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WHITTAKER-SHINTANI FUNCTIONS FOR ORTHOGONAL GROUPS

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Abstract. As generalizations of zonal spherical functions and Whittaker functions, certain special functions on p-adic orthogonal groups closely related to automorphic forms are introduced. Their multiplicity one property and explicit formula are established.

Introduction.

0.1. The object of this paper is to study certain special functions on orthogonal groups over p-adic fields, which naturally arise from the investigation of automorphic L-functions for these groups.

Let $\mathbf{G} = \mathbf{SO}_m$ be a split special orthogonal group of degree $m = m' + 2r + 1$ ($r \geq 0$) defined over a non-archimedean local field k with the ring of integers o. Let $\mathbf{Q}$ be a parabolic subgroup of $\mathbf{G}$ whose Levi subgroup is isomorphic to $\mathbf{SO}_{m'+1} \times (\mathbf{GL}_1)^r$. We embed another split special orthogonal group $\mathbf{G}' = \mathbf{SO}_{m'}$ into $\mathbf{SO}_{m'+1}$ as the stabilizer of an anisotropic vector, and regard $\mathbf{G}'$ as a subgroup of $\mathbf{G}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{Q}$. We denote by $G = \mathbf{G}(k)$ and $G' = \mathbf{G}'(k)$ the groups of k-rational points of $\mathbf{G}$ and $\mathbf{G}'$, respectively. (As above, algebraic groups are denoted in boldface letters, while the corresponding groups of k-rational points in italic letters.) We also let $K = G \cap \mathbf{GL}_m(o)$ and $K' = G' \cap \mathbf{GL}_{m'}(o)$ be maximal open compact subgroups of G and G', respectively. We choose a generic character $\psi_U : U \to \boldsymbol{C}^\times$ invariant under the action of G' on U.

Let us denote by L and R the left and the right regular representations of G on a suitable function space on G, respectively. Let $C^\infty(G, \psi_U)$ be the space of smooth functions F on G satisfying $L(u)F = \psi_U(u)F$ for $u \in U$. Under the assumption on ψ_U, the group G' acts on $C^\infty(G, \psi_U)$ via the left translation so that $C^\infty(G, \psi_U)$ becomes a $G \times G'$ module. (The G-action is the right regular one.)

Let $\mathcal{H} = \mathcal{H}(G, K)$ (resp. $\mathcal{H}' = \mathcal{H}(G', K')$) be the Hecke algebra of (G, K) (resp. (G', K')) over $\boldsymbol{C}$. They act on $C^\infty(G, \psi_U)^{K \times K'}$, the space of $K \times K'$-fixed vectors in $C^\infty(G, \psi_U)$. For $\omega \in \mathrm{Hom}_{\boldsymbol{C}\text{-alg}}(\mathcal{H}, \boldsymbol{C})$ and $\omega' \in \mathrm{Hom}_{\boldsymbol{C}\text{-alg}}(\mathcal{H}', \boldsymbol{C})$, we define the space of *Whittaker-Shintani functions* attached to (ω, ω') to be the space of (ω, ω')-eigenvectors in $C^\infty(G, \psi_U)^{K \times K'}$. Namely, a function F on G is said to be a Whittaker-Shintani function

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attached to (ω, ω') if it satisfies the following two conditions:

$$L(uk')R(k)F = \psi_U(u)F \quad (u \in U,\ k' \in K',\ k \in K)\,; \tag{0.1.1}$$

$$L(\varphi')R(\varphi)F = \omega'(\varphi')\omega(\varphi)F \quad (\varphi' \in \mathcal{H}',\ \varphi \in \mathcal{H})\,. \tag{0.1.2}$$

0.2. In this paper, with an application to the theory of automorphic L-functions in mind, we prove that the space of Whittaker-Shintani functions with arbitrary eigenvalues (ω, ω') is one-dimensional, and give an explicit formula for the Whittaker-Shintani functions in terms of the Satake parameters attached to (ω, ω'). In a subsequent paper, by using the uniqueness and the explicit formula presented here, we will show that certain Rankin-Selberg convolutions actually give integral expressions of the standard L-functions for $\mathbf{SO} \times \mathbf{GL}$ (see [KMS]). This kind of convolution is also studied in [GPR].

Our Whittaker-Shintani functions are studied by several authors. When $m' = 0$ or 1, the functions considered here are the usual Whittaker functions. The explicit formula has been given by Casselman-Shalika [CS] and one of the authors [K1] independently. In the case where $m' = 2$, Novodvorsky studied these functions, whose explicit formula is given in [BFF]. We note that $\mathbf{G}'$ is abelian for $m' \leq 2$. The case where $m' \geq 3$ is considered in [GPR]. On the other hand, if $r = 0$, the Whittaker-Shintani functions coincide with the special functions studied in [MS2], in which they are called Shintani functions.

In the course of our investigation of Whittaker-Shintani functions, it is indispensable to study the double coset decomposition $UK'\backslash G/K$, since those functions satisfy (0.1.1). We shall show that we can choose essentially a subset of maximal torus as representative for the decomposition. This result may be considered as an analogue/mixture of usual Cartan and Iwasawa decompositions for p-adic groups.

0.3. We now explain our results more precisely. Let $\mathbf{P}$ (resp. $\mathbf{T}$) be the Borel subgroup (resp. the maximal torus) of $\mathbf{G}$ consisting of upper triangular matrices (resp. diagonal matrices) in $\mathbf{G}$. We assume that $\mathbf{P} \subset \mathbf{Q}$. We denote by $\mathbf{P}'$ and $\mathbf{T}'$ the subgroups of $\mathbf{G}'$ corresponding to the above $\mathbf{P}$ and $\mathbf{T}$. We have the Cartan decompositions $G = KT^{++}K$ and $G' = KT'^{++}K'$ for some subsemigroups $T^{++} \subset T$ and $T'^{++} \subset T'$.

THEOREM 0.4 (See Theorems 5.1 and 6.1.).

(1) *There exist an element $g_{m,r} \in G$ and a subsemigroup $\tilde{T}^{++}$ of T containing T^{++} such that the decomposition $G = UK'T'^{++}g_{m,r}\tilde{T}^{++}K$ holds.*

(2) *The support of any Whittaker-Shintani function is contained in $UK'T'^{++}g_{m,r}$ $T^{++}K$.*

Thus Whittaker-Shintani functions are determined by the value on the "torus" as zonal spherical functions and Whittaker functions are.

Let (ω, ω') be a pair of "eigenvalues" as in 0.1. The Satake parameter of ω is an element Ξ of $X_{nr}(T)$, the group of unramified characters of T ([Sa]). We shall naturally identify $X_{nr}(T)$ with $(\mathbf{C}^\times)^l$, $l = \dim \mathbf{T}$ so that $\Xi = (\Xi_1, \dots, \Xi_l) \in (\mathbf{C}^\times)^l$. Similarly, we let ξ be the Satake parameter of ω'; hence $\xi = (\xi_1, \dots, \xi_{l'}) \in (\mathbf{C}^\times)^{l'} \simeq X_{nr}(T')$ ($l' = \dim \mathbf{T}'$).

The Weyl group $W = W(G, T)$ canonically acts on $X_{nr}(T)$ (via permutation of coordinates $\{\Xi_i, \Xi_i^{-1}\ (1 \le i \le l)\}$). The same holds for the action of $W' = W(G', T')$ on $X_{nr}(T')$.

Since $UK'T'^{++}g_{m,r}T^{++}K = UK'T'^{++}g_{m,r}w_\ell(T^{++})^{-1}K$, where $w_\ell \in K$ is a representative of the longest element of W, Whittaker-Shintani functions are determined by their values on $T'^{++}g_{m,r}w_\ell(T^{++})^{-1}$.

Let us define a rational function $\mathbf{c}_{\mathrm{WS}}(\Xi, \xi)$ in Ξ and ξ by

$$\mathbf{c}_{\mathrm{WS}}(\Xi, \xi) = \frac{\mathbf{b}(\Xi, \xi)}{\mathbf{d}_m(\Xi)\mathbf{d}_{m'}(\xi)}\,,$$

where

$$\mathbf{b}(\Xi, \xi) = \prod_{\substack{1 \le i \le l' \\ 1 \le j \le l}} (1 - q^{-1/2}(\xi_i^{-1}\Xi_j)^{\eta_{ij}})(1 - q^{-1/2}\xi_i \Xi_j)$$

(q=the cardinality of the residue field of k; $\eta_{ij} = 1\ (j \le r + i),\ = -1\ (j > r + i)$)

and

$$\mathbf{d}_m(\Xi) = \begin{cases} \displaystyle\prod_{1 \le i < j \le l} (1 - \Xi_i \Xi_j^{-1})(1 - \Xi_i \Xi_j) \prod_{1 \le i \le l} (1 - \Xi_i^2) & \text{if } m = 2l + 1\,, \\ \displaystyle\prod_{1 \le i < j \le l} (1 - \Xi_i \Xi_j^{-1})(1 - \Xi_i \Xi_j) & \text{if } m = 2l\,. \end{cases}$$

(The definition of $\mathbf{d}_{m'}(\xi)$ is similar.)

THEOREM 0.5 (See Theorem 10.9). *For any (ω, ω'), the space of Whittaker-Shintani functions attached to (ω, ω') is one-dimensional, and is spanned by the function F given by the following formula,*

$$F(t' g_{m,r} w_\ell t^{-1}) = \sum_{\substack{w \in W \\ w' \in W'}} \mathbf{c}_{\mathrm{WS}}(w\Xi, w'\xi)((w\Xi)^{-1}\delta^{1/2})(t)((w'\xi)^{-1}\delta'^{1/2})(t')\,.$$

Here δ (resp. δ') is the modulus character of P (resp. P').

The resemblance between this formula and that for zonal spherical functions ([Mac]) or Whittaker functions ([CS], [K1]) is obvious. These Whittaker-Shintani functions, zonal spherical functions, and Whittaker functions are interpreted as spherical functions on spherical homogeneous spaces. (This will be explained in 4.3.) Actually, this fact plays an important role in our study of Whttaker-Shintani functions. It is to be noted that Shintani functions for $\mathbf{GL}_n(k)$ ([MS3]) and Whittaker-Shintani functions for $\mathbf{Sp}_{2n}(k)$ ([Sh2], [MS1]) are also examples of those functions. We can give explicit formulas for these (Whittaker-) Shintani functions by the same method as that in this paper. Details will appear elsewhere.

0.6. This paper is organized as follows. The sections 1 through 3 are of preliminary nature. In Sections 1 and 2, we shall review several facts on unramified principal series representations of p-adic groups and give some results for our later use in the study of Whittaker-Shintani functions. In Section 3, we shall give several notation, definitions and preparatory results concerning the special orthogonal group $\mathbf{G} = \mathbf{SO}_m$ and their subgroups.

In Section 4, we shall define Whittaker-Shintani functions precisely and give some representation theoretic interpretations (including an integral expression) of these functions.

A double coset decomposition $UK'\backslash G/K$ is presented in Section 5. For some technical reasons, we first give the corresponding decomposition for the full orthogonal group $\mathbf{O}_m(k)$ and then handle the case for $G = \mathbf{SO}_m(k)$. The support of Whittaker-Shintani functions, which turns out to be a proper subset of G if $r > 0$, is studied in Section 6.

In Section 7, we shall show that the dimension of the space of Whittaker-Shintani functions (with fixed eigenvalues of Hecke algebras) is at most one. (Later we shall prove that the dimension is exactly one.) This theorem is deduced from Section 6 by using a system of difference equations as in the case of Whittaker functions [Sh1], [K1].

Section 8 is devoted to the calculation of some integrals relevant to Whittaker-Shintani functions. The calculation is done by case-by-case considerations.

Then we shall give the main results of this paper, the uniqueness (up to a scalar multiple) of Whittaker-Shintani functions and an explicit formula of them for fixed eigenvalues of Hecke algebras, in Section 10. The method employed here is similar to that in [CS]. To establish these results, we use the calculation in Section 8 together with a new rationality argument in Section 9 (see also Section 2).

In the final section 11, we shall evaluate the value of Whittaker-Shintani functions at the identity element by using a combinatorial argument.

0.7. Main results of this paper were announced at the meeting on "Automorphic forms on algebraic groups", 1996 (RIMS, Kyoto University, Japan), [KMS]. See also [M].

NOTATION. We let k be a non-archimedean local field, o the ring of integers in k and π a prime element in o. The cardinality of the residue field $o/\pi o$ is denoted by q.

We assume that the characteristic of k is different from 2 for simplicity.

The normalized absolute value on k is denoted by $|\cdot|$. The normalized additive valuation is given by $v : k^\times \to \mathbf{Z}$ so that $|x| = q^{-v(x)}$ for $x \in k^\times$.

For any algebraic group, say $\mathbf{G}$, we shall denote by G the locally compact group of its k-rational points $\mathbf{G}(k)$.

The symbols $\mathbf{Mat}_{m,n}$ and $\mathbf{Alt}_n$ denote the variety of $m \times n$-matrices and that of alternating matrices of size n over k, respectively.

If $A \subset G$, then we let ch_A be the characteristic function of A.

1. Unramified principal series representations. In this section, we shall give some preliminary results on the unramified principal series representations of reductive groups. The main references are [C1], [C2]. We follow the notation in [C2] unless otherwise stated. Throughout this and the next sections, we work with general reductive groups instead of orthogonal groups which are the main subjects of this paper.

1.1. Let $\mathbf{G}$ be a connected reductive group over k and $\mathbf{P}$ a minimal parabolic subgroup of $\mathbf{G}$. We restrict ourselves to the case where $\mathbf{G}$ is split over k for simplicity, since later we shall work only in this situation. However we remark here that all the statements given in Sections 1 and 2 are valid also for non-split groups with suitable modifications.

We fix a maximal split torus $\mathbf{T}$ in $\mathbf{P}$. The group $\mathbf{P}$ is actually a Borel subgroup from our assumption. Then we have the Levi decomposition $\mathbf{P} = \mathbf{TN}$, where $\mathbf{N}$ is the unipotent radical of $\mathbf{P}$. We denote by Σ the root system of $(\mathbf{G}, \mathbf{T})$ and by Σ^+ the set of positive roots corresponding to $\mathbf{P}$. The unipotent radical of the opposite of $\mathbf{P}$ is denoted by $\mathbf{N}^-$. Since $\mathbf{G}$ is split, we can assume that $\mathbf{G}$ and other subgroups $\mathbf{T}$, $\mathbf{P}$, $\mathbf{N}$ are defined over o.

Let $K = \mathbf{G}(o)$ be the maximal compact subgroup of $\mathbf{G}$ consisting of o-rational points of $\mathbf{G}$. Then $G = \mathbf{G}(k)$ admits the Iwasawa decomposition $G = PK = NTK$ and the Cartan decomposition $G = KT^{++}K$, where

$$T^{++} = \{t \in T \mid |\alpha(t)| \le 1 \ (\alpha \in \Sigma^+)\}\,.$$

Denote by $W = N_G(T)/T$ the Weyl group of G with respect to T. We shall often identify each element $w \in W$ with a representative in K, and regard W as a subset of K. We let $\ell : W \to \mathbf{Z}_{\ge 0}$ be the length function with respect to Σ^+. The longest element of W is denoted by w_ℓ, and the reflection associated with $\alpha \in \Sigma$ by w_α.

Let B be the Iwahori subgroup contained in K corresponding to Σ^+ so that $B \pmod{\pi} = \mathbf{P}(o/\pi o)$. We have various Bruhat-type decompositions $G = PWP$, $G = PWB$, $G = BWTB$ and $K = BWB$.

1.2. Let

$$X_{nr}(T) := \{\chi \in \mathrm{Hom}(T, \boldsymbol{C}^\times) \mid \chi|_{T\cap K} \equiv 1\}$$

be the group of unramified characters of T. We also denote $X_{nr}(T)$ simply by X. We set $\chi(tn) = \chi(t)$ for $t \in T, n \in N$ so that $\chi \in X$ defines an element of $\mathrm{Hom}(P, \boldsymbol{C}^\times)$. For $\chi \in X$, the space of unramified principal series representation $I(\chi)$ is given by

$$I(\chi) = \{f \in C^\infty(G) \mid f(pg) = (\chi\delta^{1/2})(p) f(g) \ (p \in P, g \in G)\}\,.$$

Here $\delta : P \to \boldsymbol{R}^\times_{>0}$ is the modulus character of P. The group G acts on $I(\chi)$ by the right regular action $f \mapsto R(g)f$ for $g \in G$, where $(R(g)f)(x) = f(xg)$. Note that, by the Iwasawa decomposition, $I(\chi)$ is canonically isomorphic to $C_c^\infty(P\cap K\backslash K)$ as a K-module.

We denote by $\mathcal{P}_\chi$ the G-projection from $C_c^\infty(G)$ to $I(\chi)$ defined by

$$\mathcal{P}_\chi(f)(g) = \int_P (\chi^{-1}\delta^{1/2})(p) f(pg) dp \quad (f \in C_c^\infty(G))\,.$$

Here dp is the left invariant Haar measure of P with $\int_{P\cap K} dp = 1$ (see [C2]).

1.3. Let $\mathbf{Q}$ be an algebraic subgroup of $\mathbf{G}$. Let $\mathcal{U}$ be a locally closed subset of G invariant under the left and right translations by P and Q, respectively. We denote by $I(\chi; \mathcal{U})$ the Q-module consisting of $f \in C^\infty(\mathcal{U})$ with compact support modulo P, such that $f(px) = (\chi\delta^{1/2})(p) f(x)$ for $p \in P, x \in \mathcal{U}$. If $\mathcal{U}$ is open in G, then $I(\chi; \mathcal{U})$ is a Q-submodule of $I(\chi)$ via extension by zero outside of $\mathcal{U}$.

