Tohoku Math. J.
52 (2000), 431–447

SATURATION OF THE APPROXIMATION BY SPECTRAL DECOMPOSITIONS

Dedicated to Professor Satoru Igari on his sixtieth birthday

MIHO TANIGAKI

(Received January 13, 1999)

Abstract. We shall give a saturation class for approximations by eigenfunction expansions of the Laplacian in an open domain in the Euclidean space.

1. Introduction. Let Ω be an open domain in the n dimensional Euclidean space $\boldsymbol{R}^n$. Consider the operator $A = -\Delta$ in $L^2(\Omega)$ with the domain of definition $D(A) = C_c^\infty(\Omega)$, where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is the Laplacian. Denote by $\hat{A}$ a nonnegative selfadjoint extension of A. Let $\{k_\lambda(t)\}$ be a family of bounded piecewise smooth functions on $[0, \infty)$. Suppose we have two constants $\kappa_1, \kappa_2 > 0$ such that $k_\lambda(t)\sqrt{t}^{\,n/2-2\kappa_2+1/2} \in L^1(0, \infty)$, $(k_\lambda(t) - 1)/\lambda^{-\kappa_1}t^{\kappa_2}$ are uniformly bounded in λ and $t \in [0, \infty)$, and $(k_\lambda(t) - 1)/\lambda^{-\kappa_1}t^{\kappa_2}$ converge to a nonzero constant as $\lambda \to \infty$ for any $t \in [0, \infty)$. Let

$$I_\lambda(r) = \int_0^\infty k_\lambda(t^2)J_\nu(rt)t^{\nu+1}dt\,,$$

where $\nu = n/2 - 2\kappa_2 + 1$ and J_ν is the Bessel function of order ν. We assume, furthermore, the following conditions

$$\int_0^R s^{2\kappa_2-1}ds\left|\int_s^R r^{n/2-2\kappa_2+2}I_\lambda(r)dr\right| = O(\lambda^{-\kappa_1})\,, \tag{1.1}$$

$$\left|\int_R^\infty r^{\nu+1}I_\lambda(r)dr\right| = o(\lambda^{-\kappa_1})\,, \tag{1.2}$$

and

$$\left(\sum_{T=0}^\infty T^{4\kappa_2-3}\max_{T\le s\le T+1}\left|\int_R^\infty J_\nu(sr)I_\lambda(r)rdr\right|^2\right)^{1/2} = o(\lambda^{-\kappa_1}) \tag{1.3}$$

as $\lambda \to \infty$ for any small $R > 0$.

We shall consider the approximation operator $k_\lambda(\hat{A})$ for $f \in L^2(\Omega)$. We say $\Delta f \in L^\infty_{\text{loc}}(\Omega)$ if for every compact set K in Ω there is a constant C_K such that

$$\left|\int_K f(x)\Delta g(x)dx\right| \le C_K\|g\|_{L^1(K)}$$

1991 *Mathematics Subject Classification*. Primary 41A40.

for any infinitely differentiable function g whose support is contained in K. Let $\{\varphi_\varepsilon\}$ be an infinitely differentiable approximate identity with supports contained in $\{x; |x| < \varepsilon\}$. For a function f on Ω and $x \in \Omega$, f is said to be regulated at x if $f * \varphi_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0^+$.

In 1970, Igari proved the following Theorem in [5].

THEOREM A. *Suppose that there exist a complete orthonormal system $\{u_j\}$ of smooth functions in $L^2(\Omega)$ and a numerical sequence $\{\lambda_j\}$ for which $-\Delta u_j = \lambda_j u_j$ in Ω. Let*

$$f_j = \int_\Omega f(x)\overline{u_j(x)}dx\,, \quad f \in L^2(\Omega)$$

and

$$s_\lambda^\delta f = \sum_{\lambda_j \le \lambda} \left(1 - \frac{\lambda_j}{\lambda}\right)^\delta f_j u_j\,, \quad f \in L^2(\Omega)\,.$$

Let $\delta \ge (n+3)/2$ and $f \in L^2(\Omega)$ be regulated in Ω. Then the following hold.

(i) *The following conditions are equivalent.*

(ia)

$$\|s_\lambda^\delta f - f\|_{L^\infty(K)} = O(\lambda^{-1})$$

as $\lambda \to \infty$ for every compact set K in Ω.

(ib) $\Delta f \in L^\infty_{\mathrm{loc}}(\Omega)$.

(ii) *The following conditions are equivalent.*

(iia)

$$\|s_\lambda^\delta f - f\|_{L^\infty(K)} = o(\lambda^{-1})$$

as $\lambda \to \infty$ for every compact set K in Ω.

(iib) *Δf vanishes in Ω.*

Our aim is to give a generalization of Theorem A. Let $\{k_\lambda(t)\}$ be a family of bounded Borel functions on $[0, \infty)$. We can define the bounded operator $k_\lambda(\hat{A})$ in $L^2(\Omega)$.

EXAMPLE 1. Suppose that there exist a complete orthonormal system $\{u_j\}$ of smooth functions in $L^2(\Omega)$ and a sequence $\{\lambda_j\}$ such that $-\Delta u_j = \lambda_j u_j$ in Ω. Let

$$f_j = \int_\Omega f(x)\overline{u_j(x)}dx\,, \quad f \in L^2(\Omega)\,.$$

Let $\hat{A}$ be the selfadjoint extension of $-\Delta$ defined by

$$D(\hat{A}) = \left\{ f \in L^2(\Omega); \sum_{j=1}^\infty \lambda_j^2 |f_j|^2 < \infty \right\}$$

and

$$\hat{A}f = \sum_{j=1}^\infty \lambda_j f_j u_j\,, \quad f \in D(\hat{A})\,.$$

For any $f \in L^2(\Omega)$ the spectral decomposition of f is given by

$$E((-\infty, t])f = \sum_{\lambda_j \le t} f_j u_j$$

and $k_\lambda(\hat{A})$ is defined by

$$k_\lambda(\hat{A})f = \sum_{j=1}^{\infty} k_\lambda(\lambda_j) f_j u_j\,, \quad f \in L^2(\Omega)\,.$$

EXAMPLE 2. Let $\Omega = \boldsymbol{R}^n$. Let

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\boldsymbol{R}^n} f(x) e^{-i\xi\cdot x} dx\,, \quad f \in L^2(\boldsymbol{R}^n)\,.$$

In this case, there is a unique nonnegative selfadjoint extension $\hat{A}$ of $-\Delta$ defined by

$$D(\hat{A}) = \{f \in L^2(\boldsymbol{R}^n);\ |\xi|^2 \hat{f}(\xi) \in L^2(\boldsymbol{R}^n)\}$$

and

$$\hat{A} f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\boldsymbol{R}^n} |\xi|^2 \hat{f}(\xi) e^{ix\cdot\xi} d\xi\,, \quad f \in D(\hat{A})\,.$$

