of the considered system if we do not have to.

DEFINITION 2.2 ([19]). Suppose that $D : C \to R^n$ is linear, continuous and atomic at 0. The operator D in (2.1) is said to be stable if the zero solution of the homogeneous difference equation $Dy_t = 0$, $t \geq 0$ with $y_0 = \phi \in C_D := \{\phi \in C : D\phi = 0\}$ is uniformly asymptotically stable.

LEMMA 2.1. *Suppose that $f(t+\omega, \phi) = f(t, \phi)$. If $x_t(t_0, \phi)$ is a solution of (2.1) with $x_{t_0} = x_{t_0+\omega}$, then*

$$x_t(t_0, \phi) = x_{t+\omega}(t_0, \phi), \quad t \geq t_0 .$$

By the uniqueness of solutions and periodicity of (2.1), Lemma 2.1 is obvious.

LEMMA 2.2 ([6]). *If D is a stable operator, then there exist positive constants b and c such that for any $h \in C([\tau, +\infty), R^n)$, every solution $x_t(t_0, \phi, h)$ of $Dx_t = h(t)$, $t \geq t_0$ with $x_{t_0} = \phi$ satisfies*

$$\|x_t(t_0, \phi, h)\| \leq c\exp\{-b(t-t_0)\}\|x_{t_0}\| + c \sup_{t_0 \leq u \leq t} |h(u)| .$$

LEMMA 2.3 ([6]). *Suppose that D is a stable operator, $\alpha \geq -\infty$ and $h : [\alpha, +\infty) \to R^n$ satisfies*

$$|h(t_1) - h(t_2)| \leq H|t_1 - t_2|$$

for any $t_1, t_2 \in [\alpha, +\infty)$. Then there is a positive constant $N(H)$ such that the solution $x_t(t_0, \phi)$ of $Dx_t = h(t)$, $t \geq t_0$ with $x_{t_0} = \phi$ satisfies

$$|x(t_0, \phi)(t_1) - x(t_0, \phi)(t_2)| \leq N(H)|t_1 - t_2|$$

for any $t_1, t_2 \in [t_0, +\infty)$, $t_0 > \alpha$, where $N(H) = c(c+1)H$, c being the constant in Lemma 2.2.

THEOREM 2.1. *Suppose that D is a stable operator and $f(t, \phi)$ is a convex functional in ϕ. Then there is an ω-periodic solution of (2.1) if and only if there is a bounded solution of (2.1).*

PROOF. We only need to show that the existence of a bounded solution implies the existence of an ω-periodic solution, since a periodic solution itself is a bounded solution. Let $\eta = \max\{r, \omega\}$, and define C_0 as the Banach space of continuous functions mapping $[-\eta, 0]$ into R^n with the supremum norm, i.e., $\|\phi\|_0 = \sup_{-\eta \leq \theta \leq 0} |\phi(\theta)|$ for any $\phi \in C_0$. In the following we will write $x_t(t_0, \phi)$ or $x(t_0, \phi)(t)$ for the solution of (2.1) starting at t_0 with the initial functional $\phi \in C_0$ with the understanding that if $\eta > r$ then we use only the values of ϕ on $[-r, 0]$. Suppose that the bound for the bounded solution x_t of (2.1) is h. From the local Lipschitz condition on f, there is a constant $K > 0$ such that $|f(t, \phi)| \leq K$ for $\|\phi\| \leq h$. Let $N(K) = c(c+1)K$, where c is the constant in Lemma 2.2.

Consider the set defined by

$$\Omega := \left\{ \phi \in C_0 \,\middle|\, \begin{array}{ll} \text{(a)} & \|\phi\|_0 \le h, \\ \text{(b)} & |\phi(s_1) - \phi(s_2)| \le N(K)|s_1 - s_2| \ \text{ for any } s_1, s_2 \in [-\eta, 0], \\ \text{(c)} & \|x_t(0, \phi)\|_0 \le h \ \text{ for any } t \ge 0 \end{array} \right\}.$$

First of all, we need to show that Ω is nonempty. Consider the bounded solution x_t. Since f is an ω-periodic function in t, we may assume (by a translation argument) that the bounded solution x_t is defined on the interval $[-\eta - r, +\infty)$ and it satisfies the equation on $[-\eta, +\infty)$. Define $\phi_0(s) := x_0(s) = x(s)$ for $s \in [-\eta, 0]$ and $\bar{\phi}_0(s) := x_{-\eta}(s) = x(s-\eta)$ for $s \in [-r, 0]$. By definition, $x_t = x_t(0, \phi_0) = x_t(-\eta, \bar{\phi}_0)$ and hence ϕ_0 satisfies (a) and (c) in the definition of Ω.