PROPOSITION 1.4 ([C1, 6.1.1], see also [BZ]). *Let $\mathcal{U}, \mathcal{V}$ be two $P \times Q$-invariant open subsets of G such that $\mathcal{U} \supset \mathcal{V}$. Then the sequence of Q-modules*

$$0 \longrightarrow I(\chi; \mathcal{V}) \xrightarrow{i} I(\chi; \mathcal{U}) \xrightarrow{\mathrm{res}} I(\chi; \mathcal{U} - \mathcal{V}) \longrightarrow 0$$

is exact. Here i is the natural inclusion and res *is the restriction map.*

1.5. Now we put $\mathbf{Q} = \mathbf{P}$ in the above setting. Let us put $G_w = \bigcup PyP$ ($y = w$, or $\ell(y) > \ell(w)$) for $w \in W$. It is known that G_w is open in G, and that PwP is closed in G_w. Thus we have, from 1.4, an exact sequence of P-modules,

$$0 \longrightarrow \sum_{\ell(v)>\ell(w)} I(\chi; G_v) \longrightarrow I(\chi; G_w) \longrightarrow I(\chi; PwP) \longrightarrow 0\,. \tag{1.5.1}$$

Since the Jacquet module $I(\chi; PwP)_N$ is isomorphic to the one-dimensional representation $(w^{-1}\chi)\delta^{1/2}$ of T, we have

$$I(\chi)_N \simeq \bigoplus_{w\in W} (w\chi)\delta^{1/2} \tag{1.5.2}$$

for $\chi \in X^{\mathrm{reg}}$, where $X^{\mathrm{reg}} = \{\chi \in X \mid w\chi \neq \chi \text{ for any } w \in W\}$ is the set of regular characters in X.

1.6. We assume χ to be regular until the end of 1.10. Let $T_{w,\chi} : I(\chi) \to I(w\chi)$ be the intertwining operator given by the following integral

$$T_{w,\chi}(\phi)(x) = \int_{N\cap wNw^{-1}\backslash N} \phi(w^{-1}nx)d\dot{n} \tag{1.6.1}$$

for $\phi \in I(\chi)$. Here $d\dot{n}$ is the invariant measure of $N \cap wNw^{-1}\backslash N$ with $\int_{(\text{Image of } N\cap K)} d\dot{n} = 1$. (This integral (1.6.1) converges under certain conditions on χ and is continued holomorphically to X^{reg}. See [C2], [Mat].) By the Frobenius reciprocity [C1], this $T_{w,\chi}$ corresponds to the projection $I(\chi)_N \to (w\chi)\delta^{1/2}$ arising from (1.5.2). We note that the image $T_{y^{-1},y\chi}(I(y\chi; G_{yw}))$ is contained in $I(\chi; G_w)$ if $\ell(yw) = \ell(y) + \ell(w)$ (see [C1, 6.4.3]). The next proposition will be used in Section 2.

PROPOSITION 1.7. *For any $y, w \in W$ with $\ell(yw) = \ell(y) + \ell(w)$,*

$$T_{y^{-1},y\chi}(I(y\chi; G_{yw})) + \sum_{\ell(v)>\ell(w)} I(\chi; G_v) = I(\chi; G_w)\,.$$

PROOF. In view of (1.5.1), it suffices to show that the composite of the maps

$$\mathrm{res} \circ T_{y^{-1},y\chi} \;:\; I(y\chi; G_{yw}) \xrightarrow{T_{y^{-1},y\chi}} I(\chi; G_w) \xrightarrow{\mathrm{res}} I(\chi; PwP)$$

is surjective. We note that, for any $z \in W$, $P\backslash PzP$ is naturally isomorphic to $(N \cap z^{-1}Nz)\backslash N$. Hence we have an isomorphism as vector spaces

$$\iota_{z,\chi} = \iota_z : I(\chi; PzP) \longrightarrow C_c^\infty(N \cap z^{-1}Nz\backslash N)$$

given by

$$\iota_z(\phi)(n) = \phi(zn) \quad (\phi \in I(\chi; PzP), n \in N)\,.$$

The inverse of ι_z is given by

$$\iota_z^{-1}(a)(pzn) = (\chi\delta^{1/2})(p)a(n) \quad (a \in C_c^\infty(N \cap z^{-1}Nz\backslash N),\ p \in P,\ n \in N)\,.$$

Now we put $\iota = \iota_{yw,y\chi}$ and $\iota' = \iota_{w,\chi}$. We calculate res $\circ\, T_{y^{-1},y\chi}(\phi)$ for $\phi \in I(y\chi; G_{yw})$ with $\phi|_{PywP} = \iota^{-1}(a)$ $(a \in C_c^\infty(N \cap (yw)^{-1}N(yw)\backslash N))$. We then have

$$\begin{aligned}(\iota' \circ \mathrm{res} \circ T_{y^{-1},y\chi})(\phi)(n) &= (T_{y^{-1},y\chi}\phi)(wn)\\ &= \int_{N\cap y^{-1}Ny\backslash N} \phi(yn_1wn)d\dot{n}_1\\ &= \int_{w^{-1}Nw\cap(yw)^{-1}N(yw)\backslash w^{-1}Nw} \phi(ywn_2n)d\dot{n}_2\,.\end{aligned}$$

Note that the conditions $\alpha > 0$ and $w\alpha < 0$ imply that $yw\alpha < 0$. This shows that

$$N \cap w^{-1}Nw \cap (yw)^{-1}N(yw) = N \cap (yw)^{-1}N(yw)$$

and

$$w^{-1}Nw \cap (yw)^{-1}N(yw)\backslash w^{-1}Nw = N \cap (yw)^{-1}N(yw)\backslash N \cap w^{-1}Nw\,.$$

Thus the integral in the right hand side above is written as

$$\int_{w^{-1}Nw\cap(yw)^{-1}N(yw)\backslash w^{-1}Nw} \phi(ywn_2n)d\dot{n}_2 = \int_{N\cap(yw)^{-1}N(yw)\backslash N\cap w^{-1}Nw} a(n_3n)d\dot{n}_3\,.$$

Obviously the map π from $C_c^\infty(N \cap (yw)^{-1}N(yw)\backslash N)$ to $C_c^\infty(w^{-1}Nw \cap N\backslash N)$ given by

$$\pi(a)(n) = \int_{N\cap(yw)^{-1}N(yw)\backslash N\cap w^{-1}Nw} a(n_3n)d\dot{n}_3$$

is surjective. Thus the map res $\circ\, T_{y^{-1},y\chi} = \iota'^{-1} \circ \pi \circ \iota$ is surjective. □

1.8. Let $\mathcal{H} = \mathcal{H}(G, K)$ be the Hecke algebra of (G, K). For $\chi \in X_{nr}(T)$, we let $\phi_K = \phi_{K,\chi}$ be the function on G given by $\phi_K(ntk) = (\chi\delta^{1/2})(t)$ $(n \in N,\ t \in T,\ k \in K)$. This is a basis element of the one-dimensional space $I(\chi)^K$, the space of K-fixed vectors in $I(\chi)$. After Satake [Sa], we define a $\boldsymbol{C}$-homomorphism ω_χ of $\mathcal{H}$ to $\boldsymbol{C}$ by

$$\omega_\chi(\varphi) = \int_G \phi_K(g)\varphi(g)dg \quad (\varphi \in \mathcal{H})\,,$$

where dg is the Haar measure of G with $\mathrm{vol}(K) = 1$. Hence we have

$$R(\varphi)\phi_K = \omega_\chi(\varphi)\phi_K\,,$$

where

$$(R(\varphi)\phi_K)(x) = \int_G \varphi(g)\phi_K(xg)dg$$

by definition. Then $\chi \mapsto \omega_\chi$ gives rise to a bijection between $W\backslash X_{nr}(T)$ and $\mathrm{Hom}_{\boldsymbol{C}\text{-alg}}(\mathcal{H}, \boldsymbol{C})$.

1.9. Let us put $\phi_w = \phi_{w,\chi} = \mathcal{P}_\chi(\mathrm{ch}_{BwB})$ $(w \in W)$ so that

$$\phi_w(g) = \begin{cases}(\chi\delta^{1/2})(t) & \text{if } k \in BwB\,,\\ 0 & \text{otherwise}\,,\end{cases} \tag{1.9.1}$$

for $g = ntk$ $(n \in N, t \in T, k \in K)$. Then $\{\phi_w\ (w \in W)\}$ is a basis for $I(\chi)^B$. Let $\mathbf{c}_\alpha(\chi)$ $(\alpha \in \Sigma)$ be the c-function in [C2] (see also [Mac]). According to [C2], there is another basis $\{f_w\ (w \in W)\}$ for $I(\chi)^B$ satisfying

(1.9.2) $$R(\mathrm{ch}_{BtB}) f_w = \mathrm{vol}(BtB)(w\chi)\delta^{1/2}(t) f_w \quad (t \in T^{++}),$$

(1.9.3) $$f_{w_\ell} = \phi_{w_\ell},$$

and

(1.9.4) $$\phi_K = \sum_{w \in W} \mathbf{c}_w(\chi) f_w,$$

where $\mathbf{c}_w(\chi) = \prod \mathbf{c}_\alpha(\chi)$ $(\alpha > 0,\ w\alpha < 0)$. We easily see that $T_{w^{-1},w\chi}(\phi_{w_\ell,w\chi}) = f_{w_\ell w}$.

PROPOSITION 1.10. *There is a basis $\{g_w (w \in W)\}$ for $I(\chi)^B$ satisfying the following properties*:

(1.10.1) $$R(\mathrm{ch}_{Bt^{-1}B}) g_w = \mathrm{vol}(BtB)(w\chi)^{-1}\delta^{1/2}(t) g_w \quad (t \in T^{++});$$

(1.10.2) $$g_1 = \phi_1;$$

(1.10.3) $$\phi_K = q^{\ell(w_\ell)} \sum_{w \in W} \bar{\mathbf{c}}_w(\chi) g_w,$$

where $\bar{\mathbf{c}}_w(\chi) = \prod \mathbf{c}_\alpha(\chi)$ $(\alpha > 0,\ w\alpha > 0)$.

PROOF. We note that $w_\ell(t)^{-1} \in T^{++}$ if $t \in T^{++}$. For $t \in T^{++}$, we have

(1.10.4) $$Bw_\ell B \cdot Bt^{-1}B = Bw_\ell t^{-1}B = Bw_\ell(t)^{-1}B \cdot Bw_\ell B$$

by using the Iwahori factorization $B = (B \cap N^-)(B \cap T)(B \cap N)$ and the facts $t(B \cap N)t^{-1} \subset B \cap N$ and $t^{-1}(B \cap N^-)t \subset B \cap N^-$. Let $\mathcal{H}(G, B)$ be the Hecke algebra of (G, B). This is a $\boldsymbol{C}$-algebra under the convolution product with a basis $\{\mathrm{ch}_{BwB}\ (w \in W)\}$, where $\mathrm{vol}(B)^{-1}\mathrm{ch}_B$ is the unit element. Then (1.10.4) implies that

$$\mathrm{ch}_{Bw_\ell B} \cdot \mathrm{ch}_{Bt^{-1}B} = \mathrm{vol}(B)\mathrm{ch}_{Bw_\ell t^{-1}B} = \mathrm{ch}_{Bw_\ell(t)^{-1}B} \cdot \mathrm{ch}_{Bw_\ell B}$$

in the Hecke algebra $\mathcal{H}(G, B)$. Note that basis elements ch_{BwB} $(w \in W)$ are invertible. Therefore we have (1.10.1) if we put $g_w = R(\mathrm{ch}_{Bw_\ell B})^{-1} f_{w_\ell w}$ for $w \in W$. Since $f_{w_\ell} = \phi_{w_\ell} = \mathcal{P}_\chi(\mathrm{ch}_{Bw_\ell B})$, we see that

$$g_1 = \mathrm{vol}(B) R(\mathrm{ch}_{Bw_\ell B})^{-1} \mathcal{P}_\chi(\mathrm{ch}_{Bw_\ell B}) = \mathcal{P}_\chi(\mathrm{ch}_B) = \phi_1.$$

Finally applying $\mathrm{vol}(B)R(\mathrm{ch}_{Bw_\ell B})^{-1}$ on both sides of (1.10.3), we get

$$q^{-\ell(w_\ell)}\phi_K = \sum_{w \in W} \mathbf{c}_w(\chi) g_{w_\ell w} = \sum_{w \in W} \bar{\mathbf{c}}_w(\chi) g_w. \qquad \square$$

We note that

(1.10.5) $$g_w = T_{w^{-1},w\chi}(\phi_{1,w\chi})$$

for $w \in W$ (cf. [Mat], [K2]).

1.11. For a suitable subset V_X of $X = X_{nr}(T)$, we consider an analytic family of representations $I(\chi)$ $(\chi \in V_X)$ (see [C1, 2.7]) in a certain algebraic way.

Let $\boldsymbol{C}[X]$ be the coordinate ring of the affine variety $X \simeq (\boldsymbol{C}^\times)^{\dim T}$. Since $X = \mathrm{Hom}(T/T \cap K, \boldsymbol{C}^\times)$, we see that each element t of T (modulo $T \cap K$) defines a regular function $\eta(t)$ on X by $\eta(t)(\chi) = \chi(t)$ $(\chi \in X)$. Note that $\eta : T \to \boldsymbol{C}[X]^\times$ is a homomorphism. We regard η as a homomorphism from P to $\boldsymbol{C}[X]^\times$. As in [K2], we define a G-module I over $\boldsymbol{C}[X]$ by

$$\begin{aligned} I &= \{f \in \boldsymbol{C}[X] \otimes_{\boldsymbol{C}} C^\infty(G) \mid f(pg) = (\eta\delta^{1/2})(p) f(g)\ (p \in P,\, g \in G)\} \\ &\simeq \boldsymbol{C}[X] \otimes_{\boldsymbol{C}} C^\infty(P \cap K \backslash K) \quad (\text{as } \boldsymbol{C}[X]\text{-modules}). \end{aligned}$$

This $\boldsymbol{C}[X]$-module I reduces to $I(\chi)$ under the specialization at $\chi \in X$. Also, under the notation of 1.3, we can define $\boldsymbol{C}[X]$-module $I(\mathcal{U})$ for a $P \times Q$-stable open subset $\mathcal{U}$ of G. The specialization of $I(\mathcal{U})$ at $\chi \in X$ is $I(\chi; \mathcal{U})$.

Let V_X be a Zariski open subset of X. We denote by $\boldsymbol{C}[V_X]$ the ring of regular functions on V_X. Then we define a G-module over $\boldsymbol{C}[V_X]$, the restriction of I to V_X by $I|_{V_X} := \boldsymbol{C}[V_X] \otimes_{\boldsymbol{C}[X]} I$. We use a similar notation $I(\mathcal{U})|_{V_X}$ for $I(\mathcal{U})$ above. Let X^{reg} be the set of all the regular elements in X. We know that the intertwining operator $T_{w,\chi} : I(\chi) \to I(w\chi)$ $(\chi \in X)$ is regular on X^{reg}, i.e., $T_{w,\chi}(\mathcal{P}_\chi(f))(g)$ is regular in $\chi \in X^{\mathrm{reg}}$ for any $f \in C_c^\infty(G)$ and $g \in G$ (see [C2]). This follows from the following two facts:

(1.11.1) The restriction of T_w on $I(\chi)^B$ is regular in $\chi \in X^{\mathrm{reg}}$ (see [Mat], [C2]).
(1.11.2) The space $I(\chi)^B$ generates $I(\chi)$; or more strongly, I^B generates I as a G-module over $\boldsymbol{C}[X]$ (see [Mat, 5.3.14]).

Let I_w $(w \in W)$ be the $\boldsymbol{C}[X]$-module whose specialization at χ is given by $I(w\chi)$. (Hence $I_1 = I$ by definition.) Since the intertwining operators $T_{w,\chi}$ $(w \in W)$ are regular in $\chi \in X^{\mathrm{reg}}$, we have G-homomorphisms over $\boldsymbol{C}[X^{\mathrm{reg}}]$, $T_{w,z} : I_z|_{X^{\mathrm{reg}}} \to I_{wz}|_{X^{\mathrm{reg}}}$ that induce $T_{wz^{-1},z\chi} : I(z\chi) \to I(w\chi)$ for any $w, z \in W$ and $\chi \in X^{\mathrm{reg}}$.

1.12. We say that a linear form $l_\chi : I(\chi) \to \boldsymbol{C}$ is *rational* in χ if l_χ is obtained from the specialization of a $\boldsymbol{C}[V_X]$-homomorphism $l : I|_{V_X} \to \boldsymbol{C}[V_X]$ for some Zariski open subset V_X of X. More generally, if a family of subspaces $I'(\chi)$ of $I(\chi)$ $(\chi \in V_X)$ is the specialization of a $\boldsymbol{C}[V_X]$-submodule I' of $I|_{V_X}$, we can define the rationality of a linear form $l'_\chi : I'(\chi) \to \boldsymbol{C}$ as well. Let $\mathcal{P}$ be the canonical G-map from $C_c^\infty(G)$ to I given by

$$\mathcal{P}(f)(g) = \int_P (\eta^{-1}\delta^{1/2})(p) f(pg)\,dp \quad (f \in C_c^\infty(G))$$

(see 1.11). The image of $\mathcal{P}$ generates I as a $\boldsymbol{C}[X]$-module. Hence, in order to see that a linear form $l_\chi : I(\chi) \to \boldsymbol{C}$ is rational, it is enough to check that, for any $f \in C_c^\infty(G)$, the function of $\chi \in X$ given by $l_\chi(\mathcal{P}_\chi(f))$ is in $\boldsymbol{C}[V_X]$ for some open V_X (independent of f).

Suppose that a linear form $l_{\chi,\sigma} : I(\chi) \to \boldsymbol{C}$ has a parameter $\sigma \in Y$, where Y is a parameter space (a Zariski open subset of $\boldsymbol{C}^s$, $s \geq 0$, for example). Then we say that $l_{\chi,\sigma}$ is rational in (χ, σ) if $l_{\chi,\sigma}$ is the specialization of $\boldsymbol{C}[V_{X\times Y}]$-homomorphism $\boldsymbol{C}[V_{X\times Y}] \otimes_{\boldsymbol{C}[X]} I \to \boldsymbol{C}[V_{X\times Y}]$ for some open subset $V_{X\times Y} \subset X \times Y$.

Finally, we remark here that we can formulate 1.7 as a statement for $\boldsymbol{C}[X^{\mathrm{reg}}]$-modules:

$$T_{y^{-1},y}(I_y(G_{yw})) + \sum_{\ell(v)>\ell(w)} I(G_v) = I(G_w) \tag{1.12.1}$$

for any $y, w \in W$ with $\ell(yw) = \ell(y)+\ell(w)$. This shows that a linear form $l_\chi : I(\chi; G_w) \to \boldsymbol{C}$ is rational in χ if both the restriction of l_χ to $T_{y^{-1},y\chi}(I(y\chi; G_{yw})$ and that to $\sum_{\ell(v)>\ell(w)} I(\chi; G_v)$ are rational.

2. Equivariant linear forms. In this section, we study the space $\mathrm{Hom}_Q(I(\chi), \rho)$ for a subgroup Q of G and a one-dimensional representation ρ of Q. We note that $\mathrm{Hom}_Q(I(\chi), \rho)$ is naturally isomorphic to the space of distributions F on G satisfying $L(p)R(x)F = (\chi^{-1}\delta^{1/2})(p)\rho(x)F$ for $p \in P, x \in Q$. Here L and R are respectively the left and right regular actions of G on the space of distributions.