Then the spectral decomposition of $f \in L^2(\boldsymbol{R}^n)$ is given by

$$E((-\infty, t])f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{|\xi|^2 \le t} \hat{f}(\xi) e^{ix\cdot\xi} d\xi$$

and $k_\lambda(\hat{A})$ is defined by

$$k_\lambda(\hat{A}) f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\boldsymbol{R}^n} k_\lambda(|\xi|^2) \hat{f}(\xi) e^{ix\cdot\xi} d\xi\,, \quad f \in L^2(\boldsymbol{R}^n)\,.$$

For $\kappa_2 > 0$ and $1 < p \le \infty$, we say $(-\Delta)^{\kappa_2} f$ belongs to $L^p_{\mathrm{loc}}(\Omega)$ if for every bounded open set G in Ω with the closure $\bar{G}$ contained in Ω, there is a constant C_G such that

$$\left| \int_{\bar{G}} f(x)(-\Delta)^{\kappa_2} g(x) dx \right| \le C_G \|g\|_{L^{p'}(\bar{G})}$$

for any infinitely differentiable function g with support contained in $\bar{G}$, where $1/p + 1/p' = 1$.

Our results are stated as follows.

MAIN THEOREM. *Let Ω be an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ be a nonnegative selfadjoint extension of $-\Delta$ in Ω. Let $\{k_\lambda(t)\}$ be a family of bounded piecewise smooth functions on $[0, \infty)$ and $\kappa_1, \kappa_2 > 0$ such that $k_\lambda(t)\sqrt{t}^{\,n/2-2\kappa_2+1/2} \in L^1(0, \infty)$, $\lambda^{\kappa_1} t^{-\kappa_2}[k_\lambda(t) - 1]$ are uniformly bounded in λ and $t \in [0, \infty)$, and $\lambda^{\kappa_1} t^{-\kappa_2}[k_\lambda(t) - 1]$ converge to a nonzero constant as $\lambda \to \infty$ for any $t \in [0, \infty)$.*

Suppose that $\{k_\lambda(t)\}$ satisfies the conditions (1.1), (1.2) and (1.3) *as $\lambda \to \infty$. Let f be a regulated function in $L^2(\Omega)$. Furthermore, suppose that $1 < p \le \infty$ and $f \in L^p_{\mathrm{loc}}(\Omega)$. Then the following hold.*

(i) *The following two conditions are equivalent.*

(ia)

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K)} = O(\lambda^{-\kappa_1})$$

as $\lambda \to \infty$ for every compact set K in Ω.

(ib) $(-\Delta)^{\kappa_2} f \in L^p_{\mathrm{loc}}(\Omega)$.

(ii) *Let $G \subset \Omega$ be any open set.*

(iia) *Suppose that $(-\Delta)^{\kappa_2} f$ vanishes in G. Then*

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K)} = o(\lambda^{-\kappa_1})$$

as $\lambda \to \infty$ for any compact set $K \subset G$.

(iib) *If*

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K)} = o(\lambda^{-\kappa_1})$$

as $\lambda \to \infty$ for any compact set $K \subset G$, then $(-\Delta)^{\kappa_2} f$ vanishes in G.

If $\delta > (n+3)/2$ and $k_\lambda(t) = (1 - t/\lambda^2)_+^\delta$, then the conditions (1.1), (1.2) and (1.3) are satisfied. Therefore we have the following:

COROLLARY 1. *Let Ω be an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ be a nonnegative selfadjoint extension of $-\Delta$ in Ω. Let $s_\lambda^\delta = (1 - \hat{A}/\lambda^2)_+^\delta$ and $\delta > (n+3)/2$. Let f be a regulated function in $L^2(\Omega)$. Suppose that $1 < p \leq \infty$ and $f \in L^p_{\mathrm{loc}}(\Omega)$. Then the following hold.*

(i) *The following are equivalent.*

(ia)

$$\|s_\lambda^\delta f - f\|_{L^p(K)} = O(\lambda^{-2})$$

as $\lambda \to \infty$ for every compact set K in Ω.

(ib) $\Delta f \in L^p_{\mathrm{loc}}(\Omega)$.

(ii) *Let $G \subset \Omega$ be any open set.*

(iia) *Suppose that Δf vanishes in G. Then*

$$\|s_\lambda^\delta f - f\|_{L^p(K)} = o(\lambda^{-2})$$

as $\lambda \to \infty$ for any compact set $K \subset G$.

(iib) *If*

$$\|s_\lambda^\delta f - f\|_{L^p(K)} = o(\lambda^{-2})$$

as $\lambda \to \infty$ for any compact set $K \subset G$, then Δf vanishes in G.

Our main theorem follows from Theorem 1 in §2 and Theorem 2 in §3. Corollary 1 is proved in §4.

The author is grateful to Professor Satoru Igari for his advice. The author also thanks the referee and the editor for their careful reading.

2. Saturation of the approximation. Let Ω be an open domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$. Let

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{2.1}$$

be a differential operator, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $D^\alpha = (-i)^{|\alpha|}(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ and $a_\alpha \in C^\infty(\Omega)$. We consider A as an operator in $L^2(\Omega)$ with the domain of definition $D(A) = C_c^\infty(\Omega)$. Suppose that A is formally selfadjoint and semibounded. If $\hat{A}$ is a selfadjoint extension of A with the same lower bound c, then $\hat{A}$ can be represented in the form of

$$\hat{A} = \int_c^\infty tE(dt).$$

Let $\{k_\lambda(t)\}$ be a family of bounded Borel functions on $[c, \infty)$, $\kappa_1, \kappa_2 > 0$ and

$$\psi_\lambda(t) := \frac{k_\lambda(t) - 1}{\lambda^{-\kappa_1} t^{\kappa_2}}. \tag{2.2}$$

Suppose that

(1) $\psi_\lambda(t)$ are uniformly bounded in λ and $t \in [c, \infty)$, and

(2) $\psi_\lambda(t)$ converge to a nonzero constant C as $\lambda \to \infty$ for any $t \in [c, \infty)$.