The bounded solution x_t of (2.1) satisfies

$$Dx_t = Dx_t(-\eta, \bar{\phi}_0) = \bar{\phi}_0 + \int_{-\eta}^{t} f(s, x_s(-\eta, \bar{\phi}_0))ds := h(t)\,.$$

For any $s_1, s_2 \in [-\eta, +\infty)$, by the local Lipschitz condition of f, we have

$$\begin{aligned} |h(s_1) - h(s_2)| &= \left| \int_{s_2}^{s_1} f(s, x_s(-\eta, \bar{\phi}_0)ds \right| \\ &\le \left| \int_{s_2}^{s_1} |f(s, x_s(-\eta, \bar{\phi}_0)|ds \right| \\ &\le \left| \int_{s_2}^{s_1} K ds \right| = K|s_1 - s_2|\,, \end{aligned}$$

where we use the boundedness of x_t. By Lemma 2.3, for any $s_1, s_2 \in [-\eta, +\infty)$, we have

$$|x(-\eta, \bar{\phi}_0)(s_1) - x(-\eta, \bar{\phi}_0)(s_2)| \le N(K)|s_1 - s_2|\,.$$

Hence

$$\begin{aligned} |\phi_0(s_1) - \phi_0(s_2)| &= |x_0(0, \phi_0)(s_1) - x_0(0, \phi_0)(s_2)| \\ &= |x_0(-\eta, \bar{\phi}_0)(s_1) - x_0(-\eta, \bar{\phi}_0)(s_2)| \\ &= |x(-\eta, \bar{\phi}_0)(s_1) - x(-\eta, \bar{\phi}_0)(s_2)| \\ &\le N(K)|s_1 - s_2| \end{aligned}$$

for any $s_1, s_2 \in [-\eta, 0]$. This proves that ϕ_0 satisfies condition (b) as well as (a) and (c) in the definition of Ω, so $\phi_0 \in \Omega$, and hence Ω is nonempty.

Next, we prove that Ω is convex. For any $\phi_1, \phi_2 \in \Omega$, $\alpha \in [0, 1]$, we have

$$\|\alpha\phi_1 + (1 - \alpha)\phi_2\|_0 \le \alpha\|\phi_1\|_0 + (1 - \alpha)\|\phi_2\|_0 \le h$$

and

$$\begin{aligned} &|\alpha\phi_1(s_1) + (1 - \alpha)\phi_2(s_1) - (\alpha\phi_1(s_2) + (1 - \alpha)\phi_2(s_2))| \\ &\qquad \le \alpha|\phi_1(s_1) - \phi_1(s_2)| + (1 - \alpha)|\phi_2(s_1) - \phi_2(s_2)| \\ &\qquad \le K|s_1 - s_2| \end{aligned}$$

for any $s_1, s_2 \in [-\eta, 0]$. So conditions (a) and (b) are satisfied for the convex linear combination. For condition (c), by the uniqueness of the solution and the convexity of the equation (this is the only place we really need it), for any $\phi_1, \phi_2 \in \Omega$, one obtains

$$\begin{aligned}\|x_t(0, \alpha\phi_1) + (1-\alpha)\phi_2\|_0 &= \|\alpha x_t(0, \phi_1) + (1-\alpha)x_t(0, \phi_2)\|_0 \\ &\leq \alpha\|x_t(0, \phi_1)\|_0 + (1-\alpha)\|x_t(0, \phi_2)\|_0 \leq h,\end{aligned}$$

using the condition (c) for ϕ_1 and ϕ_2, and hence condition (c) is satisfied. Now we have proved that Ω is a convex set.

Obviously, Ω is equicontinuous and uniformly bounded. By Ascoli's theorem, Ω is precompact. We conclude that Ω is closed. In fact, if $\phi_n \in \Omega$ and $\phi_n \to \phi$ as $n \to +\infty$ in the supremum norm $\|\cdot\|_0$, then ϕ clearly satisfies (a) and (b) in the definition of Ω. If the condition (c) is not satisfied, for contradiction, we suppose that $\|x_t(0, \phi)\|_0 > h$ for some $t > 0$. By the continuous dependence of the solution on the initial function, we can find an $n > 0$ such that $\|x_t(0, \phi_n)\|_0 > h$, a contradiction to the fact that $\phi_n \in \Omega$. This proves that ϕ satisfies (c) and hence $\phi \in \Omega$. Therefore Ω is closed, so it is compact.