2.1. Let $\mathbf{Q}$ be an algebraic subgroup of $\mathbf{G}$ such that Q has finitely many orbits on $P\backslash G$. We let $\{\rho = \rho_\sigma : Q \to \boldsymbol{C}^\times\}$ be a family of one-dimensional representations with a parameter $\sigma \in Y$, where the parameter space $Y = \{\sigma\}$ is a Zariski open subset of $\boldsymbol{C}^s$ for some $s \geq 0$.

LEMMA 2.2. *Let $\mathcal{O}$ be a $P \times Q$-orbit in G. Then* $\dim \mathrm{Hom}_Q(I(\chi; \mathcal{O}), \rho) \leq 1$.

PROOF. We have

$$I(\chi; \mathcal{O}) \simeq \mathrm{Ind}_c(g^{-1}(\chi\delta^{1/2}) \mid Q \cap g^{-1}Pg, Q)$$

by definition, if $\mathcal{O} = PgQ$ for some $g \in G$. Here the right hand side denotes the space of smooth functions f on Q with compact support modulo $Q \cap g^{-1}Pg$ such that $f(px) = (\chi\delta^{1/2})(gpg^{-1})f(x)$ for $p \in Q \cap g^{-1}Pg, x \in Q$. Thus, if we let δ_g be the modulus character of $Q \cap g^{-1}Pg$, we get

$$\begin{aligned}\dim \mathrm{Hom}_Q(I(\chi; \mathcal{O}), \rho) &= \dim \mathrm{Hom}_Q(\mathrm{Ind}_c(g^{-1}(\chi\delta^{1/2}) \otimes \rho^{-1} \mid Q \cap g^{-1}Pg, Q), \boldsymbol{C}) \\ &= \dim \mathrm{Hom}_{Q\cap g^{-1}Pg}(g^{-1}(\chi\delta^{1/2}) \otimes \rho^{-1}, \delta_g) \qquad \text{([C1, 2.4.3])} \\ &\leq 1\,. \end{aligned}$$

□

Now we assume the following properties on P, Q, χ and ρ.

ASSUMPTION 2.3.

(2.3.1) There exists a unique open $P \times Q$-orbit $\mathcal{O}_0$ in G.

(2.3.2) There exists an open dense subset Z of $X \times Y$ such that

$$\mathrm{Hom}_Q(I(\chi; \mathcal{O}), \rho_\sigma) = \{0\}$$

for any $P \times Q$-orbit $\mathcal{O}$ distinct from $\mathcal{O}_0$ if $(\chi, \sigma) \in Z$.

PROPOSITION 2.4. *Suppose that Assumption* 2.3 *holds. Then the restriction map from* $\mathrm{Hom}_Q(I(\chi), \rho_\sigma)$ *to* $\mathrm{Hom}_Q(I(\chi; \mathcal{O}_0), \rho_\sigma)$ *is injective for* $(\chi, \sigma) \in Z$*, and hence*

$$\dim \mathrm{Hom}_Q(I(\chi), \rho_\sigma) \leq 1\,.$$

PROOF. Let us set $\mathcal{U}_d = \bigcup \mathcal{O}$ ($\operatorname{codim} \mathcal{O} \le d$) for $d \ge 0$. Then $\mathcal{U}_d$ are $P \times Q$-stable open subsets of G for $d \ge 0$. Note that $\mathcal{U}_0 = \mathcal{O}_0$ and that $\mathcal{U}_d = G$ for d large enough. We have exact sequences of Q-modules

$$0 \longrightarrow I(\chi; \mathcal{U}_{d-1}) \longrightarrow I(\chi; \mathcal{U}_d) \longrightarrow \sum_{\operatorname{codim} \mathcal{O} = d} I(\chi; \mathcal{O}) \longrightarrow 0$$

for any $d \ge 1$ by 1.4. Thus, from (2.3.1) and (2.3.2), the restriction map is injective and

$$\dim \operatorname{Hom}_Q(I(\chi), \rho_\sigma) \le \dim \operatorname{Hom}_Q(I(\chi; \mathcal{O}_0), \rho_\sigma)\,. \tag{2.4.1}$$

□

REMARK 2.5. (1) The argument in 2.4 actually shows that

$$\dim \operatorname{Hom}_Q(I(\chi; \mathcal{U}), \rho_\sigma) \le 1$$

for any $P \times Q$-stable open subset $\mathcal{U}$ of G under the assumption 2.3.

(2) Similar result holds when there are finitely many open $P \times Q$-orbits with a suitable modification (of 2.3 and 2.4).

2.6. Now we shall work with Q satisfying $Q \subset P$ in the following situation:

(2.6.1) For some open (but not necessarily Zariski open) subset Z^+ of $X \times Y$, there exists a family of non-zero elements $l_{\chi,\sigma} \in \operatorname{Hom}_Q(I(\chi), \rho_\sigma)$ $((\chi, \sigma) \in Z^+)$.

We shall give conditions on $l_{\chi,\sigma} \in \operatorname{Hom}_Q(I(\chi), \rho_\sigma)$ to be meromorphically (rationally) continued to the whole $X \times Y$ (see 1.12). Note that $P w_\ell P$ is a $P \times Q$-stable open subvariety of G. We impose the following condition on the family of $l_{\chi,\sigma}$ for $(\chi, \sigma) \in Z^+$.

ASSUMPTION 2.7. The restriction of $l_{\chi,\sigma}$ to $I(\chi; P w_\ell P)$ depends rationally on $X \times Y$. Namely, there exists a Zariski open subset Z' of $X \times Y$ so that the function of (χ, σ) given by $l_{\chi,\sigma}(\mathcal{P}_\chi(f))$ for a fixed $f \in C_c^\infty(P w_\ell P)$ is a regular function on Z'. In particular, one can extend $l_{\chi,\sigma}|_{I(\chi; P w_\ell P)}$ to generic (χ, σ).

2.8. The Weyl group W acts on $X \times Y$ by (natural action)×(trivial action). We may suppose that Z in 2.3 is identical to Z' above, and moreover that Z is W-invariant and contained in $X^{\mathrm{reg}} \times Y$, by replacing Z by a dense subset if necessary.

Let $T_w = T_{w, w^{-1}\chi} : I(w^{-1}\chi) \to I(\chi)$ be the intertwining operator in 1.6. Then $T_w^* l_{\chi,\sigma} = l_{\chi,\sigma} \circ T_w \in \operatorname{Hom}_Q(I(w^{-1}\chi), \rho_\sigma)$ for $(\chi, \sigma) \in Z^+$.

Thus the uniqueness property 2.5 (1) shows that (under 2.3 and 2.7), if $(\chi, \sigma) \in Z \cap Z^+$,

$$T_w^* l_{\chi,\sigma}|_{I(w^{-1}\chi; P w_\ell P)} = a(w, \chi, \sigma) l_{w^{-1}\chi,\sigma}|_{I(w^{-1}\chi; P w_\ell P)} \tag{2.8.1}$$

with some scalar factor $a(w, \chi, \sigma)$. (Note that $l_{w^{-1}\chi,\sigma}|_{I(w^{-1}\chi; P w_\ell P)}$ in the right hand side is rational in (χ, σ) by Assumption 2.7.)

ASSUMPTION 2.9. The scalar factor $a(w_\alpha, \chi, \sigma)$ for any simple root α depends rationally on $(\chi, \sigma) \in X \times Y$.

PROPOSITION 2.10. *Under the assumptions* 2.7 *and* 2.9, $l_{\chi,\sigma} \in \mathrm{Hom}_Q(I(\chi), \rho_\sigma)$ *depends rationally on* $(\chi, \sigma) \in X \times Y$. *In particular, for generic* (χ, σ), $l_{\chi,\sigma}$ *is defined and satisfies* $\mathrm{Hom}_Q(I(\chi), \rho_\sigma) = \boldsymbol{C} \cdot l_{\chi,\sigma}$.

PROOF. We shall prove that the restriction of $l_{\chi,\sigma}$ to $I(\chi; G_{ww_\ell})$ depends rationally on (χ, σ) by induction on $\ell(w)$. (For the definition of G_{ww_ℓ}, see 1.5.) This is valid for $w = 1$ from the assumption 2.7. We assume that $\ell(w) > 0$ and that $l_{\chi,\sigma}|_{I(\chi; G_{yw_\ell})}$ for any $y \in W$ with $\ell(y) < \ell(w)$ depends rationally in (χ, σ). We decompose w as $w = w_\alpha y$ ($\ell(w) = \ell(y) + 1, \alpha \in \Delta$). Then, by (2.8.1),

$$T^*_{w_\alpha} l_{\chi,\sigma}|_{I(w_\alpha\chi; G_{w_\ell})} = a(w_\alpha, \chi, \sigma) l_{w_\alpha\chi,\sigma}|_{I(w_\alpha\chi; G_{w_\ell})}$$

for $(\chi, \sigma) \in Z \cap Z^+$. Since the right hand side above is defined on $I(w_\alpha\chi; G_{yw_\ell})$ for generic (χ, σ) (and is rational) by the induction hypothesis, the uniqueness 2.4 implies that

$$T^*_{w_\alpha} l_{\chi,\sigma}|_{I(w_\alpha\chi; G_{yw_\ell})} = a(w_\alpha, \chi, \sigma) l_{w_\alpha\chi,\sigma}|_{I(w_\alpha\chi; G_{yw_\ell})}\,.$$

The intertwining operator $T_w = T_{w,\chi}$ depends rationally on χ (see 1.11). Thus the restriction $l_{\chi,\sigma}|_{T_{w_\alpha}(I(w_\alpha\chi; G_{yw_\ell}))}$ depends rationally on χ. The induction hypothesis and 1.7 (see also 1.12, especially (1.12.1)) show that $l_{\chi,\sigma}|_{I(\chi; G_{ww_\ell})}$ is rational in (χ, σ). Therefore we see that $l_{\chi,\sigma}$ is rational in (χ, σ), and hence is defined for generic (χ, σ). Moreover the uniqueness argument 2.4 shows that $\mathrm{Hom}_Q(I(\chi), \rho_\sigma) = \boldsymbol{C} \cdot l_{\chi,\sigma}$ for generic (χ, σ). □

2.11. In Section 9 we shall construct a family of the equivariant linear forms $l_{\chi,\sigma}$ in the following way. Suppose that there exist an open subset Z^+ of $X \times Y$ and a family of continuous functions $Y_{\chi,\sigma}$ ($(\chi, \sigma) \in Z^+$) satisfying

$$Y_{\chi,\sigma}(pgx) = (\chi^{-1}\delta^{1/2})(p)\rho_\sigma(x)^{-1} Y_{\chi,\sigma}(g) \quad (p \in P,\ g \in G,\ x \in Q)\,. \tag{2.11.1}$$

These $Y_{\chi,\sigma}$ give elements $l_{\chi,\sigma}$ of $\mathrm{Hom}_Q(I(\chi), \rho_\sigma)$ by setting

$$l_{\chi,\sigma}(\mathcal{P}_\chi(f)) = \int_G f(g) Y_{\chi,\sigma}(g) dg \quad (f \in C_c^\infty(G))\,.$$

3. Orthogonal groups. In what follows, we shall give several notation, definitions and preliminary results concerning the split special orthogonal groups $\mathbf{G}_m = \mathbf{SO}_m$ ($m = 1, 2, \dots$) and their subgroups. We often handle the odd case (where m is odd) and the even case (where m is even) separately.

3.1. Let m be a positive integer and put $l = [m/2]$, the integral part of $m/2$. Let S_m be a symmetric matrix of degree m given by

$$S_m = \begin{cases} \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix} & \text{if } m \text{ is even}, \\[2ex] \begin{pmatrix} 0 & 0 & J_l \\ 0 & 2 & 0 \\ J_l & 0 & 0 \end{pmatrix} & \text{if } m \text{ is odd}, \end{cases}$$

where

$$J_l = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \in \mathbf{GL}_l(k) \,.$$

Denote by $\mathbf{G}_m$ (or $\mathbf{SO}_m$) the special orthogonal group of the symmetric matrix S_m:

$$\mathbf{G}_m = \mathbf{SO}_m = \mathbf{SO}(S_m) = \{g \in \mathbf{SL}_m \mid {}^t g S_m g = S_m\} \,.$$

The group $\mathbf{G}_m$ is split over k and defined over o. The rank of $\mathbf{G}_m$ is $l = [m/2]$.

3.2. Let $\mathbf{T}_m = \{\mathbf{d}_m(t_1, \dots, t_l) \mid t_1, \dots, t_l \in \mathbf{GL}_1\}$ be the subgroup of diagonal matrices in $\mathbf{G}_m$, which is a maximal split torus of $\mathbf{G}_m$. Here $\mathbf{d}_m(t_1, \dots, t_l)$ denotes the diagonal matrix $\mathrm{diag}(t_1, \dots, t_l, 1, t_l^{-1}, \dots, t_1^{-1})$ if m is odd (resp. $\mathrm{diag}(t_1, \dots, t_l, t_l^{-1}, \dots, t_1^{-1})$ if m is even).

We let $\mathbf{P}_m$ be the standard Borel subgroup consisting of all upper triangular matrices in $\mathbf{G}_m$. Then $\mathbf{P}_m = \mathbf{T}_m\mathbf{N}_m$, where $\mathbf{N}_m$ is the unipotent radical of $\mathbf{P}_m$ consisting of all upper triangular unipotent matrices in $\mathbf{G}_m$. We also denote by $\mathbf{N}_m^-$ the group of lower triangular unipotent elements in $\mathbf{G}_m$ so that the group $\mathbf{T}_m\mathbf{N}_m^-$ is the opposite of $\mathbf{P}_m$.

We let $K_m = \mathbf{G}_m(o)$ be a maximal compact subgroup of $G_m = \mathbf{G}_m(k)$. Let $\varpi : K_m \to \mathbf{G}_m(o/\pi o)$ be the reduction modulo π. Then $B_m := \varpi^{-1}(\mathbf{P}_m(o/\pi o))$ is an Iwahori subgroup of G_m. We have the Iwahori factorization $B_m = N^-_{m,(1)} T_{m,(0)} N_{m,(0)}$. Here, for any subgroup $\mathbf{V}$ of $\mathbf{G}_m$ over o, we set

$$V_{(0)} := \mathbf{V}(o) \ (= V \cap K_m)$$

and

$$V_{(1)} := \mathrm{Ker}(\varpi|_{V_{(0)}} : V_{(0)} \to \mathbf{V}(o/\pi o)) \,.$$

We denote by dk the normalized Haar measure of K_m. Let dn (resp. dt) be the Haar measure of N_m (resp. T_m) normalized so that $\mathrm{vol}(N_m \cap K_m) = 1$ (resp. $\mathrm{vol}(T_m \cap K_m) = 1$). We denote by δ_m the modulus character of T_m (or of P_m). Namely, δ_m is defined to be $\delta_m(t) = d(tnt^{-1})/dn$. For $t = \mathbf{d}_m(t_1, \dots, t_l) \in T_m$, $\delta_m(t)$ is given explicitly as $\delta_m(t) = \prod_{i=1}^l |t_i|^{m-2i}$. Then the Haar measure dg of G_m with $\mathrm{vol}(K_m) = 1$ is given by, symbolically, $dg = \delta_m(t) dn dt dk$ as usual. (See the Iwasawa decomposition given below.)

The Weyl group $W_m := N_{G_m}(T_m)/T_m$ acts on T_m. As in Section 1, we shall choose representatives of W_m in K_m and often regard W_m as a subset of K_m.

3.3. Let $\mathrm{Hom}(\mathbf{T}_m, \mathbf{GL}_1)$ be the character group of $\mathbf{T}_m$ and $\mathrm{Hom}(\mathbf{GL}_1, \mathbf{T}_m)$ the group of its one-parameter subgroups. We give $\{\varepsilon_i \ (1 \le i \le l)\}$, the standard basis of $\mathrm{Hom}(\mathbf{T}_m, \mathbf{GL}_1)$ so that $\varepsilon_i(\mathbf{d}_m(t_1, \dots, t_l)) = t_i$ for $t_1, \dots, t_l \in k^\times$. Let $\{d_i \ (1 \le i \le l)\}$ be the basis of $\mathrm{Hom}(\mathbf{GL}_1, \mathbf{T}_m)$ that is dual to $\{\varepsilon_i \ (1 \le i \le l)\}$. Namely, d_i is given by $d_i(t) = \mathbf{d}_m(1, \dots, 1, \overset{i}{t}, 1, \dots, 1)$ $(t \in \mathbf{GL}_1)$ for $1 \le i \le l$. We denote the canonical pairing on $\mathrm{Hom}(\mathbf{T}_m, \mathbf{GL}_1) \times \mathrm{Hom}(\mathbf{GL}_1, \mathbf{T}_m)$ by $\langle \ , \ \rangle$ so that $\langle \varepsilon_i, d_j \rangle = \delta_{ij}$.

Set $\Lambda_m = \mathbf{Z}^l$. For $\lambda \in \Lambda_m$, we put $t(\lambda) = \mathbf{d}_m(\pi^{\lambda_1}, \dots, \pi^{\lambda_l}) \in T_m$. We can naturally identify Λ_m with $\mathrm{Hom}(\mathbf{GL}_1, \mathbf{T}_m)$ by the map $\eta : \Lambda_m \to \mathrm{Hom}(\mathbf{GL}_1, \mathbf{T}_m)$ defined to be $\langle \gamma, \eta(\lambda) \rangle = v(\gamma(t(\lambda)))$ $(\gamma \in \mathrm{Hom}(\mathbf{T}_m, \mathbf{GL}_1), \lambda \in \Lambda_m)$. For simplicity, we identify Λ_m with

$\mathrm{Hom}(\mathbf{GL}_1, \mathbf{T}_m)$ through η so that we write $\langle \gamma, \lambda \rangle$ instead of $\langle \gamma, \eta(\lambda) \rangle$ (see Section 7). We have a bijective correspondence between Λ_m and $T_m/(T_m \cap K_m)$ given as

$$\lambda \in \Lambda_m \longleftrightarrow t(\lambda) \pmod{T_m \cap K_m} \in T_m/(T_m \cap K_m)\,.$$

The Iwasawa decomposition shows that

$$G_m = \bigsqcup_{\lambda \in \Lambda_m} N_m t(\lambda) K_m\,.$$

Let us denote by Λ_m^+ the subsemigroup of Λ_m given by

$$\Lambda_m^+ = \begin{cases} \{\lambda = (\lambda_1, \ldots, \lambda_l) \in \Lambda_m \mid \lambda_1 \geq \cdots \geq \lambda_l \geq 0\} & \text{if } m \text{ is odd}\,, \\ \{\lambda = (\lambda_1, \ldots, \lambda_l) \in \Lambda_m \mid \lambda_1 \geq \cdots \geq \lambda_{l-1} \geq |\lambda_l|\} & \text{if } m \text{ is even}\,. \end{cases}$$

Under the identification above, Λ_m^+ corresponds to the dominant coweights in $\mathrm{Hom}(\mathbf{GL}_1, \mathbf{T}_m)$. Then we have the following Cartan decomposition:

$$G_m = \bigsqcup_{\lambda \in \Lambda_m^+} K_m t(\lambda) K_m\,.$$

The Weyl group W_m acts on Λ_m in a natural manner. We may regard W_m as a subgroup of $GL(\Lambda_m)$, which induces permutations on $\{\pm\varepsilon_1, \ldots, \pm\varepsilon_l\}$.