LEMMA 1. *If $f \in L^2(\Omega)$ and $g \in D(\hat{A}^{\kappa_2})$, then $\lambda^{\kappa_1}(k_\lambda(\hat{A})f - f, g) \to C(f, \hat{A}^{\kappa_2}g)$ as $\lambda \to \infty$.*

PROOF. By the definition of $k_\lambda(\hat{A})$, we have

$$\begin{aligned}
\lambda^{\kappa_1}(k_\lambda(\hat{A})f - f, g) &= \lambda^{\kappa_1} \int_c^\infty [k_\lambda(t) - 1](E(dt)f, g) \\
&= \int_c^\infty \frac{k_\lambda(t) - 1}{\lambda^{-\kappa_1}}(f, E(dt)g) = \int_c^\infty \frac{k_\lambda(t) - 1}{\lambda^{-\kappa_1} t^{\kappa_2}} t^{\kappa_2}(f, E(dt)g) \\
&= \int_c^\infty \psi_\lambda(t) t^{\kappa_2}(f, E(dt)g) = (f, \overline{\psi_\lambda}(\hat{A})\hat{A}^{\kappa_2}g) \\
&= \int_c^\infty \psi_\lambda(t)(f, E(dt)\hat{A}^{\kappa_2}g).
\end{aligned}$$

Let $\rho = (f, E(\cdot)\hat{A}^{\kappa_2}g)$ and $|\rho|$ be the total variation of ρ. Then

$$\int_c^\infty |\rho|(dt) \le \|f\|_{L^2(\Omega)} \|\hat{A}^{\kappa_2}g\|_{L^2(\Omega)} < \infty.$$

Therefore, by Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned}
\lim_{\lambda\to\infty} \lambda^{\kappa_1}(k_\lambda(\hat{A})f - f, g) &= \lim_{\lambda\to\infty} \int_c^\infty \psi_\lambda(t)\rho(dt) \\
&= \int_c^\infty \lim_{\lambda\to\infty} \psi_\lambda(t)\rho(dt) = C\int_c^\infty \rho(dt) = C(f, \hat{A}^{\kappa_2}g).
\end{aligned}$$

Thus Lemma 1 is proved.

Let G be an open subset in Ω with compact closure $\bar{G}$ and $1 < p \le \infty$. We say $A^{\kappa_2}f \in L^p(\bar{G})$ if

$$\|A^{\kappa_2}f\|_{L^p(\bar{G})} := \sup_{\|g\|_{L^{p'}(\bar{G})}=1} \left| \int_\Omega f(x)\overline{\hat{A}^{\kappa_2}g(x)}dx \right| < \infty,$$

where $1/p + 1/p' = 1$ and g is an infinitely differentiable function whose support is contained in $\bar{G}$.

THEOREM 1. *Let Ω be an open domain in $\boldsymbol{R}^n$ and A be a formally selfadjoint semi-bounded differential operator with coefficients in $C^\infty(\Omega)$ given by* (2.1). *Suppose that $\hat{A}$ is a selfadjoint extension of A with the same lower bound c. Let $\{k_\lambda(t)\}$ be a family of bounded Borel functions on $[c, \infty)$ and $\kappa_1, \kappa_2 > 0$ such that the sequence $\{\psi_\lambda(t)\}$ of Borel functions on $[c, \infty)$ given by* (2.2) *satisfies* (1) *and* (2). *Let $f \in L^2(\Omega)$, $1 < p \leq \infty$ and G be any open set in Ω with compact closure $\bar{G}$. Then the following hold.*

(i) *If*

$$\|k_\lambda(\hat{A})f - f\|_{L^p(\bar{G})} = O(\lambda^{-\kappa_1})$$

as $\lambda \to \infty$, then $A^{\kappa_2} f \in L^p(\bar{G})$.

(ii) *If*

$$\|k_\lambda(\hat{A})f - f\|_{L^p(\bar{G})} = o(\lambda^{-\kappa_1})$$

as $\lambda \to \infty$, then $A^{\kappa_2} f$ vanishes in $\bar{G}$.

PROOF. Let g be an infinitely differentiable function and $\operatorname{supp} g$ be its support. Suppose that $\operatorname{supp} g \subset \bar{G}$. Then by Lemma 1

$$\lambda^{\kappa_1}(k_\lambda(\hat{A})f - f, g) \to C(f, \hat{A}^{\kappa_2} g) \quad \text{as} \quad \lambda \to \infty. \tag{2.3}$$

On the other hand, we have

$$|\lambda^{\kappa_1}(k_\lambda(\hat{A})f - f, g)| \leq \lambda^{\kappa_1} \|k_\lambda(\hat{A})f - f\|_{L^p(\bar{G})} \|g\|_{L^{p'}(\bar{G})}. \tag{2.4}$$

If $\|k_\lambda(\hat{A})f - f\|_{L^p(\bar{G})} = O(\lambda^{-\kappa_1})$ as $\lambda \to \infty$, then by (2.4) for any λ

$$|\lambda^{\kappa_1}(k_\lambda(\hat{A})f - f, g)| \leq C' \|g\|_{L^{p'}(\bar{G})}$$

with some constant $C' > 0$. Therefore, by (2.3), we have

$$\left| \int_\Omega f(x) \overline{\hat{A}^{\kappa_2} g(x)} dx \right| = |(f, \hat{A}^{\kappa_2} g)| \leq C^{-1} C' \|g\|_{L^{p'}(\bar{G})}$$

for any g. Thus (i) is proved.

If $\|k_\lambda(\hat{A})f - f\|_{L^p(\bar{G})} = o(\lambda^{-\kappa_1})$ as $\lambda \to \infty$, then in the same way as in (i), (ii) is proved.

EXAMPLES. (1) Riesz summation: For $\kappa > 0$ and $\delta > 0$, the Riesz summation is given by the multiplier $k_\lambda(t) = [1 - (t/\lambda^2)^\kappa]_+^\delta$. In this case, $(\lambda^2/t)^\kappa [k_\lambda(t) - 1]$ are uniformly bounded in λ and $t \in [c, \infty)$ with a constant $c > 0$ and

$$\lim_{\lambda \to \infty} \frac{k_\lambda(t) - 1}{(\lambda^{-2} t)^\kappa} = \lim_{s \to +0} \frac{(1 - s^\kappa)^\delta - 1}{s^\kappa} = - \lim_{s \to +0} \delta (1 - s)^{\delta - 1} = -\delta$$

for any $t \in [c, \infty)$. Thus $\kappa_1 = 2\kappa$, $\kappa_2 = \kappa$ and $C = -\delta$, where C is a constant in (2).

(2) Fejér-Korovkin summation is defined by

$$k_\lambda(t) = \begin{cases} \left(1 - \dfrac{t}{\lambda^2}\right) \cos \dfrac{\pi t}{\lambda^2} + \dfrac{1}{\lambda^2} \cot \dfrac{\pi}{\lambda^2} \sin \dfrac{\pi t}{\lambda^2} & t < \lambda^2, \\ 0 & t \geq \lambda^2. \end{cases}$$

In this case, $(\lambda^2/t)^2[k_\lambda(t)-1]$ are uniformly bounded in λ and $t \in [c, \infty)$ and

$$\lim_{\lambda\to\infty} \frac{k_\lambda(t)-1}{(\lambda^{-2}t)^2} = \lim_{s\to+0} \frac{\cos\pi s - 1}{s^2} = \lim_{s\to+0} \frac{\cos^2\pi s - 1}{s^2(\cos\pi s + 1)}$$
$$= -\lim_{s\to+0} \frac{\sin^2\pi s}{s^2(\cos\pi s+1)} = -\frac{\pi^2}{2}$$

for any $t \in [c, \infty)$. Thus $\kappa_1 = 4$, $\kappa_2 = 2$ and $C = -\pi^2/2$.