Now, let us define the mapping $P : \Omega \to \Omega$ by

$$P\phi := x_\omega(0, \phi),$$

namely

$$P\phi(s) := x_\omega(0, \phi)(s) = x(0, \phi)(s + \omega) \quad \text{for } s \in [-\eta, 0].$$

First we prove that P maps Ω into Ω. By (c) for $\phi \in \Omega$ we find that $P\phi$ satisfies (a). For condition (c) on $P\phi$ we note that by the periodicity of f and the uniqueness of the solution we find that

$$\begin{aligned}\|x_t(0, P\phi)\|_0 &= \|x_{t+\omega}(\omega, P\phi)\|_0 \\ &= \|x_{t+\omega}(\omega, x_\omega(0, \phi))\|_0 \\ &= \|x_{t+\omega}(0, \phi)\|_0 \leq h.\end{aligned}$$

To prove (b) for $P\phi$ we do exactly the same as we did in proving that $\phi_0 \in \Omega$. We can prove that for any $t_1, t_2 \in [0, +\infty)$, we have

$$|x(0, \phi)(t_1) - x(0, \phi)(t_2)| \leq N(K)|t_1 - t_2|. \tag{2.2}$$

Since $\phi \in \Omega$, for any $t_1, t_2 \in [-\eta, 0]$, we obtain

$$|x(0, \phi)(t_1) - x(0, \phi)(t_2)| = |\phi(t_1) - \phi(t_2)| \leq N(K)|t_1 - t_2|. \tag{2.3}$$

Now we can conclude that

$$|P\phi(s_1) - P\phi(s_2)| \leq N(K)|s_1 - s_2| \quad \text{for any } s_1, s_2 \in [-\eta, 0].$$

In fact, if $s_1, s_2 \in [-\omega, 0]$ (resp. $s_1, s_2 \in [-\eta, -\omega]$), by (2.2) (resp. by (2.3)), one gets

$$\begin{aligned}|P\phi(s_1) - P\phi(s_2)| &= |x_\omega(0, \phi)(s_1) - x_\omega(0, \phi)(s_2)| \\ &= |x(0, \phi)(s_1 + \omega) - x(0, \phi)(s_2 + \omega)| \\ &\leq N(K)|t_1 - t_2|.\end{aligned}$$

If $s_1 \in [-\eta, -\omega]$, $s_2 \in [-\omega, 0]$, then clearly $s_1 \le -\omega \le s_2$ and by (2.2) and (2.3) we have

$$\begin{aligned}|P\phi(s_1) - P\phi(s_2)| &= |x_\omega(0,\phi)(s_1) - x_\omega(0,\phi)(s_2)| \\ &= |x(0,\phi)(s_1+\omega) - x(0,\phi)(s_2+\omega)| \\ &= |x(0,\phi)(s_1+\omega) - x(0,\phi)(0)| + |x(0,\phi)(0) - x(0,\phi)(s_2+\omega)| \\ &\le N(K)|s_1+\omega| + N(K)|s_2+\omega| \\ &= -N(K)(s_1+\omega) + N(K)(s_2+\omega) = N(K)|s_1 - s_2|\,.\end{aligned}$$

Hence, $P\phi$ satisfies (a), (b) and (c), so P indeed maps Ω into Ω. Moreover P is continuous from the continuous dependence of the solution on the initial function. By Schauder's fixed point theorem P has a fixed point in Ω, i.e., there is a $\phi \in \Omega$ such that $P\phi = \phi$, which means that $x_\omega(0,\phi) = \phi = x_0(0,\phi)$, and by Lemma 2.1 we have

$$x_t(0,\phi) = x_{t+\omega}(0,\phi) \quad \text{for all } t \ge 0\,.$$

Therefore $x_t(0,\phi)$ is an ω-periodic solution of (2.1), and now the proof is complete.

REMARK 2.1. The periodic solution we find in the proof of Theorem 2.1 is bounded by the bound of the bounded solution.

REMARK 2.2. Theorem 2.1 generalizes the results obtained by Chow [15] and Makay [16] for linear RFDE with finite delay. In fact, if $D\phi = \phi(0)$, then Theorem 2.1 is Theorem 1 in [16]. If $D\phi = \phi(0)$ and $f(t,\phi) = L(t,\phi) + f(t)$, where L and f are continuous and periodic with ω in t, and L is also linear in ϕ, then Theorem 2.1 is Theorem 1 proved by Chow [15] and the assumption $\omega \ge r$ is removed.

3. Periodic solutions of convex neutral functional differential equations of D-operator type with infinite delay. In this section we consider NFDE(D, f)

$$\frac{dDx_t}{dt} = f(t, x_t) \tag{3.1}$$

with infinite delay, i.e., $x_t(s) = x(t+s)$ for $s \in (-\infty, 0]$.

The development of the theory of NFDE with infinite delay depends on a choice of a phase space. In order to overcome the difficulties caused by the infinite delay we first establish the phase space B with a strong fading memory for NFDE with infinite delay.