3.4. Let $X_{nr}(k^\times)$ be the group of unramified characters of $k^\times$. We shall identify $X_{nr}(k^\times)$ with $\boldsymbol{C}^\times$ by the correspondence $X_{nr}(k^\times) \ni \chi \leftrightarrow \chi(\pi) \in \boldsymbol{C}^\times$. Moreover, by abuse of notation, we shall often denote $\chi(\pi)$ simply by χ in the above correspondence. We denote by $X_m = X_{nr}(T_m)$ the group of unramified characters of T_m. Then, as in the above, we can identify X_m with $(\boldsymbol{C}^\times)^l$ so that $\Xi(t(\lambda)) = \Xi_1^{\lambda_1} \cdots \Xi_l^{\lambda_l}$ for $\Xi = (\Xi_1, \ldots, \Xi_l) \in (\boldsymbol{C}^\times)^l$. The Weyl group W_m acts on X_m by $w\Xi(t) = \Xi(w^{-1}(t))$ $(w \in W_m,\ \Xi \in X_m,\ t \in T_m)$. It induces permutations on $\{\Xi_1, \Xi_1^{-1}, \ldots, \Xi_l, \Xi_l^{-1}\}$.

3.5. The root system of $(\mathbf{G}_m, \mathbf{T}_m)$, which is a subset of $\mathrm{Hom}(\mathbf{T}_m, \mathbf{GL}_1)$, is denoted by $\Sigma_m = \Sigma(\mathbf{G}_m, \mathbf{T}_m)$ and given as follows:

$$\Sigma_m = \begin{cases} \{\pm\varepsilon_i \pm \varepsilon_j \ (1 \leq i < j \leq l)\,,\ \pm\varepsilon_i \ (1 \leq i \leq l)\} & \text{if } m = 2l+1\,, \\ \{\pm\varepsilon_i \pm \varepsilon_j \ (1 \leq i < j \leq l)\} & \text{if } m = 2l\,. \end{cases}$$

For $\alpha \in \Sigma_m$, we let $\mathbf{X}_\alpha$ be the corresponding root subgroup. More precisely, we choose each isomorphism $x_\alpha : \bar{k} \to \mathbf{X}_\alpha$ over $\boldsymbol{Z}$ in the following way: $x_\alpha(t)$ $(t \in \bar{k})$ is given by

$$\begin{array}{ll} I + t(E_{i,j} - E_{m-j+1,m-i+1}) & \text{if} \quad \alpha = \varepsilon_i - \varepsilon_j \ (1 \leq i \neq j \leq l)\,; \\ I + t(E_{i,m-j+1} - E_{j,m-i+1}) & \text{if} \quad \alpha = \varepsilon_i + \varepsilon_j \ (1 \leq i < j \leq l)\,; \\ I - t(E_{m-i+1,j} - E_{m-j+1,i}) & \text{if} \quad \alpha = -\varepsilon_i - \varepsilon_j \ (1 \leq i < j \leq l)\,; \\ I + t(2E_{i,l+1} - E_{l+1,m-i+1}) - t^2 E_{i,m-i+1} & \text{if} \quad \alpha = \varepsilon_i \ (1 \leq i \leq l, m = 2l+1)\,; \\ I - t(2E_{m-i+1,l+1} - E_{l+1,i}) - t^2 E_{m-i+1,i} & \text{if} \quad \alpha = -\varepsilon_i \ (1 \leq i \leq l, m = 2l+1)\,. \end{array}$$

Here E_{ij} denotes the matrix unit $(1 \leq i, j \leq m)$, and $\bar{k}$ the algebraic closure of k.

Let $\alpha^\vee \in \mathrm{Hom}(\mathbf{GL}_1, \mathbf{T}_m)$ be the coroot corresponding to $\alpha \in \Sigma_m$. We put $a_\alpha := t(\alpha^\vee) \in T_m$.

We record here a well-known formula:

$$x_\alpha(t) = x_{-\alpha}(t^{-1}) w_\alpha a_\alpha^{-v(t)} h x_{-\alpha}(t^{-1}) \quad (\alpha \in \Sigma_m, t \in k^\times) \tag{3.5.1}$$

with some element $h \in T_{m,(0)} = T_m \cap K_m$. Here w_α is the reflection associated with α. This is a consequence of the decomposition

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -t^{-1} & 0 \\ 0 & -t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix}$$

for $t \neq 0$.

Let $\Sigma_m^+ \supset \Delta_m$ be the standard sets of positive roots and simple roots, respectively, with respect to $\mathbf{P}_m$:

$$\Sigma_m^+ = \begin{cases} \{\varepsilon_i \pm \varepsilon_j \ (1 \leq i < j \leq l)\,,\ \varepsilon_i \ (1 \leq i \leq l)\} & \text{if } m = 2l+1\,, \\ \{\varepsilon_i \pm \varepsilon_j \ (1 \leq i < j \leq l)\} & \text{if } m = 2l\,, \end{cases}$$

and

$$\Delta_m = \begin{cases} \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \ (1 \leq i \leq l-1)\,,\ \alpha_l = \varepsilon_l\,\} & \text{if } m = 2l+1\,, \\ \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \ (1 \leq i \leq l-1)\,,\ \alpha_l = \varepsilon_{l-1} + \varepsilon_l\} & \text{if } m = 2l\,. \end{cases}$$

Hence the standard Borel subgroup $\mathbf{P}_m$ corresponds to $\emptyset \subset \Delta_m$, and $\mathbf{N}_m$, the unipotent radical of $\mathbf{P}_m$, is written as $\mathbf{N}_m = \prod_{\alpha>0} \mathbf{X}_\alpha$.

3.6. We let $\mathbf{Q}_{m,r}$ $(1 \leq r \leq l)$ be the standard maximal parabolic subgroup corresponding to $J = \Delta_m - \{\alpha_r\}$. When $r = 0$, we put $\mathbf{Q}_{m,0} = \mathbf{G}_m$ for convenience. The standard Levi decomposition of $\mathbf{Q}_{m,r}$ is given by $\mathbf{Q}_{m,r} = \mathbf{M}_{m,r}\mathbf{U}_{m,r}$. Here

$$\mathbf{M}_{m,r} \simeq \mathbf{GL}_r \times \mathbf{SO}_{m'+1} \quad (m = 2r + m' + 1)$$

is the standard Levi part containing $\mathbf{T}_m$, and $\mathbf{U}_{m,r}$ is the unipotent radical of $\mathbf{Q}_{m,r}$. We write $\mathbf{M}_{m,r} = \mathbf{G}^{(1)} \times \mathbf{G}^{(2)}$ where $\mathbf{G}^{(1)} \simeq \mathbf{GL}_r$ (resp. $\mathbf{G}^{(2)} \simeq \mathbf{SO}_{m'+1}$). The root systems of $\mathbf{G}^{(1)}$ and $\mathbf{G}^{(2)}$ are given by

$$\Sigma^{(1)} = \{\pm(\varepsilon_i - \varepsilon_j) \ (1 \leq i < j \leq r)\}$$

and

$$\Sigma^{(2)} = \begin{cases} \{\pm\varepsilon_i \pm \varepsilon_j \ (r+1 \leq i < j \leq r+l')\,,\ \pm\varepsilon_i \ (r+1 \leq i \leq r+l')\} & \\ & \text{if } m = 2r+2l'+1\,, \\ \{\pm\varepsilon_i \pm \varepsilon_j \ (r+1 \leq i < j \leq r+l'+1)\,\} & \text{if } m = 2r+2l'+2\,, \end{cases}$$

respectively.

Subgroups of $\mathbf{G}^{(i)}$ $(i = 1, 2)$ are denoted by $\mathbf{P}^{(i)}$ (the standard Borel subgroup of upper triangular matrices), $\mathbf{N}^{(i)}$ (the unipotent radical of $\mathbf{P}^{(i)}$), $\mathbf{T}^{(i)}$ (the standard maximal torus of diagonal matrices), etc. In matrix form, some of these subgroups are given as follows: We set

$$\nu_{m,r}(x, y) := \begin{pmatrix} 1_r & J_r{}^t x S_{m-2r} & J_r(y - \frac{1}{2}S_{m-2r}[x]) \\ 0 & 1_{m-2r} & -x \\ 0 & 0 & 1_r \end{pmatrix} \in \mathbf{Q}_{m,r}$$

for $x \in \mathbf{Mat}_{m-2r,r}$, $y \in \mathbf{Alt}_r$. Then we have

$$\mathbf{U}_{m,r} = \{\nu_{m,r}(x, y) \mid x \in \mathbf{Mat}_{m-2r,r},\ y \in \mathbf{Alt}_r\}.$$

We also have

$$\mathbf{M}_{m,r} = \left\{ \mu_{m,r}(a, h) := \begin{pmatrix} a & & 0 \\ & h & \\ 0 & & \tilde{a} \end{pmatrix} \middle| \ a \in \mathbf{GL}_r,\ h \in \mathbf{G}_{m-2r} \right\}.$$

Here $\tilde{a} = J_r {}^t a^{-1} J_r$ for $a \in \mathbf{GL}_r$. Let $\mathbf{Z}_r$ be the group of unipotent upper triangular matrices in $\mathbf{GL}_r$. Then $\mathbf{N}^{(1)} = \{\mu_{m,r}(z, 1) \mid z \in \mathbf{Z}_r\}$.

3.7. Henceforth we fix two non-negative integers m' and r satisfying $m = m' + 2r + 1$. Note that $\mathbf{M}_{m,r} \simeq \mathbf{GL}_r \times \mathbf{G}_{m'+1}$ in this setting. We set $\mathbf{G} = \mathbf{G}_m$, $\mathbf{G}' = \mathbf{G}_{m'}$ and so on. Hence we put

$$K = K_m, \quad T = T_m, \quad \mathcal{H} = \mathcal{H}_m, \quad l = [m/2],$$

$$X = X_m \ni \Xi = (\Xi_1, \ldots, \Xi_l)$$

and

$$K' = K_{m'}, \quad T' = T_{m'}, \quad \mathcal{H}' = \mathcal{H}_{m'},\ l' = [m'/2],$$

$$X' = X_{m'} \ni \xi = (\xi_1, \ldots, \xi_{l'}),$$

for example.

3.8. We define an embedding $\iota = \iota_{m'}$ of $\mathbf{G}_{m'}$ into $\mathbf{G}_{m'+1}$ as follows:

(a) If $m' = 2l'$ is even,

$$\iota\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G}_{m'}$ is the block decomposition corresponding to the partition $m' = l' + l'$.

(b) If $m' = 2l' + 1$ is odd,

$$\iota\left(\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\right) = \begin{pmatrix} a_1 & a_2/2 & a_2/2 & a_3 \\ b_1 & (b_2+1)/2 & (b_2-1)/2 & b_3 \\ b_1 & (b_2-1)/2 & (b_2+1)/2 & b_3 \\ c_1 & c_2/2 & c_2/2 & c_3 \end{pmatrix},$$

where $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in \mathbf{G}_{m'}$ is the block decomposition corresponding to the partition $m' = l' + 1 + l'$.

Note that the image of $\iota_{m'}$ is the stabilizer in $\mathbf{G}_{m'+1}$ of the anisotropic vector

$${}^t(0, \ldots, 0, \overset{l'+1}{1}, 0, \ldots, 0) \quad (\text{resp. } {}^t(0, \ldots, 0, \overset{l'+1}{1}, \overset{l'+2}{-1}, 0, \ldots, 0))$$

in $\bar{k}^{m'+1}$ for $m' = 2l'$ (resp. $m' = 2l' + 1$). Henceforth we shall regard $\mathbf{G}' = \mathbf{G}_{m'}$ as a subgroup of $\mathbf{G} = \mathbf{G}_m$ under the map $g' \mapsto \mu_{m,r}(1, \iota_{m'}(g'))$ $(g' \in \mathbf{G}')$ unless otherwise stated.

3.9. We shall call the case where $m = 2r + 2l' + 1,\ m' = 2l'$ (hence $l = r + l'$) *the odd case* and the case where $m = 2r + 2l' + 2,\ m' = 2l' + 1$ (hence $l = r + l' + 1$) *the even case*, respectively.

In the odd case (where $m' = 2l'$ is even), we take $\mathbf{T}^{(2)} = \mathbf{T} \cap \mathbf{G}^{(2)}$ as a maximal torus $\mathbf{T}'$ of $\mathbf{G}'$. Then the embedding $\mathbf{G}' \hookrightarrow \mathbf{G}^{(2)}$ corresponds to the injection

$$\Sigma' = \{\pm\varepsilon'_i \pm \varepsilon'_j\ (1 \le i < j \le l')\} \to \Sigma^{(2)}$$

given by $\varepsilon'_i \mapsto \varepsilon_{r+i}\ (1 \le i \le l')$.

In the even case (where $m' = 2l' + 1$ is odd), we take $\mathrm{Ker}(\varepsilon_{r+l'+1}) \cap \mathbf{T}^{(2)}$ as a maximal torus $\mathbf{T}'$ of $\mathbf{G}'$. The embedding $\mathbf{G}' \hookrightarrow \mathbf{G}^{(2)}$ corresponds to the surjection

$$\Sigma^{(2)} \to \Sigma' = \{\pm\varepsilon'_i \pm \varepsilon'_j\ (1 \le i < j \le l'),\ \varepsilon'_i\ (1 \le i \le l')\}$$

induced by the natural projection

$$\mathrm{Hom}(\mathbf{T}^{(2)}, \mathbf{GL}_1) = \sum_{r+1 \le i \le r+l'+1} \boldsymbol{Z}\varepsilon_i \to \mathrm{Hom}(\mathbf{T}', \mathbf{GL}_1) = \Big(\sum_{r+1 \le i \le r+l'+1} \boldsymbol{Z}\varepsilon_i\Big)\Big/\boldsymbol{Z}\varepsilon_{r+l'+1}\,.$$

(We denote the image of ε_{r+i} under this projection by ε'_i.) The root subgroups of $\mathbf{G}'$ are given by

$$\mathbf{X}_{\pm\varepsilon'_i} = \{x_{\pm\varepsilon'_i}(t) := x_{\pm\varepsilon_{r+i} \mp \varepsilon_{r+l'+1}}(t) x_{\pm\varepsilon_{r+i} \pm \varepsilon_{r+l'+1}}(t) \mid t \in \bar{k}\}\,;\ \mathbf{X}_{\pm\varepsilon'_i \pm \varepsilon'_j} = \mathbf{X}_{\pm\varepsilon_{r+i} \pm \varepsilon_{r+j}}\,.$$

As in the case of $\mathbf{G}^{(i)}$ $(i = 1, 2)$, we denote by $\mathbf{P}',\ \mathbf{T}',\ \mathbf{N}'$ etc., the counterparts of the objects for $\mathbf{G}$.

3.10. Let $\mathbf{Q}$ be the parabolic subgroup of $\mathbf{G}$ with $\mathbf{P} \subset \mathbf{Q} \subset \mathbf{Q}_{m,r}$ whose Levi factor is $T^{(1)} \times G^{(2)} \simeq (\mathbf{GL}_1)^r \times \mathbf{SO}_{m'+1}$. The unipotent radical of $\mathbf{Q}$ is given by $\mathbf{U} := \mathbf{N}^{(1)}\mathbf{U}_{m,r}$. Then the group $\mathbf{G}'$ normalizes $\mathbf{U}$ (see 0.1). Let us denote by $\mathbf{H}$ the semidirect product of $\mathbf{G}'$ and $\mathbf{U}$. Obviously the unipotent radical of $\mathbf{H}$ is $\mathbf{U}$.

Let ψ be an additive character of k with conductor o. We define a character ψ_U of U by

$$\psi_U(\nu_{m,r}(x, y)\mu_{m,r}(z, 1)) = \psi\Big(x_{l'+1,1} - \epsilon_m x_{l'+2,1} + \sum_{i=1}^{r-1} z_{i,i+1}\Big)$$

for $x \in \mathbf{Mat}_{m-2r,r}(k),\ y \in \mathbf{Alt}_r(k)$ and $z \in Z_r$, where we put

$$\epsilon_m = \begin{cases} 1 & \text{if } m \text{ is even}\,, \\ 0 & \text{if } m \text{ is odd}\,. \end{cases}$$

The character ψ_U is invariant under the conjugation by G'. (This is a consequence of the fact that G' is the stabilizer in $G^{(2)}$ of certain anisotropic vector, see 3.8. See also [GP] for the definition of ψ_U in an algebraic way.) Thus we can extend ψ_U to the character of H, which we denote by the same symbol ψ_U, by putting $\psi_U|_{G'} = 1$.