(3) Rogosinski summation is given by

$$k_\lambda(t) = \begin{cases} \cos\dfrac{\pi t}{2\lambda^2} & t < \lambda^2, \\ 0 & t \geq \lambda^2. \end{cases}$$

In this case, $(\lambda^2/t)^2[k_\lambda(t)-1]$ are uniformly bounded in λ and $t \in [c, \infty)$ and

$$\lim_{\lambda\to\infty} \frac{k_\lambda(t)-1}{(\lambda^{-2}t)^2} = \lim_{s\to+0} \frac{\cos\frac{\pi}{2}s - 1}{s^2} = -\lim_{s\to+0} \frac{\sin^2\frac{\pi}{2}s}{s^2\left(\cos\frac{\pi}{2}s+1\right)} = -\left(\frac{\pi}{2}\right)^2\cdot\frac{1}{2} = -\frac{\pi^2}{8}$$

for any $t \in [c, \infty)$. Thus $\kappa_1 = 4$, $\kappa_2 = 2$ and $C = -\pi^2/8$.

(4) Jackson summation is given by

$$k_\lambda(t) = \begin{cases} 1 - \dfrac{3}{2}\left(\dfrac{t}{\lambda^2}\right)^2 + \dfrac{3}{4}\left(\dfrac{t}{\lambda^2}\right)^3 & t < \lambda^2, \\ \dfrac{1}{4}\left(2 - \dfrac{t}{\lambda^2}\right)^3 & \lambda^2 \leq t < 2\lambda^2, \\ 0 & t \geq 2\lambda^2. \end{cases}$$

In this case, $(\lambda^2/t)^2[k_\lambda(t)-1]$ are uniformly bounded in λ and $t \in [c, \infty)$ and $\lim_{\lambda\to\infty}(\lambda^2/t)^2[k_\lambda(t)-1] = -3/2$. Thus $\kappa_1 = 4$, $\kappa_2 = 2$ and $C = -3/2$.

(5) Gauss-Weierstrass summation: We consider the multiplier $k_\lambda^W(t) = \exp(-t/\lambda)$. The function of t $(\lambda/t)[k_\lambda(t)-1]$ is bounded uniformly in λ, and we have

$$\lim_{\lambda\to\infty} \frac{k_\lambda(t)-1}{\lambda^{-1}t} = \lim_{s\to+0} \frac{e^{-s}-1}{s} = -\lim_{s\to+0} e^{-s} = -1.$$

Thus $\kappa_1 = \kappa_2 = 1$ and $C = -1$. Poisson summation is given by the function $k_\lambda^P(t) = \exp(-\sqrt{t}/\lambda)$, and we have $\kappa_1 = 1$ and $\kappa_2 = 1/2$.

3. Estimates of $k_\lambda(\hat{A})f - f$. The aim of this section is to prove the following theorem.

THEOREM 2. *Let Ω be an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ be a nonnegative selfadjoint extension of $-\Delta$ in Ω. Suppose that K is a compact set in Ω and K' is a closed subset of K with* $\operatorname{dist}(K', K^c) > 0$. *Let $\{k_\lambda(t)\}$ be a family of bounded piecewise smooth functions on $[0, \infty)$ such that $k_\lambda(t)\sqrt{t}^{\,n/2-2\kappa_2+1/2} \in L^1(0, \infty)$ with a constant $\kappa_2 > 0$ and $k_\lambda(0) = 1$ for any λ.*

Suppose that $\{k_\lambda(t)\}$ *satisfies the conditions* (1.1), (1.2) *and* (1.3) *with a constant* $\kappa_1 > 0$ *and* $0 < R < \operatorname{dist}(K', K^c)$. *Let f be a regulated function in* $L^2(\Omega)$. *Suppose that* $1 < p \le \infty$ *and* $f \in L^p(K)$. *Then the following hold.*

(i) *If* $(-\Delta)^{\kappa_2} f \in L^p(K)$, *then*

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K')} = O(\lambda^{-\kappa_1}) \quad as \quad \lambda \to \infty .$$

(ii) *If* $(-\Delta)^{\kappa_2} f$ *vanishes in* K, *then*

$$\|k_\lambda(\hat{A})f - f\|_{L^p(K')} = o(\lambda^{-\kappa_1}) \quad as \quad \lambda \to \infty .$$

3.1 Generalized eigenfunction system. In order to prove Theorem 2, we shall use the generalized eigenfunction system corresponding to an ordered representation of $L^2(\Omega)$ associated with the Laplace operator.

We shall begin with several definitions. We consider $A = -\Delta$ as an operator in $L^2(\Omega)$ with the domain of definition $D(A) = C_c^\infty(\Omega)$. Let $\hat{A}$ be a nonnegative selfadjoint extension of A. Let $\mathfrak{B}$ be the Borel field on $\boldsymbol{R}$ and E be the unique spectral measure corresponding to $\hat{A}$. For $h \in L^2(\Omega)$, we define the following closed subspace of $L^2(\Omega)$:

$$\begin{aligned} H(h) :&= \{F(\hat{A})h;\ F \text{ is a Borel function on } \boldsymbol{R} \text{ and } h \in D(F(\hat{A}))\} \\ &= \{F(\hat{A})h;\ F \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h, h))\} . \end{aligned}$$

If $f \in H(h)$, then we can write uniquely $f = F(\hat{A})h$, where $F \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h, h))$ and

$$\|f\|_{L^2(\Omega)} = \left(\int_{\boldsymbol{R}} |F(t)|^2 (E(dt)h, h) \right)^{1/2} .$$

Therefore we can define an isomorphism U_h from $H(h)$ onto $L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h, h))$ by $U_h f := F$, which preserves inner products.

There exist a sequence of functions $\{h_j\} \subset L^2(\Omega)$ and a sequence of sets $\{e_j\} \subset \mathfrak{B}$, called the set of multiplicity, with the following properties (see [3, XII.3.16] or [4, Chap. 14]):

(I)

$$L^2(\Omega) = \bigoplus_j H(h_j) .$$

That is, $H(h_j)$ are mutually orthogonal and span $L^2(\Omega)$.

(II) $\boldsymbol{R} = e_1 \supseteq e_2 \supseteq \cdots$.

(III) $(E(e)h_j, h_j) = (E(e \cap e_j)h_1, h_1)$ for any $e \in \mathfrak{B}$.