Suppose that $(B, |\cdot|_B)$ is a Banach space of continuous functions mapping $(-\infty, 0]$ into R^n with norm $|\cdot|_B$, i.e., $B = C((-\infty,0], R^n)$. For any given $\alpha > 0$ and $\phi \in B$, $t_0 \in R$, we define

$$\begin{aligned}\mathcal{F}_{\alpha,t_0}(\phi) &= \{x \mid x : (-\infty, t_0+\alpha] \to R^n,\ x_{t_0} = \phi,\ x \text{ is continuous on } [t_0, t_0+\alpha]\}\,, \\ \mathcal{F}_{\alpha,t_0}(B) &= \bigcup_{\phi\in B} \mathcal{F}_{\alpha,t_0}(\phi)\,.\end{aligned}$$

We assume that the space B satisfies the following hypotheses:

(β_1) There is a positive constant $k > 0$ such that

$$|\phi(0)| \le k|\phi|_B$$

for any $\phi \in B$.

(β_2) If $x \in \mathcal{F}_{\alpha,t_0}(B)$, then $x_t \in B$ for any $t \geq t_0$ and x_t is continuous in t on $[t_0, t_0+\alpha]$.

(β_3) If $x \in \mathcal{F}_{\alpha,t_0}(B)$, then there exist a constant $k_1 > 0$ and a continuous function $k_2 : R^+ \times B \to R^+$ such that

$$|x_{t_0+\alpha}|_B \leq k_1 \sup_{s\in[t_0,t_0+\alpha]} |x(s)| + k_2(\alpha, x_{t_0}),$$

and satisfies

$$k_2(\alpha, 0) = 0, \quad k_2(\alpha, \phi) \to 0 \ \text{as}\ \alpha \to +\infty, \quad k_2(\alpha, \phi) \leq \bar{k}_2(a) \ \text{for}\ |\phi|_B \leq a,$$

where $\bar{k}_2 : R^+ \to R^+$ is continuous.

The phase space B defined here is a special case of the phase space defined in [20]. The phase space $(C_h, |\cdot|_h)$ defined in [5] satisfies our assumptions $(\beta_1) - (\beta_3)$ on B.

DEFINITION 3.1. $(B, |\cdot|_B)$ is said to be with a strong fading memory, if for any compact set $I \subset R^-$ and $\phi_n \in B$, $n = 1, 2, \ldots$, the condition that $|\phi_n - \phi|^I \to 0$, as $n \to +\infty$ and $|\phi_n|_B$ is bounded imply that $\phi \in B$ and $|\phi_n - \phi|_B \to 0$ as $n \to +\infty$, where $|\phi|^I := \sup_{\theta\in I} |\phi(\theta)|$.

The phase space $(C_h, |\cdot|_h)$ defined in [5] has a strong fading memory.

DEFINITION 3.2. The operator D in (3.1) is said to be B-uniformly stable, if there exist constants $k_3 > 0$, $k_4 > 0$ such that the solution $x_t(t_0, \phi)$ of $Dx_t = g(t)$, $t \geq t_0$ with $x_{t_0} = \phi$ satisfies

$$|x(t_0, \phi)(t)| \leq k_3 \sup_{\theta\in[t_0,t]} |g(s)| + k_4 |x_{t_0}|_B,$$

where $\phi \in B$, $x_t \in B$ and $g \in C([t_0, +\infty), R^n)$.

In (3.1), we assume that D is linear and B-uniformly stable; $f : R \times B \to R^n$ is continuous functional and satisfies $f(t+\omega, \phi) = f(t, \phi)$ for some $\omega > 0$, and there exists a constant $L > 0$ such that

$$|f(t, \phi_1) - f(t, \phi_2)| \leq L|\phi_1 - \phi_2|_B. \tag{3.2}$$

Wu [20] establishes the fundamental theory for NFDE with infinite delay. Since (3.2) is satisfied, we can conclude from [20] that for each $(t_0, \phi) \in R \times B$ there is a unique solution $x_t(t_0, \phi)$, which depends continuously on the initial data, and if the solution is bounded, then it can be continued for all future time.

LEMMA 3.1. *Suppose that D is linear, continuous and B-uniformly stable, and there is a constant $G > 0$ such that*

$$|g(t_1) - g(t_2)| \leq G|t_1 - t_2|$$

for $g \in C([\alpha, +\infty), R^n)$ $(\alpha \geq -\infty)$ and $t_1, t_2 \in [\alpha, +\infty)$. Let $x_t(t_0, \phi)$ be a solution of $Dx_t = g(t)$, $t \geq t_0$ with $x_{t_0} = \phi$, $\phi \in B$. Then there is a constant $N(G) > 0$ such that

$$|x(t_0, \phi)(t_1) - x(t_0, \phi)(t_2)| \leq N(G)|t_1 - t_2|$$

for any $t_1, t_2 \in [t_0, +\infty)$ $(t_0 \geq \alpha)$, where $N(G) = (1 + k_1k_4)k_3G$.