It is convenient to see the restriction of ψ_U to each root subgroups in U for our later use. The set of roots appearing in $\mathbf{U}$ is $\Psi := \Sigma^+ \setminus \Sigma^{(2)+}$. Let us define a character $\psi_\alpha : X_\alpha \to \boldsymbol{C}^\times$ by $\psi_\alpha(x_\alpha(t)) = \psi(t)$ $(t \in k)$. Then we have

$$\begin{aligned}
&\psi_U|_{X_{\varepsilon_i-\varepsilon_{i+1}}} = \psi_{\varepsilon_i-\varepsilon_{i+1}} \quad (1 \le i \le r-1)\,,\\
&\psi_U|_{X_{\varepsilon_r}} = \psi_{\varepsilon_r} \quad \text{(in the odd case)}\,,\\
&\psi_U|_{X_{\varepsilon_r-\varepsilon_n}} = \psi_{\varepsilon_r-\varepsilon_n}\,,\ \psi_U|_{X_{\varepsilon_r+\varepsilon'_l}} = \psi^{-1}_{\varepsilon_r+\varepsilon'_l} \quad \text{(in the even case)}\,,
\end{aligned}$$

and

$$\psi_U|_{X_\alpha} = 1 \quad \text{(otherwise)}\,.$$

3.11. Set $\mathbf{P_H} = \mathbf{P}'\mathbf{U} = \mathbf{P}'\mathbf{N}^{(1)}\mathbf{U}_{m,r}$. This is a Borel subgroup of $\mathbf{H}$. The unipotent radical of $\mathbf{P_H}$ is $\mathbf{N_H} = \mathbf{N}'\mathbf{U}$ and hence $\mathbf{P_H} = \mathbf{T}'\mathbf{N_H}$, where $\mathbf{T}' = \mathbf{T}_{m'}$ is a maximal torus of $\mathbf{G}'$.

We are concerned with the open orbit in $P\backslash G/P_H$, where $P_H = \mathbf{P_H}(k)$. Henceforth we restrict ourselves to the case where $l' > 0$. We can easily modify the argument below in the case where $l' = 0$; we put $g_{m,r} = 1$ in that case, for example. For $\mathbf{y} = {}^t(y_1, \dots, y_{l'}) \in k^{l'}$, let $g_{m,r}(\mathbf{y})$ be an element of G given by

$$g_{m,r}(\mathbf{y}) = \begin{cases} \mu_{m,r}\left(1_r, \begin{pmatrix} 1_{l'} & 2\mathbf{y} & -J_{l'}\mathbf{y}{}^t\mathbf{y} \\ 0 & 1 & -{}^t\mathbf{y}J_{l'} \\ 0 & 0 & 1_{l'} \end{pmatrix}\right) & \text{if } m \text{ is odd}\,, \\[2ex] \mu_{m,r}\left(1_r, \begin{pmatrix} a(\mathbf{y}) & 0 \\ 0 & \tilde{a}(\mathbf{y}) \end{pmatrix}\right) & \text{if } m \text{ is even}\,, \end{cases}$$

where $a(\mathbf{y}) = \begin{pmatrix} 1_{l'} & \mathbf{y} \\ 0 & 1 \end{pmatrix} \in \mathbf{GL}_{l'+1}(k)$. We put

$$g_{m,r} = g_{m,r}(\mathbf{1}) \quad (\mathbf{1} := {}^t(1, \dots, 1) \in k^{l'})\,.$$

In the odd case,

$$g_{m,r}(\mathbf{y}) = x_{\varepsilon_{r+1}}(y_1)\cdots x_{\varepsilon_{r+l'}}(y_{l'})n' \tag{3.11.1}$$

for some $n' \in N'$. Thus we have

$$\{\, g_{m,r}(\mathbf{y}) \mid \mathbf{y} \in k^{l'} \} \times N_H \simeq N \quad ((g_{m,r}(\mathbf{y}), n_H) \leftrightarrow g_{m,r}(\mathbf{y})n_H) \tag{3.11.2}$$

(as topological spaces). Note that, for any permutation σ of $1, \dots, l'$, there exists $n'' \in N'$ (depending on $\mathbf{y}$ and σ) such that

$$x_{\varepsilon_{r+\sigma(1)}}(y_{\sigma(1)})\cdots x_{\varepsilon_{r+\sigma(l')}}(y_{\sigma(l')}) = x_{\varepsilon_{r+1}}(y_1)\cdots x_{\varepsilon_{r+l'}}(y_{l'})n''\,.$$

On the other hand, in the even case,

$$g_{m,r}(\mathbf{y}) = x_{\varepsilon_{r+1}-\varepsilon_{r+l'+1}}(y_1)\cdots x_{\varepsilon_{r+l'}-\varepsilon_{r+l'+1}}(y_{l'})\,.$$

(Observe that the factors in the right hand side are mutually commutative.) We note that

$$g_{m,r}(\mathbf{y})N' = x_{\varepsilon_{r+1}+\varepsilon_{r+l'+1}}(y_1)\cdots x_{\varepsilon_{r+l'}+\varepsilon_{r+l'+1}}(y_{l'})N'\,, \tag{3.11.3}$$

since $x_{\varepsilon_{r+i}-\varepsilon_{r+l'+1}}(t)x_{\varepsilon_{r+i}+\varepsilon_{r+l'+1}}(t) \in N'$. Hence we also have (3.11.2) in the even case.

PROPOSITION 3.12. (1) *One has*

$$G = \bigcup_{w \in W} \bigcup_{\mathbf{y} \in \{0,1\}^{l'}} Pwg_{m,r}(\mathbf{y})P_H .$$

(2) *The orbit* $\mathcal{O}_0 = Pw_\ell g_{m,r} P_H$ *is open dense in* G *.*
(3) $\mathcal{O}_0 \simeq P \times P_H \simeq P \times P' \times U$ *.*

PROOF. The Bruhat decomposition of G shows that

$$G = \bigcup_{w \in W,\ \mathbf{y} \in k^{l'}} Pwg_{m,r}(\mathbf{y})P_H .$$

We know that $Pwg_{m,r}(\mathbf{y})P_H = Pwg_{m,r}(\mathbf{y}_\epsilon)P_H$, where $\mathbf{y}_\epsilon = {}^t(\epsilon_1, \dots \epsilon_{l'}) \in \{0,1\}^{l'} \subset k^{l'}$ is defined to be $\epsilon_i = 0$ if and only if $y_i = 0$; see for example, the equality

$$g_{m,r}(\mathbf{y}) = d \cdot g_{m,r}(\mathbf{1}) \cdot d^{-1} \tag{3.12.1}$$

with

$$d = \begin{cases} \mathbf{d}_m(\overbrace{1, \dots, 1}^{r}, \overbrace{y_1, \dots, y_{l'}}^{l'}) & \text{(in the odd case; } m = 2r + 2l' + 1), \\ \mathbf{d}_m(\overbrace{1, \dots, 1}^{r}, \overbrace{y_1, \dots, y_{l'}, 1}^{l'+1}) & \text{(in the even case; } m = 2r + 2l' + 2) \end{cases}$$

for $\mathbf{y} = (y_1, \dots, y_{l'})$ with $y_1 \cdots y_{l'} \neq 0$. Thus (1) is proved. Since $\mathcal{O}_0$ is the open subset of the big cell $Pw_\ell P \simeq P \times N$ given by

$$\mathcal{O}_0 = \{pw_\ell n \in Pw_\ell N \mid n = g_{m,r}(\mathbf{y})n_H \ (y_1 \cdots y_{l'} \neq 0,\ n_H \in N_H)\}, \tag{3.12.2}$$

(3.12.1) shows (2) and (3.11.2) does (3) of the proposition. □

REMARK 3.13. Obviously, the proof of this proposition works over $\bar{k}$ instead of k. In particular, we see that $\mathbf{P}w_\ell g_{m,r}\mathbf{P_H} \subset \mathbf{G}$ is Zariski open.

3.14. We construct some relative invariants on $\mathbf{G}$ under the action of $\mathbf{P} \times \mathbf{P_H}$, and describe the open orbit $\mathcal{O}_0 = Pw_\ell g_{m,r} P_H$ (or $\mathcal{O}_0{}^{-1} = P_H g_{m,r} w_\ell P$) in terms of these relative invariants.

From now on, we shall fix w_ℓ, a representative of the longest element of W, as follows:

$$\begin{aligned} w_\ell &= \begin{pmatrix} & & J_l \\ & (-1)^l & \\ J_l & & \end{pmatrix} && \text{if } m = 2l + 1, \\ &= \begin{pmatrix} & J_l \\ J_l & \end{pmatrix} && \text{if } m = 2l,\ l \text{ even}, \\ &= \begin{pmatrix} & & & J_{l-1} \\ & 1 & 0 & \\ & 0 & 1 & \\ J_{l-1} & & & \end{pmatrix} && \text{if } m = 2l,\ l \text{ odd}. \end{aligned}$$

Let $I = \{i_1, \dots, i_s\}$ and $J = \{j_1, \dots, j_s\}$ be two subsets of $\{1, \dots, m\}$ with cardinality s. For $g \in \mathbf{Mat}_m(k)$, we define a polynomial function $\Delta_{I,J}$ on $\mathbf{Mat}_m(k)$ by

$$\Delta_{I,J}(g) = \det(g_{I,J}),$$

where $g_{I,J} = (g_{i_k, j_l})_{1 \le k, l \le s} \in \mathbf{Mat}_s(k)$.

Now we define polynomial functions α_i on $\mathbf{G} = \mathbf{SO}_m$ by the following formula

$$\alpha_i(g) = \Delta_{\{1,\dots,i\},\{1,\dots,i\}}(w_\ell g) \quad (1 \le i \le l).$$

We put $\alpha_0(g) = 1$ for convenience. Then $g \in G$ is contained in the big cell $P w_\ell P$ if and only if $\alpha_i(g) \neq 0$ for any $i = 1, \dots, l$. Obviously,

$$\alpha_i(g_{m,r}(\mathbf{y}) w_\ell) = 1 \quad (\mathbf{y} \in k^{l'})$$

and

$$\alpha_i(p^{(1)} g p^{(2)}) = (t_1^{(1)} \cdots t_i^{(1)})^{-1} (t_1^{(2)} \cdots t_i^{(2)}) \alpha_i(g)$$

for $p^{(a)} = \mathbf{d}_m(t_1^{(a)}, \dots, t_l^{(a)}) \cdot n^{(a)} \in P$ $(a = 1, 2)$ with $t_i^{(a)} \in k^\times, n^{(a)} \in N$. Set $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \in \mathrm{Hom}(\mathbf{T}, \mathbf{GL}_1)$ $(1 \le i \le l)$ and $\varpi'_j = \varepsilon'_1 + \cdots + \varepsilon'_j \in \mathrm{Hom}(\mathbf{T}', \mathbf{GL}_1)$ $(1 \le j \le l')$. These are dominant weights of $\mathbf{G}$ and $\mathbf{G}'$ (relative to $\mathbf{P}$ and $\mathbf{P}'$) respectively. Then the above formula shows that α_i has a highest weight

$$\begin{array}{ll} (\varpi_i, 0) & (1 \le i \le r), \\ (\varpi_i, \varpi'_{i-r}) & (r+1 \le i \le r+l'), \end{array}$$

or

$$(\varpi_{r+l'+1}, \varpi'_{l'}) \quad (i = r + l' + 1 = l;\ \text{in the even case})$$

under the $\mathbf{P} \times \mathbf{P}'$ action.

To obtain the open orbit $\mathcal{O}_0$, we need other functions: For $g \in G$, we set

$$\beta_j(g) = \Delta_{\{1,\dots,r+j-1,r+l'+1\},\{1,\dots,r+j\}}(w_\ell g) \quad (1 \le j \le l')$$

in the odd case ($m = 2r + 2l' + 1$) and

$$\begin{aligned} \beta_j(g) &= \Delta_{\{1,\dots,r+j-1,r+l'+1\},\{1,\dots,r+j\}}(w_\ell g) \\ &\quad - \Delta_{\{1,\dots,r+j-1,r+l'+2\},\{1,\dots,r+j\}}(w_\ell g) \quad (1 \le j \le l'+1) \end{aligned}$$

in the even case ($m = 2r + 2l' + 2$), respectively. For each j with $1 \le j \le l'$ in the odd case and $1 \le j \le l' + 1$ in the even case, it is easily checked that

$$|\beta_j(g_{m,r}(\mathbf{y}) w_\ell)| = \begin{cases} 1 & \text{if } j = l' + 1 \text{ in the even case}, \\ |y_j| & \text{otherwise} \end{cases}$$

for $\mathbf{y} \in k^{l'}$, and

$$\beta_k(p_H g p) = (t'_1 \cdots t'_{j-1})^{-1} (t_1 \cdots t_{r+j}) \beta_j(g)$$

for $p_H = \mathbf{d}_{m'}(t'_1, \dots, t'_{l'}) \cdot n_H \in P_H,\ p = \mathbf{d}_m(t_1, \dots, t_l) \cdot n \in P$ with $t_i, t'_i \in k^\times,\ n_H \in N_H,\ n \in N$. This formula shows that β_j has a highest weight

$$(\varpi_{r+j}, \varpi'_{j-1})$$

under the $\mathbf{P} \times \mathbf{P}'$ action. Here we put $\varpi'_0 = 0$ for convenience.

Then we easily have the following lemmas.

LEMMA 3.15. *For $g \in G$, $g \in P_H g_{m,r} w_\ell P$ if and only if*

$$\alpha_i(g) \neq 0, \quad \beta_j(g) \neq 0$$

for any i, j.

LEMMA 3.16. *Suppose that $g \in P_H g_{m,r} w_\ell P$ is written in the form*

$$g = n_H \cdot \mathbf{d}_{m'}(t'_1, \ldots, t'_{l'}) g_{m,r} w_\ell \mathbf{d}_m(t_1, \ldots, t_l) \cdot n$$

for some $n_H \in N_H$, $n \in N$, and t_i, $t'_j \in k^\times$ $(1 \leq i \leq l,\ 1 \leq j \leq l')$. Then the absolute values of t_i, t'_j $(1 \leq i \leq l,\ 1 \leq j \leq l')$ are given by

$$|t_i| = \left|\frac{\alpha_i(g)}{\alpha_{i-1}(g)}\right| \qquad (1 \leq i \leq r),$$

$$|t_{r+i}| = \left|\frac{\beta_i(g)}{\alpha_{r+i-1}(g)}\right| \qquad \begin{pmatrix} 1 \leq i \leq l' & \textit{in the odd case} \\ 1 \leq i \leq l'+1 & \textit{in the even case} \end{pmatrix},$$

and

$$|t'_j| = \left|\frac{\beta_j(g)}{\alpha_{r+j}(g)}\right| \qquad (1 \leq j \leq l').$$

4. Whittaker-Shintani functions. In this section, we shall introduce the Whittaker-Shintani functions on orthogonal groups that are the main subject of this paper. Then we shall give an integral expression of these functions through a representation-theoretic interpretation.

DEFINITION 4.1. For $(\varXi, \xi) \in X \times X'$, a function $F \in C^\infty(G)$ is said to be a *Whittaker-Shintani function* attached to $(\varXi, \xi)$, if the following two conditions hold:

$$L(uk')R(k)F = \psi_U(u)F \quad (u \in U,\ k' \in K',\ k \in K), \tag{4.1.1}$$

$$L(\varphi')R(\varphi)F = \omega_\xi(\varphi')\omega_\varXi(\varphi)F \quad (\varphi' \in \mathcal{H}',\ \varphi \in \mathcal{H}). \tag{4.1.2}$$

Here L (resp. R) denotes the left (resp. right) regular representation of G (or its restriction to subgroups) on $C^\infty(G)$ so that $(L(g_1)R(g_2)f)(x) = f(g_1^{-1}xg_2)$ $(g_1, g_2, x \in G)$. We denote the space of Whittaker-Shintani functions attached to $(\varXi, \xi)$ by $WS(\varXi, \xi)$.

REMARK 4.2. These functions are the special functions on G already studied in the following cases. When $r = 0$ (hence U is trivial), they coincide with the Shintani functions first introduced and studied in [MS2]. On the other hand, when $m' = 0$ or 1 so that $U = N_m$ (a maximal unipotent subgroup of G), they turn out to be the class-1 (or unramified) Whittaker functions of $G = \mathbf{SO}_m(k)$ (see [CS], [K1]). In the case $m' = 2$, they appear in the context of Bessel models (see [BFF]).

REMARK 4.3. These functions are examples of spherical functions on spherical homogeneous spaces. To explain this, let $\mathbf{G}_1$ be a reductive group defined over k and $\mathbf{H}_1$ an algebraic subgroup of $\mathbf{G}_1$. Let $K_1 = \mathbf{G}_1(o)$ be a "good" maximal open compact subgroup of G_1 and $\mathcal{H}_1 = \mathcal{H}(G_1, K_1)$ the corresponding Hecke algebra. For a character ψ_1 of H_1, we set

$$C^\infty(G_1, \psi_1) = \{f \in C^\infty(G) \mid L(h)f = \psi_1(h)f \ (h \in H_1)\},$$

on which G acts on the right, as above. Then for $\omega_1 \in \mathrm{Hom}_{\boldsymbol{C}\text{-alg}}(\mathcal{H}(G_1, K_1), \boldsymbol{C})$, we call a function $f \in C^\infty(G_1, \psi_1)^{K_1}$ satisfying

$$R(\varphi_1)f = \omega_1(\varphi_1)f \quad (\varphi_1 \in \mathcal{H}_1) \tag{4.3.1}$$

a *spherical function* of the homogeneous space $H_1\backslash G_1$ (with the representation ψ_1) attached to ω_1.

If $\mathbf{H}_1\backslash\mathbf{G}_1$ is spherical, namely a Borel subgroup of $\mathbf{G}_1$ has an open dense orbit on $\mathbf{H}_1\backslash\mathbf{G}_1$, then we can expect that spherical functions on $H_1\backslash G_1$ have good properties, such as multiplicity-one (-finite), an explicit formula, and so on. Zonal spherical functions and Whittaker functions are well-known examples of them. Such spherical functions, which are of interest in representation theory in its own right, have often been playing important roles in number theory in various context, especially in the theory of automorphic L-functions. (See, e.g., [K3] and [M].) We refer [HS1], [HS2] and [H] for spherical functions on symmetric spaces (which form an important family of spherical homogeneous spaces) and other number theoretic applications of these spherical functions.

Now we return to our case. Let us define a subgroup $\mathbf{H}$ of $\mathbf{G} = \mathbf{SO}_m$ to be the semi-direct product of $\mathbf{G}' \simeq \mathbf{SO}_{m'}$ and $\mathbf{U}$, as in 3.10. (Note that $\mathbf{H}$ is not reductive when $r > 0$.) We set $\mathbf{G}_1 = \mathbf{G} \times \mathbf{G}'$,

$$\mathbf{H}_1 = \{(h, p(h)) \in \mathbf{G}_1 \mid h \in \mathbf{H}\} \simeq \mathbf{H},$$

where $p : \mathbf{H} \to \mathbf{G}'$ is the natural projection. Then 3.13 shows that $\mathbf{H}_1\backslash\mathbf{G}_1$ is spherical. Since $\psi_U : U \to \boldsymbol{C}^\times$ is G'-invariant, ψ_U naturally defines a character $\psi_1 : H_1 \to \boldsymbol{C}^\times$ by $\psi_1((h, p(h))) = \psi_U(u)$ for $h = g'u \in H$ $(g' \in G', u \in U)$. Note that $\mathbf{H}_1\backslash\mathbf{G}_1 \simeq \mathbf{U}\backslash\mathbf{G}$. Thus we can see that our Whittaker-Shintani functions are spherical functions on a spherical homogeneous space $H_1\backslash G_1$.