By (I), for $f \in L^2(\Omega)$ we can write uniquely

$$f = \sum_j F_j(\hat{A})h_j ,$$

where $F_j \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h_j, h_j))$ and

$$\left(\sum_j \int_{\boldsymbol{R}} |F_j(t)|^2 (E(dt)h_j, h_j) \right)^{1/2} = \left(\sum_j \|F_j(\hat{A})h_j\|^2_{L^2(\Omega)} \right)^{1/2} = \|f\|_{L^2(\Omega)} < \infty .$$

Therefore we can define an isometry U from $L^2(\Omega)$ onto $\bigoplus_j L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h_j, h_j))$, which is equivalent to say

$$L^2(\Omega) \leftrightarrow \left\{ \{F_j\};\ F_j \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h_j, h_j)) \text{ and } \sum_j \int_{\boldsymbol{R}} |F_j(t)|^2 (E(dt)h_j, h_j) < \infty \right\},$$

and the correspondence is given by $Uf := \{F_j\}$. We denote $F_j =: (Uf)_j$.

By (III) we have

$$\bigoplus_j L^2(\boldsymbol{R}, (E(\cdot)h_j, h_j)) = \bigoplus_j L^2(e_j, (E(\cdot)h_1, h_1)).$$

Let $\rho := (E(\cdot)h_1, h_1)$. Then U is an isomorphism from $L^2(\Omega)$ onto $\bigoplus_j L^2(e_j, \rho)$ which preserves inner products, that is, for any $f, g \in L^2(\Omega)$ it holds that

$$(f, g)_{L^2(\Omega)} = \sum_j \int_{e_j} (Uf)_j(t)\overline{(Ug)_j(t)}\rho(dt). \tag{3.1}$$

U is called an ordered representation of $L^2(\Omega)$ with respect to $\hat{A}$.

With these understood, there exists a sequence of functions $\{u_j(x, t)\}$ defined on the product space of $\Omega \times \boldsymbol{R}$ such that the following conditions are satisfied (see [3, XII.3 and XIV.6] or [4, Chap. 15]):

(i) The functions $u_j(x, t)$ are $dx \times d\rho(t)$-measurable and vanish outside $\Omega \times e_j$, where dx is the Lebesgue measure.

(ii) For any fixed $t \in \boldsymbol{R}$, each $u_j(x, t)$ belongs $C^\infty(\Omega)$ and satisfies

$$-\Delta u_j(x, t) = t u_j(x, t), \quad x \in \Omega. \tag{3.2}$$

(iii) For each compact subset K of Ω and each bounded Borel set e in $\boldsymbol{R}$

$$\operatorname*{ess\,sup}_{x \in K} \int_e |u_j(x, t)|^2 \rho(dt) < \infty.$$

(iv) For each $f \in L^2(\Omega)$

$$(Uf)_j(t) = \int_\Omega f(x)\overline{u_j(x, t)}dx, \tag{3.3}$$

where the integral exists in the sense of $L^2(e_j, \rho)$.

(v) For each $f \in L^2(\Omega)$ and each $e \in \mathfrak{B}$

$$E(e)f(x) = \sum_j \int_e (Uf)_j(t) u_j(x, t)\rho(dt), \tag{3.4}$$

where the integral exists and the series converges in the sense of $L^2(\Omega)$.

$\{u_j\}$ is called the generalized eigenfunction system of $\hat{A}$ corresponding to U. By (v), for $f \in L^2(\Omega)$ we have

$$f(x) = \sum_j \int_{\boldsymbol{R}} (Uf)_j(t) u_j(x, t) \rho(dt) \tag{3.5}$$

and

$$k_\lambda(\hat{A}) f(x) = \sum_j \int_{\boldsymbol{R}} k_\lambda(t) (Uf)_j(t) u_j(x, t) \rho(dt)\,. \tag{3.6}$$

3.2 Decomposition of $k_\lambda(\hat{A})f - f$. Throughout what follows, Ω denotes an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ is a nonnegative selfadjoint extension of $-\Delta$. Let U denote an ordered representation of $L^2(\Omega)$ with respect to $\hat{A}$, $\{u_j\}$ the generalized eigenfunction system and ρ the measure associated with U. We denote the gamma function by Γ, the unit sphere in $\boldsymbol{R}^n$ by S^{n-1}, the Lebesgue measure on the unit sphere S^{n-1} by σ and the surface area $2\sqrt{\pi}^{\,n}/\Gamma(n/2)$ of S^{n-1} by ω_n. Let κ_2 be a constant in (1.1), (1.2) and (1.3), and $\nu = n/2 - 2\kappa_2 + 1$.

LEMMA 2. *Let $f \in L^2(\Omega)$, $x \in \Omega$ and $R > 0$. Then*

$$\begin{aligned}
&k_\lambda(\hat{A}) f(x) - f(x) \\
&\quad = -\sum_j \int_0^\infty t (Uf)_j(t) u_j(x, t) \rho(dt) \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} s ds \\
&\qquad + \sum_j \int_0^\infty \frac{(Uf)_j(t) u_j(x, t)}{\sqrt{t}^{\,\nu}} \rho(dt) \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t}r) r dr \\
&\qquad - f(x) \times \frac{1}{2^\nu \Gamma(\nu+1)} \int_R^\infty I_\lambda(r) r^{\nu+1} dr\,,
\end{aligned}$$

where

$$I_\lambda(r) = \int_0^\infty k_\lambda(s^2) J_\nu(rs) s^{\nu+1} ds\,.$$

PROOF. First observe that the function $k_\lambda(t)$ is piecewise smooth on $[0, \infty)$ and $k_\lambda(t)\sqrt{t}^{\,\nu-1}$ is integrable on $(0, \infty)$. By Hankel's integral formula ([2, p. 73, (60)]), we have

$$\begin{aligned}
k_\lambda(t) &= \frac{1}{\sqrt{t}^{\,\nu}} \int_0^\infty J_\nu(\sqrt{t}r) r dr \int_0^\infty k_\lambda(s^2) J_\nu(rs) s^{\nu+1} ds \\
&= \frac{1}{\sqrt{t}^{\,\nu}} \int_0^\infty I_\lambda(r) J_\nu(\sqrt{t}r) r dr\,.
\end{aligned}$$

Then, by (3.5), (3.6) and the fact that $k_\lambda(0) = 1$, we have

$$\begin{aligned} &k_\lambda(\hat{A})f(x) - f(x) \\ &= \sum_j \int_0^\infty \{k_\lambda(t) - k_\lambda(0)\}(Uf)_j(t)u_j(x,t)\rho(dt) \\ &= \sum_j \int_0^\infty (Uf)_j(t)u_j(x,t)\rho(dt) \int_0^\infty \left\{ \frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda(r) r dr \\ &= \sum_j \int_0^\infty (Uf)_j(t)u_j(x,t)\rho(dt) \int_0^R \left\{ \frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda(r) r dr \\ &\quad + \sum_j \int_0^\infty (Uf)_j(t)u_j(x,t)\rho(dt) \int_R^\infty \left\{ \frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda(r) r dr . \end{aligned}$$

Now apply the formula ([7, p. 45])

$$\frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} = -tr^\nu \int_0^r \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} s ds .$$

Note that for the second term, we have

$$\begin{aligned} &\sum_j \int_0^\infty (Uf)_j(t)u_j(x,t)\rho(dt) \int_R^\infty \left\{ \frac{J_\nu(\sqrt{t}r)}{\sqrt{t}^\nu} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda(r) r dr \\ &\quad = \sum_j \int_0^\infty \frac{(Uf)_j(t)u_j(x,t)}{\sqrt{t}^\nu} \rho(dt) \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t}r) r dr \\ &\quad - f(x) \times \frac{1}{2^\nu \Gamma(\nu+1)} \int_R^\infty I_\lambda(r) r^{\nu+1} dr . \end{aligned}$$

Thus we get Lemma 2.