PROOF. First we note that by the linearity of D we can find that

$$D(x_{t+\Delta}(t_0,\phi)-x_t(t_0,\phi)) = Dx_{t+\Delta}(t_0,\phi) - Dx_t(t_0,\phi) = g(t+\Delta)-g(t)$$

for any $t \in [t_0,+\infty)$ and $\Delta > 0$. By the B-uniform stability of D one can obtain

$$\begin{aligned} |x(t_0,\phi)(t+\Delta)-x(t_0,\phi)(t)| &\le k_3 \sup_{\theta\in[t_0,t]} |g(\theta+\Delta)-g(\theta)| + k_4|x_{t_0+\Delta}-x_{t_0}|_B \\ &\le k_3 G\Delta + k_4|x_{t_0+\Delta}-x_{t_0}|_B\,. \end{aligned} \tag{3.3}$$

For any $s \in [t_0,t]$, by the B-uniform stability of D and the assumption (β_3) on B, we also have

$$\begin{aligned} |x(t_0,\phi)(s)-x(t_0,\phi)(t_0)| &\le k_3 \sup_{\tau\in[t_0,s]} |g(\tau)-g(t_0)| + k_4|x_{t_0}-x_{t_0}|_B \\ &\le k_3 G|s-t_0|\,, \\ |x_{t_0}(t_0,\phi)-x_{t_0}(t_0,\phi)|_B &\le k_1 \sup_{\tau\in[t_0,s]} |x(t_0,\phi)(\tau)-x(t_0,\phi)(t_0)| + k_2(s-t_0,0) \\ &\le k_1k_3G|s-t_0|\,. \end{aligned}$$

Hence

$$|x_{t_0+\Delta}-x_{t_0}|_B \le k_1k_3G\Delta\,. \tag{3.4}$$

By (3.3) and (3.4), we have

$$|x(t_0,\phi)(t+\Delta)-x(t_0,\phi)(t)| \le k_3G\Delta + k_1k_3k_4G\Delta := N(G)\Delta\,.$$

Therefore for any $t_1, t_2 \in [t_0,+\infty)$, there is a constant $N(G) > 0$ such that

$$|x(t_0,\phi)(t_1)-x(t_0,\phi)(t_2)| \le N(G)|t_1-t_2|\,,$$

where $N(G) = (1+k_1k_4)k_3G$. This completes the proof.

DEFINITION 3.3. The solution $x_t(t_0,\phi)$ of (3.1) through $(t_0,\phi) \in R\times B$ is said to be B-bounded, if there exists a positive constant $M > 0$ such that

$$|x_t(t_0,\phi)|_B \le M \quad \text{for } t \ge t_0\,.$$

THEOREM 3.1. *Suppose that $(B, \|\cdot\|_B)$ has a strong fading memory, D is B-uniformly stable and $f(t,\phi)$ is a convex functional in ϕ (see Definition 2.1). Then there is an ω-periodic solution of (3.1) if and only if there is a B-bounded solution of (3.1) defined on R.*

PROOF. As before, we only need to show that the existence of a bounded solution implies the existence of an ω-periodic solution.

Suppose that the bound for the bounded solution of (3.1) is h. By (3.2), for $\phi \in B$, $|\phi|_B \le h$ we have

$$|f(t,\phi)| \le L|\phi|_B \le Lh := G\,.$$

Let $N(G) = (1+k_1k_4)k_3G$. Now we consider the subset

$$\Omega := \left\{ \phi \in B \;\middle|\; \begin{array}{ll} \text{(a)} & |\phi|_B \le h, \\ \text{(b)} & |\phi(s_1)-\phi(s_2)| \le N(G)|s_1-s_2| \text{ for any } s_1,s_2 \in (-\infty,0], \\ \text{(c)} & |x_t(0,\phi)|_B \le h \text{ for any } t \ge 0 \end{array} \right\}.$$