As is noted in the introduction, Shintani functions for $\mathbf{GL}_n(k)$ ([MS3]) and Whittaker-Shintani functions for $\mathbf{Sp}_{2n}(k)$ ([Sh2], [MS1]) are also examples of those functions. Explicit formulas for these functions are obtained in a similar manner.

4.4. Let $I(\xi)$ be the unramified principal series representation of G' for $\xi \in X'$. The group $H = G' \cdot U$ (semidirect product) acts on $I(\xi)$ via $H \to G' = H/U$. On the other hand, we have a character ψ_U of H (see Sect.3). Thus we can define "the unramified principal series representation of H", $I(\xi, \psi_U) := I(\xi) \otimes \psi_U$ $(= \mathrm{Ind}(\xi\delta'^{1/2} \otimes \psi_U \mid P_H, H))$. Note that the underlying G'-space of $I(\xi, \psi_U)$ is the same as $I(\xi)$. The action of $g'u \in G'U = H$ on $I(\xi)$ is given by $\phi_0 \mapsto \psi_U(u)R(g')\phi_0$ $(\phi_0 \in I(\xi))$.

Denote by $\langle\ ,\ \rangle_0 = \langle\ ,\ \rangle_{0,\xi}$ the canonical G'-invariant pairing on $I(\xi) \times I(\xi^{-1})$ given by

$$\langle\phi_0, \phi_0'\rangle_0 = \int_{K'} \phi_0(k')\phi_0'(k')dk' \quad (\phi_0 \in I(\xi), \phi_0' \in I(\xi^{-1}))\,.$$

This $\langle\ ,\ \rangle_0$ naturally defines an H-invariant pairing on $I(\xi, \psi_U) \times I(\xi^{-1}, \psi_U^{-1})$ by the same formula. (We still denote this extension by $\langle\ ,\ \rangle_0$.) Let T be an element of $\mathrm{Hom}_H(I(\Xi), I(\xi^{-1}, \psi_U^{-1}))$. Then the function S_T on G given by

$$S_T(g) = \langle\phi_{K',\xi}, T(R(g)\phi_{K,\Xi})\rangle_0 \tag{4.4.1}$$

is a Whittaker-Shintani function attached to (ξ, Ξ). (Recall 1.8.)

Let $\Omega : I(\xi, \psi_U) \times I(\Xi) \to \mathbf{C}$ be an H-invariant bilinear form. Namely, Ω is a bilinear form on $I(\xi) \times I(\Xi)$ satisfying $\Omega(R'(g')\phi_0, R(g'u)\phi) = \psi_U(u)\Omega(\phi_0, \phi)$ for $\phi_0 \in I(\xi), \phi \in I(\Xi), g' \in G'$ and $u \in U$. Then the function S_Ω on G given by

$$S_\Omega(g) = \Omega(\phi_{K',\xi}, R(g)\phi_{K,\Xi}) \tag{4.4.2}$$

is a Whittaker-Shintani function attached to (ξ, Ξ).

It is easy to see that the construction of (4.4.1) and (4.4.2) are equivalent. Actually, T and Ω above correspond each other in the following way. If we have $T \in \mathrm{Hom}_H(I(\Xi), I(\xi^{-1}, \psi_U^{-1}))$, then the bilinear form Ω_T on $I(\xi, \psi_U) \times I(\Xi)$ given by $\Omega_T(\phi_0, \phi) = \langle\phi_0, T(\phi)\rangle_0$ is H-invariant. Conversely, let Ω be an H-invariant bilinear form on $I(\xi, \psi_U) \times I(\Xi)$. We can define $T_\Omega \in \mathrm{Hom}_H(I(\Xi), I(\xi, \psi_U)^*)$ by $T_\Omega(\phi)(\phi_0) = \Omega(\phi_0, \phi)$. (Here $I(\xi, \psi_U)^*$ is the dual of $I(\xi, \psi_U)$.) Since $I(\Xi)$ is a smooth G'-module, the image of T_Ω is also smooth. Hence we may regard $T \in \mathrm{Hom}_H(I(\Xi), I(\xi^{-1}, \psi_U^{-1}))$.

4.5. Suppose that $Y = Y_{\Xi,\xi}$ $(\Xi \in X, \xi \in X')$ is a continuous function (or a distribution) on G satisfying

$$Y(pgp'u) = (\Xi^{-1}\delta^{1/2})(p)(\xi\delta'^{-1/2})(p')\psi_U(u)Y(g) \quad (p \in P, p' \in P', u \in U)\,. \tag{4.5.1}$$

Then we have an equivariant linear form $l_{\Xi,\xi} \in \mathrm{Hom}_{P_H}(I(\Xi), \xi^{-1}\delta'^{1/2} \otimes \psi_U^{-1})$ defined from $Y_{\Xi,\xi}$ as

$$l_{\Xi,\xi}(\mathcal{P}_\Xi(f)) = \int_G f(g)Y(g)dg \quad (f \in C_c^\infty(G))\,.$$

(See 1.2 for the definition of $\mathcal{P}_\Xi : C_c^\infty(G) \to I(\Xi)$.) The intertwining operator $T_{\Xi,\xi} \in \mathrm{Hom}_H(I(\Xi), I(\xi^{-1}, \psi_U^{-1}))$ corresponding to $l_{\Xi,\xi}$ via Frobenius reciprocity is given by

$$\begin{aligned} T_{\Xi,\xi}(\mathcal{P}_\Xi(f))(x') &= l_{\Xi,\xi}(R(x')\mathcal{P}_\Xi(f)) \\ &= \int_G f(xx')Y(x)dx \quad (f \in C_c^\infty(G),\ x' \in G')\,. \end{aligned}$$

Hence the H-invariant bilinear form $\Omega_{\Xi,\xi} = \Omega_{T_{\Xi,\xi}}$ attached to $T_{\Xi,\xi}$ is given by

$$\Omega_{\Xi,\xi}(\mathcal{P}_\xi(f_0), \mathcal{P}_\Xi(f)) = \int_{G'\times G} f_0(x')f(x)Y(x(x')^{-1})dx'dx \tag{4.5.2}$$

for $f_0 \in C_c^\infty(G')$, $f \in C_c^\infty(G)$. Here we identified the space $I(\xi)$, the image of $\mathcal{P}_\xi$, with $I(\xi, \psi_U)$. In particular, the function $S_{T_{\Xi,\xi}}$ is given by the integral

$$S_{T_{\Xi,\xi}}(g) = \int_{K' \times K} Y(kg^{-1}k')dk'dk \tag{4.5.3}$$

(see also [MS3, 4.8 – 4.9]).

4.6. In the rest of this section, we shall show how to construct a function $Y_{\Xi,\xi}$ on G satisfying (4.5.1). Consider the function $\Upsilon_{\Xi,\xi}$ on $P_H g_{m,r} w_\ell P = UP' g_{m,r} w_\ell P$ given by

$$\Upsilon_{\Xi,\xi}(up' g_{m,r} w_\ell p) = \psi_U(u)^{-1}(\Xi\delta^{-1/2})(p)(\xi^{-1}\delta'^{1/2})(p') \quad (u \in U,\ p \in P,\ p' \in P').$$

We extend this $\Upsilon_{\Xi,\xi}$ to the whole G by putting $\Upsilon_{\Xi,\xi}(g) = 0$ if $g \notin UP' g_{m,r} w_\ell P$.

LEMMA 4.7. *If $g \in UP' g_{m,r} w_\ell P$, then*

$$\begin{aligned} \Upsilon_{\Xi,\xi}(g) =& \psi_U(u(g))^{-1} \prod_{i=1}^{r} (\Xi_i \Xi_{i+1}^{-1} |\cdot|^{-1})(\alpha_i(g)) \\ & \times \prod_{j=1}^{l'-1} (\xi_j \Xi_{r+j+1}^{-1} |\cdot|^{-1/2})(\alpha_{r+j}(g)) \cdot \xi_{l'}(\alpha_{r+l'}(g)) \\ & \times \prod_{k=1}^{l'} (\xi_k^{-1} \Xi_{r+k} |\cdot|^{-1/2})(\beta_k(g)) \end{aligned} \tag{4.7.1}$$

in the odd case ($m = 2r + 2l' + 1,\ m' = 2l'$), *and*

$$\begin{aligned} \Upsilon_{\Xi,\xi}(g) =& \psi_U(u(g))^{-1} \prod_{i=1}^{r} (\Xi_i \Xi_{i+1}^{-1} |\cdot|^{-1})(\alpha_i(g)) \\ & \times \prod_{j=1}^{l'-1} (\xi_j \Xi_{r+j+1}^{-1} |\cdot|^{-1/2})(\alpha_{r+j}(g)) \\ & \times \prod_{k=1}^{l'} (\xi_k^{-1} \Xi_{r+k} |\cdot|^{-1/2})(\beta_k(g)) \cdot \Xi_{r+l'+1}(\beta_{l'+1}(g)) \end{aligned} \tag{4.7.2}$$

in the even case ($m = 2r + 2l' + 2,\ m' = 2l' + 1$). *Here $u(g)$ is the U-component of g.*

PROOF. This is a consequence of 3.14. □

Then the lemma above shows the following proposition.

PROPOSITION 4.8. *Let Z_c be the nonempty open subset of $X \times X'$ given by*

$$Z_c = \left\{ (\Xi, \xi) \in X \times X' \;\middle|\; \begin{array}{ll} |\Xi_i \Xi_{i+1}^{-1}| < q^{-1} & (1 \le i \le r) \\ |\xi_j \Xi_{r+j+1}^{-1}| < q^{-1/2} & (1 \le j \le l') \\ |\xi_k^{-1} \Xi_{r+k}| < q^{-1/2} & (1 \le k \le l') \\ |\xi_{l'}| < 1 & \end{array} \right\} \tag{4.8.1}$$

in the odd case $(m = 2r + 2l' + 1,\ m' = 2l')$, *and*

$$Z_c = \left\{ (\Xi, \xi) \in X \times X' \;\middle|\; \begin{array}{ll} |\Xi_i \Xi_{i+1}^{-1}| < q^{-1} & (1 \le i \le r) \\ |\xi_j \Xi_{r+j+1}^{-1}| < q^{-1/2} & (1 \le j \le l') \\ |\xi_k^{-1} \Xi_{r+k}| < q^{-1/2} & (1 \le k \le l') \\ |\Xi_{r+l'+1}| < 1 & \end{array} \right\} \tag{4.8.2}$$

in the even case $(m = 2r + 2l' + 2,\ m' = 2l' + 1)$. *Then the function* $\Upsilon_{\Xi,\xi}$ *on* G *is continuous for* $(\Xi, \xi) \in Z_c$.

4.9. Now let us set

$$Y_{\Xi,\xi}(g) = \Upsilon_{\Xi,\xi}(g^{-1}) \quad (g \in G)\,.$$

For $(\Xi, \xi) \in Z_c$, this $Y_{\Xi,\xi}$ is a continuous function on G. Moreover it satisfies the condition

$$Y_{\Xi,\xi}(pg\,p'u) = \psi_U(u)(\Xi^{-1}\delta^{1/2})(p)(\xi\delta'^{-1/2})(p')Y_{\Xi,\xi}(g)$$

for $u \in U$, $p \in P$, $p' \in P'$ with

$$Y_{\Xi,\xi}(w_\ell g_{m,r}) = 1\,.$$

(Note that $g_{m,r}^{-1} \in N'T'_{(0)}g_{m,r}T_{(0)}$.) Thus we can construct a Whittaker-Shintani function $S_{\Xi,\xi}$ from this $Y_{\Xi,\xi}$ as in (4.5.3) for $(\Xi, \xi) \in Z_c$.

5. Cartan-type decompositions. In this section, we shall give a double coset decomposition $UK'\backslash G/K$ explicitly, where UK' is a subgroup of $H = UG'$. This decomposition is indispensable for our study of Whittaker-Shintani functions.

Let $g_{m,r} = g_{m,r}(\mathbf{1})$ be an element of G defined in 3.11.

THEOREM 5.1. *The double coset decomposition*

$$G = \bigsqcup UK't'(\lambda')g_{m,r}t(\lambda)K$$

holds, *where* λ *runs over* $\mathbf{Z}^r \times \Lambda^+_{m-2r} \subset \Lambda_m$ *and* λ' *over* $\Lambda^+_{m'}$.

First we shall show that this theorem can be reduced to the special case of the theorem where $r = 0$, that is, $m' = m - 1$:

THEOREM 5.2. *The double coset decomposition*

$$G_m = \bigsqcup K_{m-1}t_{m-1}(\lambda')g_{m,0}t_m(\lambda)K_m$$

holds, *where* λ *runs over* Λ^+_m *and* λ' *over* Λ^+_{m-1}.

5.3. PROOF OF 5.1 BY USING 5.2. Recall the definition of the parabolic subgroup $Q_{m,r}$ introduced in 3.6. By the Iwasawa decomposition, we have

$$G_m = Q_{m,r}K_m = U_{m,r}M_{m,r}K_m\,. \tag{5.3.1}$$

Since $M_{m,r} \simeq \mathbf{GL}_r(k) \times G_{m-2r}$,

$$M_{m,r}/(K_m \cap M_{m,r}) \simeq \mathbf{GL}_r(k)/\mathbf{GL}_r(o) \times G_{m-2r}/K_{m-2r}\,.$$

We know that

$$\mathbf{GL}_r(k) = \bigsqcup_{\kappa=(\kappa_1,\dots,\kappa_r)\in\mathbf{Z}^r} Z_r \operatorname{diag}(\pi^{\kappa_1},\dots,\pi^{\kappa_r})\mathbf{GL}_r(o) \tag{5.3.2}$$

from the Iwasawa decomposition for $\mathbf{GL}_r$, and that

$$G_{m-2r} = \bigsqcup_{\substack{\lambda\in\Lambda_{m-2r}\\ \lambda'\in\Lambda_{m-2r-1}}} K_{m-2r-1}t_{m-2r-1}(\lambda')g_{m-2r,0}t_{m-2r}(\lambda)K_{m-2r} \tag{5.3.3}$$

from 5.2. Hence, by applying $\mu_{m,r}$ to (5.3.2) and (5.3.3), we get the decomposition

$$G = \bigsqcup (U_{m,r}N^{(1)})K_{m-2r-1}t_{m-2r-1}(\lambda')g_{m,r}\mu_{m,r}(\operatorname{diag}(\pi^{\kappa_1},\dots,\pi^{\kappa_r}), t_{m-2r}(\lambda))K_m$$

from (5.3.1), where λ, λ' and κ run over Λ^+_{m-2r}, $\Lambda^+_{m'}$ and $\mathbf{Z}^r$, respectively. This is nothing but the decomposition of 5.1,

$$G = \bigsqcup UK't'(\lambda')g_{m,r}t(\lambda)K \quad (\lambda\in\mathbf{Z}^r\times\ \Lambda^+_{m-2r}\subset\Lambda_m;\ l'\in\Lambda^+_{m'}). \qquad \square$$

5.4. In order to prove Theorem 5.2, we need a variant of the theorem for orthogonal groups

$$\mathbf{O}_m = \{g\in\mathbf{GL}_m \mid {}^t g S_m g = S_m\}.$$

Set $G^*_m = \mathbf{O}_m(k)$ and $K^*_m = \mathbf{O}_m(o)$. Hence G_m and K_m are subgroups of G^*_m and K^*_m, respectively. Define Λ^{*+}_m, a subset of Λ_m, by

$$\Lambda^{*+}_m = \{\lambda = (\lambda_1,\dots,\lambda_l) \mid \lambda_1\ge\cdots\ge\lambda_l\ge 0\}.$$

We embed $\mathbf{O}_{m-1}$ into $\mathbf{O}_m$ as in 3.8.

THEOREM 5.5. *The double coset decomposition*

$$G^*_m = \bigsqcup K^*_{m-1}t_{m-1}(\lambda')g_{m,0}t_m(\lambda)K^*_m$$

holds, where λ runs over Λ^{*+}_m and λ' over Λ^{*+}_{m-1}.

REMARK 5.6. We shall not give a proof for the disjointness of the decompositions appearing in these theorems 5.1, 5.2 and 5.5 in this section. The disjointness of 5.1 will be shown in Section 7. (That for 5.5 follows similarly.)

5.7. Subsections 5.7 through 5.11 are devoted to a proof of Theorem 5.5. We put $G^* = G^*_m$, $G^{*\prime} = G^*_{m-1}$, $K^* = K^*_m$ and $K^{*\prime} = K^*_{m-1}$.

Let $W(\mathrm{B}_l)$ be the Weyl group of type B_l. We regard this $W(\mathrm{B}_l)$ as a subgroup of $GL(\Lambda_m)$ as in 3.3. We remark that the "Weyl group of G^*", $W^* = N_{G^*}(T)/Z_{G^*}(T)$ is naturally isomorphic to

$$W^* \simeq \begin{cases} W = W(\mathrm{B}_l) & \text{if } m \text{ is odd},\\ W\cdot\langle\gamma_m\rangle \text{ (semidirect product)} \simeq W(\mathrm{B}_l) & \text{if } m \text{ is even},\end{cases}$$

where $\gamma_m\in GL(\Lambda_m)$ is an involution given by

$$\gamma_m(\varepsilon_l) = -\varepsilon_l, \quad \gamma_m(\varepsilon_i) = \varepsilon_i\ (i\ne l).$$

As in the case of G, we identify all the elements in W^* with their representatives in K^*.

Note first that the Cartan decomposition

$$G^* = K^* T^{*++} K^* ,$$

where

$$T^{*++} = \{\mathbf{d}_m(t_1, \ldots, t_l) \mid v(t_1) \geq \cdots \geq v(t_l) \geq 0 \ (t_i \in k^\times)\}$$

yields the decomposition

(5.7.1) $$G^* = B W^* T^{*++} K^* .$$

Let us define $\mathcal{V}$, a subset of G^*, by

(5.7.2) $$\mathcal{V} := K^{*\prime} \cdot \{ g_{m,0}(\mathbf{y}) \mid \mathbf{y} \in o^n \} .$$

Then we have

(5.7.3) $$N^+_{(0)} \subset \mathcal{V}$$

from 3.11.2. (Note that the decomposition in 3.11.2 is defined over o). In the even case, we also remark that

(5.7.4) $$\mathcal{V} = K^{*\prime} \cdot \{ \gamma_m(g_{m,0}(\mathbf{y})) \mid \mathbf{y} \in o^{l'} \}$$

(see (3.11.3)). Set

$$\mathcal{U}_w = \mathcal{V} N^-_{w,(1)} w T^{*++} K^*$$

for $w \in W^*$, where

$$N^-_{w,(1)} = \prod_{\alpha > 0,\ w^{-1}\alpha < 0} X_{-\alpha,(1)} .$$

In particular, $\mathcal{U}_1 = \mathcal{V} T^{*++} K^*$.