3.3 Proof of Theorem 2. Let f be a regulated function in $L^2(\Omega)$. Let K be a compact set in Ω and K' be a closed set in K with $\operatorname{dist}(K', K^c) > 0$. We choose $0 < R < \operatorname{dist}(K', K^c)$. Let κ_1 and κ_2 be constants in (1.1), (1.2) and (1.3). Let $\nu = n/2 - 2\kappa_2 + 1$ and $1 < p \le \infty$. Suppose that $f \in L^p(K)$ and $(-\Delta)^{\kappa_2} f \in L^p(K)$. By Lemma 2, we have

$$\begin{aligned} &\|k_\lambda(\hat{A})f - f\|_{L^p(K')} \le \|f\|_{L^p(K')} \times \frac{1}{2^\nu \Gamma(\nu+1)} \left| \int_R^\infty I_\lambda(r) r^{\nu+1} dr \right| \\ &\quad + \left\| \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r s ds \sum_j \int_0^\infty t (Uf)_j(t) u_j(\cdot, t) \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} \rho(dt) \right\|_{L^p(K')} \\ &\quad + \left\| \sum_j \int_0^\infty \frac{(Uf)_j(t)u_j(\cdot,t)}{\sqrt{t}^\nu} \rho(dt) \int_R^\infty I_\lambda(r) J_\nu(\sqrt{t}r) r dr \right\|_{L^\infty(K')} \end{aligned} \tag{3.7}$$

LEMMA 3. *We have*

$$\left\| \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r s ds \sum_j \int_0^\infty t(Uf)_j(t) u_j(\cdot, t) \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} \rho(dt) \right\|_{L^p(K')}$$
$$\leq C\lambda^{-\kappa_1} \|(-\Delta)^{\kappa_2} f\|_{L^p(K)} .$$

PROOF. Let $x \in K'$ and $0 < s < R$. Put

$$g_s(y) = \frac{1}{s^{\nu+1}|y|^{n/2-1}} \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(|y|r) dr ,$$
$$g_s^x(y) = g_s(x-y) .$$

If $|y| > s$, then $g_s(y) = 0$ ([7, p. 404, (6)]). Therefore $\operatorname{supp} g_s^x \subset K \subset \Omega$. Then, by (3.3), we have

$$\begin{aligned}
(Ug_s^x)_j(t) &= \int_\Omega g_s^x(y) \overline{u_j(y,t)} dy = \int_\Omega g_s(y) \overline{u_j(x-y,t)} dy \\
&= \frac{1}{s^{\nu+1}} \int \overline{u_j(x-y,t)} dy \frac{1}{|y|^{n/2-1}} \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(|y|r) dr \\
&= \frac{1}{s^{\nu+1}} \int_0^\infty q^{n/2} dq \int_{S^{n-1}} \overline{u_j(x-qw,t)} \sigma(dw) \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(qr) dr .
\end{aligned}$$

On the other hand, by (3.2), $u_j(y,t) \in C^\infty(\Omega)$, and we have $-\Delta u_j(y,t) = t u_j(y,t)$ for $y \in \Omega$. Therefore, by the mean-value formula, we have

$$\int_{S^{n-1}} u_j(x-qw,t) \sigma(dw) = \sqrt{2\pi}^n \frac{J_{n/2-1}(\sqrt{t}q)}{(\sqrt{t}q)^{n/2-1}} u_j(x,t) .$$

Thus, by Hankel's formula, we have

$$\begin{aligned}
(Ug_s^x)_j(t) &= \frac{\sqrt{2\pi}^n}{\sqrt{t}^{n/2-1} s^{\nu+1}} \overline{u_j(x,t)} \int_0^\infty J_{n/2-1}(\sqrt{t}q) q dq \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(qr) dr \\
&= \frac{\sqrt{2\pi}^n J_{\nu+1}(\sqrt{t}s)}{\sqrt{t}^{n/2} s^{\nu+1}} \overline{u_j(x,t)} .
\end{aligned}$$

We can assume that $f \in C_c^\infty(\Omega)$ by approximation. Then, by (3.1), we have

$$\begin{aligned}
&\sum_j \int_0^\infty t(Uf)_j(t) u_j(x,t) \frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}} \rho(dt) \\
&= \frac{1}{\sqrt{2\pi}^n} \sum_j \int_{e_j} t^{\kappa_2} (Uf)_j(t) \overline{(Ug_s^x)_j(t)} \rho(dt) \\
&= \frac{1}{\sqrt{2\pi}^n} \int_\Omega [(-\Delta)^{\kappa_2} f(y)] g_s^x(y) dy = \frac{1}{\sqrt{2\pi}^n} \int_\Omega [(-\Delta)^{\kappa_2} f(y)] g_s(x-y) dy .
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\int_0^R I_\lambda(r)r^{\nu+1}dr\int_0^r sds\sum_j\int_0^\infty t(Uf)_j(t)u_j(x,t)\frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}}\rho(dt)\\
&\quad=\frac{1}{\sqrt{2\pi}^n}\int_0^R I_\lambda(r)r^{\nu+1}dr\int_0^r sds\int_{|y|<s}[(-\Delta)^{\kappa_2}f(x-y)]g_s(y)dy\\
&\quad=\frac{1}{\sqrt{2\pi}^n}\int_0^R sds\int_s^R I_\lambda(r)r^{\nu+1}dr\int_{|y|<s}[(-\Delta)^{\kappa_2}f(x-y)]g_s(y)dy\,.
\end{aligned}$$