We do exactly the same as in Theorem 2.1, with only slight changes. The proof of the nonempty and the convexity of Ω is very similar to that of Theorem 2.1 with $\eta = +\infty$. To prove that Ω is compact we cannot use Ascoli's theorem, because we have an infinite interval. Instead, consider a sequence $\{\phi_n\} \subset \Omega$. Obviously $\{\phi_n\}$ is equicontinuous. First we point out that $\{\phi_n\}$ is uniformly bounded on $[-m, 0]$ (m is any positive integer) in the supremum norm. Since $\phi_n \in \Omega$, by the assumption (β_1) on the phase space B, for any $\theta \in [-m, 0]$, we have

$$|\phi_n(\theta)| - |\phi_n(0)| \leq |\phi_n(\theta) - \phi_n(0)| \leq N(G)|\theta|,$$

moreover

$$\begin{aligned} |\phi_n(\theta)| &\leq |\phi_n(0)| + N(G)|\theta| \\ &\leq k|\phi_n|_B + N(G)|\theta| \leq kh + N(G)|\theta|. \end{aligned}$$

Hence $\|\phi_n\|^{[-m,0]} \leq kh + mN(G)$, which means that $\{\phi_n\}$ is uniformly bounded in the supremum norm on $[-m, 0]$. Using Ascoli's theorem on the interval $[-1, 0]$, we find a subsequence $\{\phi_n^{(1)}\} \subset \{\phi_n\}$ converging in the supremum norm. For this sequence we find a subsequence $\{\phi_n^{(2)}\}$ converging on $[-2, 0]$, etc. Let $\psi_n := \phi_n^{(n)}$. Then ψ_n converges to a continuous function ϕ uniformly on any compact subinterval of $(-\infty, 0]$. In particular, it converges pointwisely to ϕ, since $(B, |\cdot|_B)$ has a strong fading memory (this is the only place where we really need it). Hence $\phi \in B$ and $|\phi_n^{(n)} - \phi|_B \to 0$ as $n \to +\infty$. Moeover we conclude that $\phi \in \Omega$. In fact, since

$$|\phi|_B \leq |\phi - \phi_n^{(n)}|_B + |\phi_n^{(n)}|_B \leq |\phi - \phi_n^{(n)}|_B + h,$$

letting $n \to +\infty$, we have $|\phi|_B \leq h$, namely, the condition (a) for ϕ is satisfied. For any $s_1, s_2 \in (-\infty, 0]$, we have

$$\begin{aligned} |\phi(s_1) - \phi(s_2)| &\leq |\phi(s_1) - \phi_n^{(n)}(s_1)| + |\phi_n^{(n)}(s_1) - \phi_n^{(n)}(s_2)| + |\phi_n^{(n)}(s_2) - \phi(s_2)| \\ &\leq |\phi(s_1) - \phi_n^{(n)}(s_1)| + N(G)|s_1 - s_2| + |\phi_n^{(n)}(s_2) - \phi(s_2)|. \end{aligned}$$

By letting $n \to +\infty$, one obtains $|\phi(s_1) - \phi(s_2)| \leq N(G)|s_1 - s_2|$, so (b) is satisfied. By the continuous dependence of the solution on the initial function, (c) is clearly satisfied. Now we reach $\phi \in \Omega$. Since Ω is a closed set (this can be proved as before), Ω is compact.

Define $P : \Omega \to \Omega$ by

$$P\phi := x_\omega(0, \phi).$$

By the continuous dependence of the solution on the initial function, we know that P is continuous. Next, we show that P maps Ω into Ω. By (c) for ϕ we find that $P\phi$ satisfies (a) in the definition of Ω. For condition (c) on $P\phi$ we note that by the ω-periodicity of f and the uniqueness of the solution we find that

$$|x_t(0, P\phi)|_B = |x_t(0, x_\omega(0, \phi))|_B = |x_{t+\omega}(\omega, x_\omega(0, \phi))|_B = |x_{t+\omega}(0, \phi)|_B \leq h, \quad t \geq 0.$$

To prove (b), we proceed as follows. We know that (3.1) with $x_0 = \phi$ is equivalent to

$$Dx_t = D\phi + \int_0^t f(s, x_s(0, \phi))ds := g(t), \quad t \geq 0.$$

For any $t_1, t_2 \in [0, +\infty)$, we have

$$\begin{aligned} |g(t_1) - g(t_2)| &= \left| \int_{t_1}^{t_2} f(s, x_s(0, \phi)) ds \right| \\ &\leq \left| \int_{t_1}^{t_2} |f(s, x_s(0, \phi))| ds \right| \leq G|t_1 - t_2|, \end{aligned}$$

where we used the boundedness of $x_t(0, \phi)$. By Lemma 3.1, the solution $x_t(0, \phi)$ satisfies

$$|x(0, \phi)(t_1) - x(0, \phi)(t_2)| \leq N(G)|t_1 - t_2|, \tag{3.5}$$

for $t_1, t_2 \in [0, +\infty)$. Since $\phi \in \Omega$, for any $t_1, t_2 \in (-\infty, 0]$, we have

$$|x(0, \phi)(t_1) - x(0, \phi)(t_2)| \leq N(G)|t_1 - t_2|. \tag{3.6}$$

We conclude that

$$|P\phi(s_1) - P\phi(s_2)| \leq N(G)|s_1 - s_2|, \quad \text{for any } s_1, s_2 \in (-\infty, 0].$$