Now we prove the following proposition.

PROPOSITION 5.8. *For any $w \in W^*$, $\mathcal{U}_w$ is a subset of $\mathcal{U}_1$.*

This proposition implies the following factorization.

COROLLARY 5.9. *One has $G^* = \mathcal{V} \cdot T^{*++} \cdot K^*$.*

5.10. PROOF OF PROPOSITION 5.8. We proceed by induction on $\ell(w) := \#\{\alpha > 0 \mid w^{-1}\alpha < 0\}$ for $w \in W^*$.

First consider the case $\ell(w) = 0$. If $w = 1$, then 5.8 is obvious. Otherwise we have $w = \gamma_m$. (Hence m should be even.) In this case, we may assume that γ_m is represented by the matrix

$$\begin{pmatrix} 1_{l'} & & & \\ & 0 & -1 & \\ & -1 & 0 & \\ & & & 1_{l'} \end{pmatrix} \quad (m = 2l' + 2) ,$$

which is in the image of the embedding of $G^{*\prime}$ in G^* (see 3.8). Hence $\mathcal{U}_{\gamma_m} = \mathcal{U}_1$ by (3.11.3).

To prove the proposition 5.8, it suffices to show that

$$\mathcal{U}_w \subset \mathcal{U}_y$$

for some $y \in W^*$ with $\ell(y) < \ell(w)$ from the assumption of the induction.

Suppose that $\ell(w) \neq 0$. Then there exists a simple root α so that $w^{-1}\alpha < 0$. This implies that w is written as $w = w_\alpha w'$ with $\ell(w') < \ell(w)$. In this setting, we note that

$$N^-_{w,(1)} w = X_{-\alpha,(1)} \cdot w_\alpha \cdot N^-_{w',(1)} w' \tag{5.10.1}$$

and that

$$X_{-\alpha,(0)} \cdot N^-_{w',(1)} w' \subset N^-_{w',(1)} w' N^-_{(0)} \,, \tag{5.10.2}$$

since $(w')^{-1}(-\alpha) < 0$.

We now consider the odd case (case A; $m = 2l' + 1$) and the even case (case B; $m = 2l' + 2$) separately. Furthermore, we divide each case into several subcases.

• Case A-1: $\alpha = \alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le l' - 1)$

In this case, we have

$$\mathcal{V} X_{-\alpha,(1)} w_\alpha \subset \mathcal{V}\,,$$

since $X_{-\alpha,(1)}$ $(\subset K^{*\prime})$ and w_α $(\in K^{*\prime})$ normalize $K^{*\prime} \cdot \{g_{m,0}(\mathbf{y}) \mid \mathbf{y} \in o^{l'}\}$. Thus, by using (5.10.1), we have $\mathcal{U}_w \subset \mathcal{U}_{w'}$.

• Case A-2: $\alpha = \alpha_{l'} = \varepsilon_{l'}$

In this case, we have

$$\begin{aligned}
\mathcal{U}_w &= \mathcal{V} w_\alpha \cdot X_{\alpha,(1)} \cdot N^-_{w',(1)} w' T^{*++} K^* \\
&\subset \mathcal{V} X_{-\alpha,(0)} X_{\alpha,(1)} \cdot N^-_{w',(1)} w' T^{*++} K^* && \left(\begin{array}{l}\text{since we may assume that} \\ w_\alpha = \gamma_{m-1} \in K^{*\prime} = \mathbf{O}_{m-1}(o)\end{array}\right) \\
&= \mathcal{V} X_{\alpha,(1)} X_{-\alpha,(0)} \cdot N^-_{w',(1)} w' T^{*++} K^* \\
&\subset \mathcal{U}_{w'} && \text{(by (5.10.2))}\,.
\end{aligned}$$

• Case B-1: $\alpha = \alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le l' - 1)$

We can show that $\mathcal{U}_w \subset \mathcal{U}_{w'}$ exactly in the same way as in the case A-1.

• Case B-2: $\alpha = \alpha_n = \varepsilon_n - \varepsilon_{l'+1}$ and $w'^{-1}(\alpha_{l'+1}) < 0$

In this case, we have the decomposition $w = w_{\alpha_{l'}} w_{\alpha_{l'+1}} w''$ with $\ell(w'') = \ell(w) - 2$. We may assume that $w_{\alpha_{l'}} w_{\alpha_{l'+1}} \in K^{*\prime}$. Here $w_{\alpha_{l'}} w_{\alpha_{l'+1}}$ gives a permutation $\varepsilon_{l'} \to -\varepsilon_{l'},\ \varepsilon_{l'+1} \to \varepsilon_{l'+1}$ in Σ, which induces a permutation $\varepsilon'_{l'} \to -\varepsilon'_{l'}$ in Σ'. Namely, $w_{\alpha_{l'-1}} w_{\alpha_{l'}}$ corresponds to $w_{\varepsilon'_{l'}}$ in $W^{*\prime}$. Thus

$$\begin{aligned}
\mathcal{U}_w &= \mathcal{V} w_{\alpha_{l'}} w_{\alpha_{l'+1}} \cdot X_{\alpha_{l'},(1)} X_{\alpha_{l'+1},(1)} \cdot N^-_{w'',(1)} w'' T^{*++} K^* \\
&\subset \mathcal{V} X_{-\alpha_{l'},(0)} X_{-\alpha_{l'+1},(0)} X_{\alpha_{l'},(1)} X_{\alpha_{l'+1},(1)} \cdot N^-_{w'',(1)} w'' T^{*++} K^* \\
&= \mathcal{V} X_{\alpha_{l'},(1)} X_{\alpha_{l'+1},(1)} X_{-\alpha_{l'},(0)} X_{-\alpha_{l'+1},(0)} \cdot N^-_{w'',(1)} w'' T^{*++} K^* \\
&\subset \mathcal{U}_{w''} \quad \text{(by (5.10.2))}\,.
\end{aligned}$$

• Case B-3: $\alpha = \alpha_{l'} = \varepsilon_{l'} - \varepsilon_{l'+1}$ and $w'^{-1}(\alpha_{l'+1}) > 0$

Since $\mathcal{V}X_{-\alpha_{l'},(1)}(T\cap K)=\mathcal{V}X_{-\alpha_{l'+1},(1)}(T\cap K)$, we have

$$\begin{aligned}\mathcal{U}_w &= \mathcal{V}w_{\alpha_{l'}}\cdot X_{-\alpha_{l'+1},(1)}\cdot N^-_{w',(1)}w'T^{*++}K^*\\ &\subset \mathcal{V}w_{\alpha_{l'}}\cdot N^-_{w',(1)}w'T^{*++}K^*.\end{aligned}$$

Hence we get, by using (5.10.2),

$$\mathcal{U}_w\subset K^{*\prime}\mathcal{N}w_{\alpha_{l'}}\cdot N^-_{w',(1)}w'T^{*++}K^*, \tag{5.10.4}$$

where we put $\mathcal{N}=X_{\varepsilon_1-\varepsilon_{l'+1},(0)}\cdots X_{\varepsilon_{l'-1}-\varepsilon_{l'+1},(0)}$ so that $\mathcal{V}=K'\mathcal{N}X_{\alpha_{l'},(0)}$. Now recall that we can decompose $w_{\alpha_{l'}}$ in the form $w_{\alpha_{l'}}=x_-x_+x_-$ (mod $T\cap K$), $x_\pm\in X_{\pm\alpha_{l'},(0)}$ from (3.5.1). Substituting this in (5.10.4), we have

$$\mathcal{U}_w\subset K^{*\prime}\mathcal{N}x_-x_+\cdot N^-_{w',(1)}w'T^{*++}K^*.$$

Note that there exists an $\bar{x}_-\in X_{-\alpha_{l'},(0)}$ such that

$$K^{*\prime}\mathcal{N}x_-=K^{*\prime}x_-\mathcal{N}=K^{*\prime}\bar{x}_-\mathcal{N}=K^{*\prime}\mathcal{N}\bar{x}_-.$$

Hence we finally see that

$$\begin{aligned}\mathcal{U}_w &\subset K^{*\prime}\mathcal{N}x_+\bar{x}_-\cdot N^-_{w',(1)}w'T^{*++}K^*\\ &\subset K^{*\prime}\mathcal{N}x_+\cdot N^-_{w',(1)}w'T^{*++}K^* \quad \text{(by (5.10.2))}\\ &\subset \mathcal{V}\cdot N^-_{w',(1)}w'T^{*++}K^*=\mathcal{U}_{w'}.\end{aligned}$$

- Case B-4: $\alpha=\alpha_{l'}=\varepsilon_{l'-1}+\varepsilon'_l$ and $(w')^{-1}\alpha_{l'-1}<0$

  We can show that $\mathcal{U}_w\subset\mathcal{U}_{w'}$ exactly in the same way as in the case B-2.
- Case B-5: $\alpha=\alpha_{l'}=\varepsilon_{l'-1}+\varepsilon'_l$ and $(w')^{-1}\alpha_{l'-1}>0$

  We can show that $\mathcal{U}_w\subset\mathcal{U}_{w'}$ exactly in the same way as in the case B-3.

Combining all of these, we have completed the proof of Proposition 5.8. □

5.11. Proof of Theorem 5.5. For $g_1,\ g_2\in G^*$, let us write $g_1\sim g_2$ if $g_1=k'g_2k$ for some $k\in K^*,\ k'\in K^{*\prime}$. Then, by 5.9, proof of Theorem 5.5 (except the disjointness) is reduced to the following lemma. Recall that $g_{m,0}=g_{m,0}(\mathbf{1})$.

Lemma 5.12. *For any $\mathbf{y}\in o^{l'}$ and $\nu\in\Lambda_m^{*++}$, there exist $\lambda\in\Lambda_m^{*++}$ and $\lambda'\in\Lambda_{m-1}^{*++}$ such that*

$$g_{m,0}(\mathbf{y})t(\nu)\sim t(\lambda')g_{m,0}(\mathbf{1})t(\lambda). \tag{5.12.1}$$

Proof. We prove the lemma in the case where $m=2l'+1$ is odd. The proof in the even case is almost similar and is omitted. Recall that

$$N'_{(0)}g_{m,0}(\mathbf{y})=N'_{(0)}x_{\varepsilon_i}(y_1)\cdots x_{\varepsilon_{l'}}(y_{l'})$$

for $\mathbf{y}={}^t(y_1,\ldots,y_{l'})$. We may assume that $\mathbf{y}={}^t(\pi^{\mu_1},\ldots,\pi^{\mu_{l'}}),\ \mu_1,\ldots,\mu_{l'}\geq 0$.

Suppose first that $\mu_1 \geq \cdots \geq \mu_i$ and $\mu_i < \mu_{i+1}$ for some i, $1 \leq i \leq l'-1$. Then, by commutator relations, we have

$$\begin{aligned} & x_{\varepsilon_{i+1}-\varepsilon_i}(1-\pi^{\mu_{i+1}-\mu_i})g_{m,0}(\mathbf{y})t(\nu) \\ & \qquad \in N'_{(0)}g_{m,0}(\mathbf{y}_1)t(\nu)x_{\varepsilon_{i+1}-\varepsilon_i}(\pi^{\nu_i-\nu_{i+1}}(1-\pi^{\mu_{i+1}-\mu_i}))\,, \end{aligned}$$

where $\mathbf{y}_1$ is the element of $o^{l'}$ obtained by substituting the $(i+1)$-st entry of $\mathbf{y}$ by π^{μ_i}, that is, $\mathbf{y}_1 = {}^t(\pi^{\mu_1}, \dots, \overset{i}{\pi^{\mu_i}}, \overset{i+1}{\pi^{\mu_i}}, \dots, \pi^{\mu_{l'}})$. Therefore $g_{m,0}(\mathbf{y})t(\nu) \sim g_{m,0}(\mathbf{y}_1)t(\nu)$, which implies that we can assume $\mu_1 \geq \cdots \geq \mu_{l'}$. Next, we shall show that we may assume $\nu_{l'} - \mu_{l'} \geq 0$. Actually, if $\nu_{l'} - \mu_{l'} < 0$, we have

$$g_{m,0}(\mathbf{y})t(\nu)x_{\varepsilon_{l'}}(1-\pi^{\mu_{l'}-\nu_{l'}}) \in N'_{(0)}g_{m,0}(\mathbf{y}_2)t(\nu)$$

with $\mathbf{y}_2 = {}^t(\pi^{\mu_1}, \dots, \pi^{\mu_{l'-1}}, \pi^{\nu_{l'}})$. Hence $g_{m,0}(\mathbf{y})t(\nu) \sim g_{m,0}(\mathbf{y}_2)t(\nu)$. Now suppose that

$$\nu_i - \mu_i < \nu_{i+1} - \mu_{i+1}\,, \quad \nu_{i+1} - \mu_{i+1} \geq \cdots \geq \nu_{l'} - \mu_{l'}$$

for some i, $1 \leq i \leq l'-1$. Then

$$\begin{aligned} & g_{m,0}(\mathbf{y})t(\nu)x_{\varepsilon_i-\varepsilon_{i+1}}(-1+\pi^{\mu_i-\mu_{i+1}-\nu_i+\nu_{i+1}}) \\ & \qquad \in N'_{(0)}x_{\varepsilon_i-\varepsilon_{i+1}}(\pi^{\mu_i-\mu_{i+1}} - \pi^{\nu_i-\nu_{i+1}})g_{m,0}(\mathbf{y}_3)t(\nu)\,, \end{aligned}$$

where $\mathbf{y}_3$ is the element of $o^{l'}$ obtained by substituting the i-th entry of $\mathbf{y}$ by $\pi^{\mu_{i+1}+\nu_i-\nu_{i+1}}$, that is, $\mathbf{y}_3 = {}^t(\pi^{\mu_1}, \dots, \overset{i}{\pi^{\mu_{i+1}+\nu_i-\nu_{i+1}}}, \dots, \pi^{\mu_{l'}})$. Therefore, if we put

$$\lambda'_i = \mu_{i+1} + \nu_i - \nu_{i+1}\,, \quad \lambda'_{i+1} = \mu_{i+1}, \dots, \lambda'_{l'} = \mu_{l'}\,,$$

we have

$$\nu_i - \lambda'_i = \nu_{i+1} - \lambda'_{i+1} \geq \cdots \geq \nu_{l'} - \lambda'_{l'} \geq 0$$

and

$$g_{m,0}(\mathbf{y})t(\nu) \sim g_{m,0}(\mathbf{y}_4)t(\nu)$$

with $\mathbf{y}_4 = {}^t(\pi^{\mu_1}, \dots, \pi^{\mu_{i-1}}, \pi^{\lambda'_i}, \dots, \pi^{\lambda'_{l'}})$. Since $\mu_1 \geq \cdots \geq \mu_{i-1} \geq \lambda'_i \geq \cdots \geq \lambda'_{l'} \geq 0$ from $\mu_i > \lambda'_i \geq \mu_{i+1}$, we have

$$\lambda'_1 \geq \cdots \geq \lambda'_{l'} \geq 0\,, \quad \nu_1 - \lambda'_1 \geq \cdots \geq \nu_{l'} - \lambda'_{l'} \geq 0$$

by repeating this argument. Thus we finally get

$$g_{m,0}(\mathbf{y})t(\nu) \sim g_{m,0}(\mathbf{y}^*)t(\nu) = t'(\lambda')g_{m,0}(\mathbf{1})t(\lambda)\,,$$

where $\mathbf{y}^* = {}^t(\pi^{\lambda'_1}, \dots, \pi^{\lambda'_{l'}})$ and $\lambda_1 = \nu_1 - \lambda'_1, \dots, \lambda_{l'} = \nu_{l'} - \lambda'_{l'}$. □

5.13. PROOF OF THEOREM 5.2. Now we shall give a proof of Theorem 5.2 by using its variant for $\mathbf{O}_m$, Theorem 5.5.

Suppose that $g \in G$ is decomposed as $g = k't'(\lambda')g_{m,0}t(\lambda)k$ for $\lambda \in \Lambda_m^{*++}$, $\lambda' \in \Lambda_{m-1}^{*++}$, $k \in K^*$, $k' \in K^*$ with $\det k = \det k' = -1$. (We have nothing to do for the case $\det k = \det k' = 1$.)

We first handle the case where m is odd ($m = 2l' + 1$). Set

$$s_{\mathrm{odd}} = \begin{pmatrix} 1_{l'-1} & & & & \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ & & & & 1_{l'-1} \end{pmatrix} \in K^*_{m-1} \subset K^*_m .$$

This s_{odd} corresponds to $\gamma_{m-1} \in W^*_{m-1}$ so that we have $s_{\mathrm{odd}} t'(\lambda') s_{\mathrm{odd}}^{-1} = t(\gamma_{m-1}(\lambda'))$ and $\gamma_{m-1}(\lambda') \in \Lambda^{++}_{m-1}$. Then s_{odd} is written as

$$s_{\mathrm{odd}} = x_{-\varepsilon_{l'}}(-1) x_{\varepsilon_{l'}}(1) x_{-\varepsilon_{l'}}(-1) h ,$$

where

$$h = \begin{pmatrix} 1_{l'-1} & & \\ & -1_3 & \\ & & 1_{l'-1} \end{pmatrix} \in T^* \cap K^* \quad (\det h = -1 = \det s_{\mathrm{odd}}) .$$

Since $g_{m,0} = u x_{\varepsilon_1}(1) \cdots x_{\varepsilon_{l'}}(1)$ for some $u \in N'_{(0)}$, we have

$$\begin{aligned} s_{\mathrm{odd}} g_{m,0} &= (s_{\mathrm{odd}} u s_{\mathrm{odd}}^{-1}) x_{\varepsilon_1}(1) \cdots x_{\varepsilon_{l'-1}}(1) x_{-\varepsilon_{l'}}(1) s_{\mathrm{odd}} \\ &= (s_{\mathrm{odd}} u s_{\mathrm{odd}}^{-1}) x_{\varepsilon_1}(1) \cdots x_{\varepsilon_{l'-1}}(1) x_{\varepsilon_{l'}}(1) x_{-\varepsilon_{l'}}(-1) h . \end{aligned}$$

Note that $s_{\mathrm{odd}} u s_{\mathrm{odd}}^{-1} \in N'_{(0)}$. Thus we have

$$\begin{aligned} g &= (k' s_{\mathrm{odd}})(s_{\mathrm{odd}} t'(\lambda') s_{\mathrm{odd}}^{-1}) s_{\mathrm{odd}} g_{m,0} t(\lambda) k \\ &= (k' s_{\mathrm{odd}}) t'(\gamma_{m-1}(\lambda')) x_{\varepsilon_1}(1) \cdots x_{\varepsilon_{l'-1}}(1) x_{\varepsilon_{l'}}(1) x_{-\varepsilon_{l'}}(-1) t(\lambda)(hk) \\ &= (k' s_{\mathrm{odd}}) t'(\gamma_{m-1}(\lambda')) g_{m,0} t(\lambda)(t(\lambda)^{-1} x_{-\varepsilon_{l'}}(-1) t(\lambda) hk) . \end{aligned}$$

Since $\det(k' s_{\mathrm{odd}}) = \det(t(\lambda)^{-1} x_{-\varepsilon_{l'}}(-1) t(\lambda) hk) = 1$, we see that $k' s_{\mathrm{odd}} \in K'$ and that $t(\lambda)^{-1} x_{-\varepsilon_{l'}}(-1) t(\lambda) hk \in K$. Therefore we are done in this case.