Applying successively Minkowski's inequality for integral, we have

$$\begin{aligned}
&\left\|\int_0^R I_\lambda(r)r^{\nu+1}dr\int_0^r sds\sum_j\int_0^\infty t(Uf)_j(t)u_j(\cdot,t)\frac{J_{\nu+1}(\sqrt{t}s)}{(\sqrt{t}s)^{\nu+1}}\rho(dt)\right\|_{L^p(K')}\\
&\leq\frac{1}{\sqrt{2\pi}^n}\int_0^R sds\left\|\int_s^R I_\lambda(r)r^{\nu+1}dr\int_{|y|<s}[(-\Delta)^{\kappa_2}f(\cdot-y)]g_s(y)dy\right\|_{L^p(K')}\\
&=\frac{1}{\sqrt{2\pi}^n}\int_0^R sds\left|\int_s^R I_\lambda(r)r^{\nu+1}dr\right|\left\|\int_{|y|<s}[(-\Delta)^{\kappa_2}f(\cdot-y)]g_s(y)dy\right\|_{L^p(K')}\\
&\leq\frac{1}{\sqrt{2\pi}^n}\int_0^R sds\left|\int_s^R I_\lambda(r)r^{\nu+1}dr\right|\int_{|y|<s}\|(-\Delta)^{\kappa_2}f(\cdot-y)\|_{L^p(K')}|g_s(y)|dy\\
&\leq\frac{1}{\sqrt{2\pi}^n}\|(-\Delta)^{\kappa_2}f\|_{L^p(K)}\int_0^R sds\left|\int_s^R I_\lambda(r)r^{\nu+1}dr\right|\int_{|y|<s}|g_s(y)|dy\,.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{|y|<s}|g_s(y)|dy&=\frac{1}{s^{\nu+1}}\int_{|y|<s}\frac{1}{|y|^{n/2-1}}dy\left|\int_0^\infty J_{\nu+1}(sr)J_{n/2-1}(|y|r)dr\right|\\
&=\frac{\omega_n}{s^{\nu+1}}\int_0^s q^{n/2}dq\left|\int_0^\infty J_{\nu+1}(sr)J_{n/2-1}(qr)dr\right|\\
&=\frac{\omega_n\Gamma((2\nu+n+2)/4)}{\Gamma(n/2)\Gamma((2\nu-n+6)/4)s^{\nu+n/2+1}}\\
&\quad\times\int_0^s|{}_2F_1((2\nu+n+2)/4,-(2\nu-n+2)/4;n/2;q^2/s^2)|q^{n-1}dq\\
&\leq\frac{C_{\kappa_2}}{s^{\nu-n/2+1}}\,,
\end{aligned}$$

where ${}_2F_1(\alpha,\beta;\gamma;z)$ is Gauss' hypergeometric function. Therefore the last term is bounded by

$$C_{\kappa_2}\|(-\Delta)^{\kappa_2}f\|_{L^p(K)}\int_0^R s^{2\kappa_2-1}ds\left|\int_s^R I_\lambda(r)r^{\nu+1}dr\right|\,.$$

By the condition (1.1), we get the bound $C\lambda^{-\kappa_1}\|(-\Delta)^{\kappa_2}f\|_{L^p(K)}$ for the last term. Thus Lemma 3 is proved.

We shall use the following lemma ([1, p. 655]).

LEMMA 4. *Under the assumptions above, if K is a compact set contained in Ω, then*

$$\left(\sum_j \int_{T\le\sqrt{t}\le T+1} |u_j(x,t)|^2 \rho(dt)\right)^{1/2} \le C_K (T+1)^{(n-1)/2},$$

where C_K is a constant independent of $T \ge 0$ and $x \in K$.

LEMMA 5. *We have*

$$\left\| \sum_j \int_0^\infty \frac{(Uf)_j(t)u_j(\cdot,t)}{\sqrt{t}^\nu}\rho(dt)\int_R^\infty I_\lambda(r)J_\nu(\sqrt{t}r)r\,dr \right\|_{L^\infty(K)} = o(\lambda^{-\kappa_1})$$

as $\lambda \to \infty$.

PROOF. We have, by Schwarz's inequality,

$$\begin{aligned}&\left| \sum_j \int_0^\infty \frac{(Uf)_j(t)u_j(x,t)}{\sqrt{t}^\nu}\rho(dt)\int_R^\infty I_\lambda(r)J_\nu(\sqrt{t}r)r\,dr \right| \\ &\quad\le \left(\sum_j \int_{e_j} |(Uf)_j(t)|^2\rho(dt)\right)^{1/2} \\ &\qquad\times \left(\sum_j \int_0^\infty \frac{|u_j(x,t)|^2}{t^\nu}\rho(dt)\left|\int_R^\infty I_\lambda(r)J_\nu(\sqrt{t}r)r\,dr\right|^2\right)^{1/2}.\end{aligned}$$

Now, by (3.1), we have

$$\left(\sum_j \int_{e_j} |(Uf)_j(t)|^2\rho(dt)\right)^{1/2} = \|f\|_{L^2(\Omega)}.$$

By Lemma 4, there exists a constant C_K such that

$$\begin{aligned}&\left(\sum_j \int_0^\infty \frac{|u_j(x,t)|^2}{t^\nu}\rho(dt)\left|\int_R^\infty I_\lambda(r)J_\nu(\sqrt{t}r)r\,dr\right|^2\right)^{1/2} \\ &\quad\le C_K \left(\sum_{T=0}^\infty T^{4\kappa_2-3} \max_{T\le s\le T+1}\left|\int_R^\infty I_\lambda(r)J_\nu(sr)r\,dr\right|^2\right)^{1/2}\end{aligned}$$

uniformly in $x \in K$. Therefore, by (1.3), we have

$$\left| \sum_j \int_0^\infty \frac{(Uf)_j(t)u_j(x,t)}{\sqrt{t}^\nu}\rho(dt)\int_R^\infty I_\lambda(r)J_\nu(\sqrt{t}r)r\,dr \right| = o(\lambda^{-\kappa_1})$$

uniformly in $x \in K$ as $\lambda \to \infty$. Thus Lemma 5 is proved.

We remark that $|\int_R^\infty I_\lambda(r)r^{\nu+1}dr| = o(\lambda^{-\kappa_1})$ by the assumption (1.2).

By (3.7) together with Lemmas 3 and 5, $\|k_\lambda(\hat{A})f - f\|_{L^p(K')} = O(\lambda^{-\kappa_1})$ as $\lambda \to \infty$. If $(-\Delta)^{\kappa_2} f$ vanishes in K, then by Lemma 3.

$$\left\| \int_0^R I_\lambda(r) r^{\nu+1} dr \int_0^r s ds \sum_j \int_0^\infty t(Uf)_j(t) u_j(\cdot, t) \frac{J_{\nu+1}(\sqrt{ts})}{(\sqrt{ts})^{\nu+1}} \rho(dt) \right\|_{L^p(K')} = 0 .$$

Therefore, by (3.7) and Lemma 5, we have $\|k_\lambda(\hat{A})f - f\|_{L^p(K')} = o(\lambda^{-\kappa_1})$ as $\lambda \to \infty$. Consequently, Theorem 2 is proved.