In fact, if $s_1, s_2 \in [-\omega, 0]$ (resp. $s_1, s_2 \in (-\infty, -\omega]$), by (3.5) (resp. by (3.6)), one obtains

$$\begin{aligned} |P\phi(s_1) - P\phi(s_2)| &\leq |x_\omega(0, \phi)(s_1) - x_\omega(0, \phi)(s_2)| \\ &= |x(0, \phi)(s_1 + \omega) - x(0, \phi)(s_2 + \omega)| \\ &\leq N(G)|s_1 - s_2|. \end{aligned}$$

If $s_1 \in (-\infty, -\omega]$, $s_2 \in [-\omega, 0]$, obviously $s_1 \leq -\omega \leq s_2$, by (3.5) and (3.6), we obtain

$$\begin{aligned} |P\phi(s_1) - P\phi(s_2)| &= |x_\omega(0, \phi)(s_1) - x_\omega(0, \phi)(s_2)| \\ &= |x(0, \phi)(s_1 + \omega) - x(0, \phi)(s_2 + \omega)| \\ &= |x(0, \phi)(s_1 + \omega) - x(0, \phi)(0)| + |x(0, \phi)(0) - x(0, \phi)(s_2 + \omega)| \\ &\leq N(K)|s_1 + \omega| + N(K)|s_2 + \omega| \\ &= -N(K)(s_1 + \omega) + N(K)(s_2 + \omega) = N(K)|s_1 - s_2|. \end{aligned}$$

Now we have proved that P maps Ω into Ω. By Schauder's fixed point theorem, P has a fixed point in Ω, i.e., there is a $\psi \in \Omega$ with $P\psi = \psi$. This means that $x_\omega(0, \psi) = x_0(0, \psi)$, and hence by the periodicity of f and the uniqueness of the solution we find that

$$x_{t+\omega}(0, \psi) = x_t(0, \psi), \quad t \geq 0.$$

We find an ω-periodic solution and the proof is complete.

REMARK 3.1. The periodic solution we find in the proof of Theorem 3.1 is bounded by the bound of the bounded solution.

REMARK 3.2. If $D\phi = \phi(0)$, then Theorem 3.1 is Theorem 2 proved by Makay [16] for RFDE with infinite delay.

4. Periodic solution of convex hyperneutral functional differential equations with finite delay. In this section we consider the existence of periodic solutions for convex hyperneutral functional differential equations with finie delay.

Let

$$C := \{\phi \mid \phi : [-r, 0] \to R^n, \phi \text{ is continuous}\},$$
$$C^1 := \{\phi \in C \mid \dot{\phi} \text{ exists and } \dot{\phi} \in C\}.$$

We define

$$\|\phi\|_r^1 := \max\left\{ \sup_{-r\le\theta\le 0} |\phi(\theta)|, \quad \sup_{-r\le\theta\le 0} |\dot{\phi}(\theta)| \right\}$$

for any $\phi \in C^1$. For any $H \geq 0$, let

$$C_H^1 := \left\{\phi \in C^1 \,\middle|\, \|\phi\|_r^1 \le H\right\}.$$

Consider the hyperneutral functional differential equation

$$x(t) = f(t, x_t, \dot{x}_t) \tag{4.1}$$

with finite delay, i.e., $x_t(\theta) = x(t+\theta)$, $\dot{x}_t(\theta) = \dot{x}(t+\theta)$ for $\theta \in [-r, 0]$, where $f : R \times C^1 \times C \to R^n$ is continuous and ω-periodic in t. We assume that (4.1) has a unique solution $x_t(t_0, \phi)$ (or $x(t_0, \phi)(t)$) for any $(t_0, \phi) \in R \times C^1$, which depends continuously on the initial data.

THEOREM 4.1. *Suppose that for any $H > 0$ there is a continuous function $W^H : [0, +\infty) \to [0, +\infty)$ with $W^H(0) = 0$ such that*

$$|f(t, x_{t_1}, \dot{x}_{t_1}) - f(t, x_{t_2}, \dot{x}_{t_2})| \le W^H(|t_1 - t_2|)$$

for any $t_1, t_2 \in R$ and for any $x_t \in C_H^1$ (we may call $f(t, x_t, \dot{x}_t)$ equicontinuous in t uniformly for $x_t \in C_H^1$). Assume, also, that $f(t, \phi, \dot{\phi})$ is a convex functional (see Definition 2.1). If there is a bounded continuous solution of (4.1), then there is an ω-periodic solution.

PROOF. We do exactly the same as in the proof of Theorems 2.1 and 3.1 with only small changes. Let h be the bound for the existing bounded solution and W^h be the function defined above for h. Let S be the set of continuous differentiable functions mapping $[-\eta, 0]$ into R^n with the norm $\|\phi\|_\eta^1 = \max\{\sup_{-\eta\le\theta\le 0} |\phi(\theta)|, \ \sup_{-\eta\le\theta\le 0} |\dot{\phi}(\theta)|\}$, where $\eta = \max\{\omega, r\}$. In the following we will write $x_t(t_0, \phi)$ for the solution of (4.1) through $(t_0, \phi) \in R \times S$ with the understanding that if $\eta > r$ then we use only the values of ϕ and $\dot{\phi}$ on $[-r, 0]$. We define Ω in a little bit different manner this time.

Let

$$\Omega := \left\{ \phi \in S \;\middle|\; \begin{array}{ll} \text{(a)} & \|\phi\|_\eta^1 \le h, \\ \text{(b)} & |\phi(s_1) - \phi(s_2)| \le W(|s_1 - s_2|) \text{ for any } s_1, s_2 \in [-\eta, 0], \\ \text{(c)} & x_t(0, \phi) \in C^1 \text{ and } \|x_t(0, \phi)\|_\eta^1 \le h \text{ for any } t \ge 0 \end{array} \right\}.$$

For the rest of the proof we need to be careful when we use the Lipschitz condition on f in the proof of Theorem 2.1 or 3.1. Instead, we say (when proving that Ω is nonempty, for example) that

$$\begin{aligned}|\phi_0(s_1) - \phi_0(s_2)| &= |x(-\eta, \bar{\phi}_0)(s_1) - x(-\eta, \bar{\phi}_0)(s_2)| \\ &= |f(s_1, x_{s_1}(-\eta, \bar{\phi}_0), \dot{x}_{s_1}(-\eta, \bar{\phi}_0)) - f(s_2, x_{s_2}(-\eta, \bar{\phi}_0), \dot{x}_{s_2}(-\eta, \bar{\phi}_0))| \\ &\leq W^h(|s_1 - s_2|)\end{aligned}$$

for any $s_1, s_2 \in [-\eta, 0]$. Also, Ascoli's theorem can be applied, since condition (b) implies the equicontinuity of the function in Ω. The remainder of the proof is exactly the same as before. The proof is complete.

The equicontinuity of f in t uniformly for ϕ may seem to be a strong condition, since there are examples (the equation $x(t) = a(t) + b(t)x(t - r) + \dot{x}(t - r)$, for example), for which this condition is not satisfied. For these equations we can prove the following.

THEOREM 4.2. *Suppose that $f(t, \phi)$ is a convex functional. If there is a bounded uniformly continuous solution $\bar{x}_t$ (or $\bar{x}(t)$) of* (4.1), *then there is an ω-periodic solution of* (4.1).

To prove the theorem, we need a definition.

DEFINITION 4.1 ([16]). We say that the function $y : R \to R^n$ is in the equicontinuity class of the uniformly continuous function $x : R \to R^n$, if y is also uniformly continuous, and if $\delta > 0$ is of the uniform continuity for $\varepsilon > 0$ for x, then the same δ works for y in the definition of the uniform continuity for ε.

Let h be the bound of the uniformly continuous solution $\bar{x}(t)$, and define

$$\Omega := \left\{ \phi \in S \;\middle|\; \begin{array}{ll} \text{(a)} & \|\phi\|_\eta^1 \leq h, \\ \text{(b)} & \text{the solution } x(0, \phi)(t) \text{ is in the equicontinuity class of } \bar{x}(t), \\ \text{(c)} & \|x_t(0, \phi)\|_\eta^1 \leq h \ \text{ for any } t \geq 0 \end{array} \right\}.$$

Clearly, the functions in Ω are equicontinuous. The rest of the proof is the same as that of Theorem 4.1.

Finally, we point out the following.

1. From the proof of our theorems it is clear that the convexity condition on the equations cannot be dropped, because we use it to prove the convexity of the set Ω. But we use the convexity only at that particular point; one might be able to find some condition to ensurre the convexity in another way.

2. The condition in Section 3 that $(B, \|\cdot\|_B)$ has a strong fading memory cannot be dropped too, since we use it to prove the compactness of Ω. But one might be able to find other conditions to guarantee the compactness.

3. In order to find a bounded solution for convex differential equations, we can use conditions implying that all solutions are bounded (see [16], [21] and their references).

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