4. Applications of main theorem.

4.1 Proof of Corollary 1. Let $k_\lambda(t) = (1 - t/\lambda^2)_+^\delta$. Then we have the formula (see [2, p. 92, (34)])

$$k_\lambda(t) = \frac{2^\delta \Gamma(\delta+1)}{\lambda^{\delta - n/2} \sqrt{t}^{\,n/2-1}} \int_0^\infty \frac{J_{n/2+\delta}(\lambda r) J_{n/2-1}(\sqrt{t} r)}{r^\delta} dr ,$$

and can take $\kappa_2 = 1$. We have

$$I_\lambda(r) = \int_0^\infty k_\lambda(t^2) J_{n/2-1}(rt) t^{n/2} dt = 2^\delta \Gamma(\delta+1) \lambda^{n/2-\delta} J_{n/2+\delta}(\lambda r) r^{-\delta-1} .$$

To check the conditions (1.1), (1.2) and (1.3), let $R > 0$ and $\delta > (n-3)/2$. Then we have

$$\left| \int_R^\infty I_\lambda(r) r^{n/2} dr \right| = 2^\delta \Gamma(\delta+1) \lambda^{n/2-\delta} \left| \int_R^\infty \frac{J_{n/2+\delta}(\lambda r)}{r^{\delta - n/2+1}} dr \right| \le C_{\delta,R} \lambda^{(n-3)/2-\delta} .$$

On the other hand, we have

$$\begin{aligned} \int_0^R s ds \left| \int_s^R I_\lambda(r) r^{n/2} dr \right| &= 2^\delta \Gamma(\delta+1) \lambda^{n/2-\delta} \int_0^R s ds \left| \int_s^R \frac{J_{n/2+\delta}(\lambda r)}{r^{\delta-n/2+1}} dr \right| \\ &\le \begin{cases} C_\delta \lambda^{(n-3)/2-\delta} & \text{if} \quad (n-3)/2 < \delta < (n+1)/2 , \\ C_\delta \lambda^{(n-3)/2-\delta} \log \lambda & \text{if} \quad \delta = (n+1)/2 , \\ C_\delta \lambda^{-2} & \text{if} \quad \delta > (n+1)/2 . \end{cases} \end{aligned}$$

We now apply the estimates (see [6, p. 202, Lemma 18.10 a])

$$\begin{aligned} &\left| \int_R^\infty \frac{J_{n/2+\delta}(\lambda r) J_{n/2-1}(sr)}{r^\delta} dr \right| \\ &\quad \le \begin{cases} C_{\delta,R} \lambda^{-1/2} s^{-1/2} & \text{if} \quad s, \lambda > 0 , \\ C_{\delta,R} \dfrac{\lambda^{-3/2} s^{1/2}}{\lambda - s} + C_{\delta,R} \lambda^{-3/2} s^{-1/2} & \text{if} \quad 0 < s < \lambda , \\ C_{\delta,R} \dfrac{\lambda^{1/2} s^{-3/2}}{s - \lambda} + C_{\delta,R} \lambda^{-1/2} s^{-3/2} & \text{if} \quad 0 < \lambda < s . \end{cases} \end{aligned}$$

Then we have

$$\left(\sum_{T=0}^{\infty} T \max_{T\le s\le T+1}\left|\int_R^{\infty} I_\lambda(r)J_{n/2-1}(sr)rdr\right|^2\right)^{1/2}$$
$$= 2^\delta \Gamma(\delta+1)\lambda^{n/2-\delta}\left(\sum_{T=0}^{\infty} T \max_{T\le s\le T+1}\left|\int_R^{\infty} \frac{J_{n/2+\delta}(\lambda r)J_{n/2-1}(sr)}{r^\delta}dr\right|^2\right)^{1/2}$$
$$\le C_{\delta,R}\lambda^{(n-1)/2-\delta}.$$

If $\delta > (n+3)/2$, then the last term is $o(\lambda^{-2})$. Thus Corollary 1 follows from Main theorem.

4.2 The Gauss-Weierstrass summation. Let $k_\lambda^W(t) = e^{-t/\lambda}(\lambda \to \infty)$. We then have

$$\int_0^\infty k_\lambda^W(t^2)J_\nu(rt)t^{\nu+1}dt = \int_0^\infty e^{-t^2/\lambda}J_\nu(rt)t^{\nu+1}dt = \frac{\lambda^{\nu+1}r^\nu}{2^{\nu+1}}\exp\left(-\frac{\lambda r^2}{4}\right) \tag{4.1}$$

(cf. [2, 7.7.3]). Let Ω be an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ be a nonnegative selfadjoint extension of $-\Delta$ in Ω.

COROLLARY 2. *Let f be a regulated function in $L^2(\Omega)$. Suppose that $1 < p \le \infty$ and $f \in L^p_{\mathrm{loc}}(\Omega)$. Then the following hold.*

(i) *The following are equivalent.*

(ia)

$$\|k_\lambda^W(\hat{A})f - f\|_{L^p(K)} = O(\lambda^{-1})$$

as $\lambda \to \infty$ for every compact set K in Ω.

(ib) $\Delta f \in L^p_{\mathrm{loc}}(\Omega)$.

(ii) *Let $G \subset \Omega$ be any open set.*

(iia) *Suppose that Δf vanishes in G. Then*

$$\|k_\lambda^W(\hat{A})f - f\|_{L^p(K)} = o(\lambda^{-1})$$

as $\lambda \to \infty$ for any compact set $K \subset G$.

(iib) *If*

$$\|k_\lambda^W(\hat{A})f - f\|_{L^p(K)} = o(\lambda^{-1})$$

as $\lambda \to \infty$ for any compact set $K \subset G$, then Δf vanishes in G.

PROOF. For the Gauss-Weierstrass summation method we take $\kappa_2 = 1$. Let R be a small positive number. By (4.1), we have

$$\int_R^\infty r^{n/2}dr\int_0^\infty k_\lambda^W(t^2)J_\nu(rt)t^{\nu+1}dt = \left(\frac{\lambda}{2}\right)^{n/2}\int_R^\infty r^{n-1}\exp\left(-\frac{\lambda r^2}{4}\right)dr = o(\lambda^{-1}),$$

$$\int_0^R sds\left|\int_s^R r^{n/2}dr\int_0^\infty k_\lambda^W(t^2)J_\nu(rt)t^{\nu+1}dt\right|$$
$$= \left(\frac{\lambda}{2}\right)^{n/2}\int_0^R sds\int_s^R r^{n-1}\exp\left(-\frac{\lambda r^2}{4}\right)dr = O(\lambda^{-1})$$

and

$$\left(\sum_{T=0}^{\infty} T \max_{T\le s\le T+1}\left|\int_R^\infty J_{n/2-1}(sr)rdr\int_0^\infty k_\lambda^W(t^2)J_\nu(rt)t^{\nu+1}dt\right|^2\right)^{1/2}$$
$$=\left(\frac{\lambda}{2}\right)^{n/2}\left(\int_R^\infty r^{n-1}\exp\left(-\frac{\lambda r^2}{2}\right)dr\right)^{1/2}=o(\lambda^{-1}).$$

Thus Corollary 2 follows from Main theorem.

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MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI 980–8578
JAPAN

E-mail address: tanigaki@math.tohoku.ac.jp