Now we shall consider the remaining even case $m = 2l' + 2$. Set

$$s_{\mathrm{even}} = \begin{pmatrix} 1_{l'} & & & \\ & & -1 & \\ & -1 & & \\ & & & 1_{l'} \end{pmatrix} \in K^*_{m-1} \subset K^*_m .$$

This $s_{\mathrm{even}} \in K^*$ corresponds to $\gamma_m \in W^*_m$. Then we see that, since $s_{\mathrm{even}} t'(\lambda') s_{\mathrm{even}}^{-1} = t'(\lambda')$ and $k' s_{\mathrm{even}} \in K'$, $g = (k' s_{\mathrm{even}}) t'(\lambda') s_{\mathrm{even}} g_{m,0} t(\lambda) k$ is contained in

$$\begin{aligned} & K' t'(\lambda') x_{\varepsilon_1 + \varepsilon_{l'+1}}(-1) \cdots x_{\varepsilon_{l'} + \varepsilon_{l'+1}}(-1)(s_{\mathrm{even}} t(\lambda) s_{\mathrm{even}}^{-1})(s_{\mathrm{even}} k) \\ &\quad = K' t'(\lambda') x_{\varepsilon_1 - \varepsilon_{l'+1}}(-1) \cdots x_{\varepsilon_{l'} - \varepsilon_{l'+1}}(-1) t(\gamma_m(\lambda))(s_{\mathrm{even}} k) \quad \text{(by (3.11.3))} \\ &\quad \subset K' t'(\lambda') g_{m,0} t(\gamma_m(\lambda)) K. \end{aligned}$$

Thus we have completed the proof of Theorem 5.2 (except the disjointness of the decomposition). $\square$

6. Support of Whittaker-Shintani functions. The following theorem gives the support of Whittaker-Shintani functions.

THEOREM 6.1. *For $F \in WS(\Xi, \xi)$,*

$$\mathrm{supp}\ F \subset \bigsqcup UK't'(\lambda')g_{m,r}t(\lambda)K\,,$$

where λ runs over Λ_m^+ and λ' over $\Lambda_{m'}^+$.

PROOF. In what follows, we shall give a proof of this theorem in the odd case. The proof in the even case is similar and is omitted. Recall the decomposition 5.1,

$$G = \bigsqcup UK't'(\lambda')g_{m,r}t(\lambda)K\,,$$

where λ runs over $\mathbf{Z}^r \times \Lambda_{m-2r}^+ \subset \Lambda_m$ and λ' over $\Lambda_{m'}^+$. We shall show that $F(t'(\lambda')g_{m,r}t(\lambda)) = 0$ unless $\lambda_1 \geq \ldots \geq \lambda_r \geq \lambda_{r+1}$ for $\lambda = (\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_l) \in \mathbf{Z}^r \times \Lambda_{m-2r}^+$. Let $\alpha = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq r-1)$. Then, for $u \in o$,

$$\begin{aligned} F(t'(\lambda')g_{m,r}t(\lambda)) &= F(t'(\lambda')g_{m,r}t(\lambda)x_\alpha(u)) \\ &= F(x_\alpha(\pi^{\lambda_i-\lambda_{i+1}}u)t'(\lambda')g_{m,r}t(\lambda)) \\ &= \psi(\pi^{\lambda_i-\lambda_{i+1}}u)F(t'(\lambda')g_{m,r}t(\lambda))\,. \end{aligned}$$

Since the conductor of ψ is o, this implies that $F(t'(\lambda')g_{m,r}t(\lambda)) = 0$ if $\lambda_i < \lambda_{i+1}$. Next, let $\alpha = \varepsilon_r - \varepsilon_{r+1}$. We note that

$$x_\alpha(u) = \nu_{m,r}(x_u, 0)$$

for $x_u = (\mathbf{u}, \mathbf{0}, \ldots, \mathbf{0}) \in \mathbf{Mat}_{m-2r,r}(k)$ with $\mathbf{u} = {}^t(0, \ldots, 0, u) \in k^{2l'+1}$ $(m-2r = 2l'+1)$. Then we have, by a direct calculation,

$$\begin{aligned} t'(\lambda')g_{m,r}t(\lambda)x_\alpha(u) &= t'(\lambda')g_{m,r}x_\alpha(\pi^{\lambda_r-\lambda_{r+1}}u)t(\lambda) \\ &= t'(\lambda')\nu_{m,r}(\pi^{\lambda_r-\lambda_{r+1}}x'_u, 0)g_{m,r}t(\lambda) \\ &= \nu_{m,r}(\pi^{\lambda_r-\lambda_{r+1}}\mathbf{d}_{2l'+1}(\pi^{\lambda'_1}, \ldots, \pi^{\lambda'_{l'}})x'_u, 0)t'(\lambda')g_{m,r}t(\lambda)\,, \end{aligned}$$

where $x'_u = (\mathbf{u}', \mathbf{0}, \ldots, \mathbf{0}) \in \mathbf{Mat}_{m-2r,r}(k)$ with $\mathbf{u}' = {}^t(-u, \ldots, \overset{l'+1}{-u}, 0, \ldots, 0, u) \in k^{2l'+1}$. Therefore, as in the first case, the definition of the character ψ_U of U shows that

$$F(t'(\lambda')g_{m,r}t(\lambda)) = F(t'(\lambda')g_{m,r}t(\lambda)x_\alpha(u)) = \psi(-\pi^{\lambda_r-\lambda_{r+1}}u)F(t'(\lambda')g_{m,r}t(\lambda))$$

for $u \in o$. This implies that $F(t'(\lambda')g_{m,r}t(\lambda)) = 0$ if $\lambda_r < \lambda_{r+1}$. □

7. Multiplicity one. In this section, we shall prove the following theorem that shows the multiplicity one property of Whittaker-Shintani functions.

THEOREM 7.1. *Suppose that $F \in WS(\Xi, \xi)$ for $(\Xi, \xi) \in X \times X'$. Then $F = 0$ if $F(1) = 0$. In particular,* $\dim_{\mathbf{C}} WS(\Xi, \xi) \leq 1$ *for any $(\Xi, \xi) \in X \times X'$.*

To prove this theorem, we shall study closely the double cosets in Theorem 5.1. For the purpose, we introduce a partial order "$\geq_{WS}$" on the set $\Lambda_m \times \Lambda'_m$.

DEFINITION 7.2. For any $(\mu, \mu'), (\lambda, \lambda') \in \Lambda_m \times \Lambda_{m'}$, we write $(\mu, \mu') \geq_{WS} (\lambda, \lambda')$ if the following conditions hold:

$$\mu_i = \lambda_i \qquad (1 \leq i \leq r)\,,$$
$$\sum_{s=1}^{j} \mu_{r+s} + \sum_{t=1}^{j} \mu'_t \geq \sum_{s=1}^{j} \lambda_{r+s} + \sum_{t=1}^{j} \lambda'_t \qquad (1 \leq j \leq l')\,,$$
$$\sum_{s=1}^{j} \mu_{r+s} + \sum_{t=1}^{j-1} \mu'_t \geq \sum_{s=1}^{j} \lambda_{r+s} + \sum_{t=1}^{j-1} \lambda'_t \qquad \begin{pmatrix} 1 \leq j \leq l' \text{ in the odd case} \\ 1 \leq j \leq l'+1 \text{ in the even case} \end{pmatrix}.$$

We can rewrite the conditions in 7.2 above by using dominant weights $\varpi_i,\ \varpi'_j$ in 3.14 as follows:

$$\langle \varpi_i, \mu \rangle = \langle \varpi_i, \lambda \rangle \qquad (1 \leq i \leq r)\,;$$
$$\langle \varpi_{r+j}, \mu \rangle + \langle \varpi'_j, \mu' \rangle \geq \langle \varpi_{r+j}, \lambda \rangle + \langle \varpi'_j, \lambda' \rangle \qquad (1 \leq j \leq l')\,;$$
$$\langle \varpi_{r+j}, \mu \rangle + \langle \varpi'_{j-1}, \mu' \rangle \geq \langle \varpi_{r+j}, \lambda \rangle + \langle \varpi'_{j-1}, \lambda' \rangle \qquad \begin{pmatrix} 1 \leq j \leq l' \text{ in the odd case} \\ 1 \leq j \leq l'+1 \text{ in the even case} \end{pmatrix}.$$

Incidentally, we recall the usual order "$\geq$" on the character group Hom($\mathbf{T}$, $\mathbf{GL}_1$); $\sigma \geq \tau$ ($\sigma, \tau \in$ Hom($\mathbf{T}$, $\mathbf{GL}_1$)) if $\sigma - \tau$ is a linear combination of positive roots with nonnegative coefficients. (We denote the corresponding order on Hom($\mathbf{T}'$, $\mathbf{GL}_1$) by the same symbol "$\geq$".)

Now we can state the following theorem. (Compare with [BT; (4.4.4) (i), (ii)].)

THEOREM 7.3. *Suppose $\mu \in \Lambda_m^+$ and $\mu' \in \Lambda_{m'}^+$.*

(1) *If $\lambda \in \Lambda_m$ and $\lambda' \in \Lambda_{m'}$ satisfy the condition*

$$K't'(\mu')Kt(\mu)^{-1}K \cap UK't'(\lambda')g_{m,r}w_\ell t(\lambda)^{-1}K \neq \emptyset\,,$$

then $(\mu, \mu') \geq_{WS} (\lambda, \lambda')$.

(2) *If $u \in U$ satisfies the condition*

$$K't'(\mu')Kt(\mu)^{-1}K \cap uK't'(\mu')g_{m,r}w_\ell t(\mu)^{-1}K \neq \emptyset\,,$$

then $\psi_U(u) = 1$.

PROOF. (1) By the assumption, the element $g = t'(\lambda')g_{m,r}w_\ell t(\lambda)^{-1}$ is written as $g = uk't'(\mu')k_1 t(\mu)^{-1}k$ for some $u \in U, k, k_1 \in K, k' \in K'$. Let $f = \alpha_{r+j}$ $(1 \leq j \leq l')$ (see 3.14). Then $f \in k[\mathbf{G}]$ is a highest weight vector under the right $\mathbf{G}$-action with highest weight ϖ_j (resp. highest weight vector under the left $\mathbf{G}'$-action with highest weight ϖ'_j). Since

$$\begin{aligned} f(t'(\lambda')g_{m,r}w_\ell t(\lambda)^{-1}) &= \varpi_{r+j}(t(\lambda))^{-1}\varpi'_j(t'(\lambda'))^{-1}f(g_{m,r}w_\ell) \\ &= \varpi_{r+j}(t(\lambda))^{-1}\varpi'_j(t'(\lambda'))^{-1}\,, \end{aligned}$$

we have

$$v(f(g)) = -\langle \varpi_{r+j}, \lambda \rangle - \langle \varpi'_j, \lambda' \rangle\,.$$

On the other hand, we know that $f \in o[\mathbf{G}]$, the coordinate ring of $\mathbf{G}$ over o. Since $o[\mathbf{G}]$ is a Hopf algebra, we have

$$f(uk't'(\mu')k_1t(\mu)^{-1}k) = \sum f_{(1)}(k')f_{(2)}(t'(\mu')k_1t(\mu)^{-1})f_{(3)}(k)$$

by using the comultiplication in $o[\mathbf{G}]$. Here we may assume that all $f_{(2)} \in o[\mathbf{G}]$ above are weight vectors under both the left $\mathbf{G}'$ and the right $\mathbf{G}$-actions so that

$$f_{(2)}(t'(\mu')k_1t(\mu)^{-1}) = \sigma'(t'(\mu'))^{-1}\sigma(t(\mu))^{-1}f_{(2)}(k_1)$$

for some $\sigma \in \mathrm{Hom}(\mathbf{T}, \mathbf{GL}_1)$ and $\sigma' \in \mathrm{Hom}(\mathbf{T}', \mathbf{GL}_1)$. Note that $\sigma \leq \varpi_{r+j}$ and $\sigma' \leq \varpi'_j$. Therefore

$$\begin{aligned} v(f(g)) &= v\Big(\sum f_{(1)}(k')f_{(2)}(t'(\mu')k_1t(\mu)^{-1})f_{(3)}(k)\Big) \\ &\geq \inf_{k\in K, k'\in K'}(v(f_{(1)}(k')f_{(2)}(t'(\mu')k_1t(\mu)^{-1})f_{(3)}(k))) \\ &\geq \inf_{\sigma,\sigma'}(-\langle\sigma,\mu\rangle - \langle\sigma',\mu'\rangle) \\ &\geq -\langle\varpi_{r+j},\mu\rangle - \langle\varpi'_j,\mu'\rangle\,. \end{aligned}$$

This shows that

$$\langle\varpi_{r+j},\lambda\rangle + \langle\varpi'_j,\lambda'\rangle \leq \langle\varpi_{r+j},\mu\rangle + \langle\varpi'_j,\mu'\rangle\,.$$

Similarly, if we apply the same argument for $f = \beta_j$ ($1 \leq j \leq l'$ in the odd case and $1 \leq j \leq l'+1$ in the even case), then we have

$$\langle\varpi_{r+j},\lambda\rangle + \langle\varpi'_{j-1},\lambda'\rangle \leq \langle\varpi_{r+j},\mu\rangle + \langle\varpi'_{j-1},\mu'\rangle\,,$$

since $\beta_j \in o[\mathbf{G}]$ is a highest weight vector with highest weight ϖ_{r+j} under the right $\mathbf{G}$-action (resp. that with highest weight ϖ'_{j-1} under the left $\mathbf{G}'$-action). Here we put $\varpi'_0 = 0$. Also, in the case where $f = \alpha_i$ ($1 \leq i \leq r$), we have

$$\langle\varpi_i,\lambda\rangle = \langle\varpi_i,\mu\rangle\,.$$

These prove (1).

(2) It is sufficient to show that

$$t'(\mu')kt(\mu) = uk't'(\mu')g_{m,r}t(\mu)k_1 \ \ (k, k_1 \in K, k' \in K', u \in U) \Longrightarrow \psi_U(u) = 1\,.$$

We shall prove this by induction on r. If $r = 0$, the group U is trivial so that there is nothing to prove. We shall assume that $r \geq 1$. Set $g = t'(\mu')kt(\mu)$. Then for $(0, \ldots, 0, 1) \in {}^t(o^m)$,

$$(0, \ldots, 0, 1)g = (0, \ldots, 0, 1)kt(\mu) = {}^t\mathbf{k}_{[m]}t(\mu)\,.$$

Here ${}^t\mathbf{k}_{[i]}$ is the i-th row of the $m \times m$ matrix k. On the other hand, the expression $g = uk't'(\mu')g_{m,r}t(\mu)k_1$ shows that

$$(0, \ldots, 0, 1)g = (0, \ldots, 0, 1)t(\mu)k_1 = \pi^{-\mu_1}({}^t\mathbf{k}_{1,[m]})\,.$$

Therefore the vector ${}^t\mathbf{v} = \pi^{\mu_1}({}^t\mathbf{k}_{[m]})t(\mu)$ $(= {}^t\mathbf{k}_{1,[m]})$ is primitive, i.e., $\mathbf{v} \in o^m$ and $\mathbf{v} \pmod{\pi} \neq 0$. Suppose that $\mu = (\mu_1, \ldots, \mu_a, \mu_{a+1}, \ldots, \mu_l)$ satisfies the condition $\mu_1 = \cdots = \mu_a > \mu_{a+1} \geq \cdots \geq \mu_l$. If we put ${}^t\mathbf{v} = (v_1, \ldots, v_m)$, we see that at least one of

$v_{m-a+1}, \ldots, v_m$ (say, v_{m-i+1}) is in $o^\times$. Let w be an element of W transposing 1 and i. Then the (m,m)-coefficient of kw^{-1} is in $o^\times$. Let us set

$$\begin{aligned}
S^\dagger &= S_{m-2}\,;\\
\mathbf{G}^\dagger &= \mathbf{SO}(S^\dagger)\,, \quad K^\dagger = \mathbf{G}^\dagger(o) = G^\dagger \cap \mathbf{GL}_{m-2}(o)\,;\\
n_{\mathbf{x}} &= \nu_{m,1}(\mathbf{x},0) = \begin{pmatrix} 1 & -{}^t\mathbf{x}S^\dagger & -\frac{1}{2}S^\dagger[\mathbf{x}] \\ 0 & 1_{m-2} & \mathbf{x} \\ 0 & 0 & 1 \end{pmatrix} \quad (\mathbf{x} \in o^{m-2})\,;\\
\overline{n_{\mathbf{y}}} &= w_\ell(n_{\mathbf{y}}) \quad (\mathbf{y} \in o^{m-2})\,.
\end{aligned}$$

The Bruhat decomposition of $K \pmod{\pi}$ implies that kw^{-1} is written as

$$kw^{-1} = n_{\mathbf{x}} \begin{pmatrix} \epsilon & & \\ & k^\dagger & \\ & & \epsilon^{-1} \end{pmatrix} \overline{n_{\mathbf{y}}}$$

for some $k^\dagger \in K^\dagger$, $\mathbf{x}, \mathbf{y} \in o^{m-2}$, and $\epsilon \in o^\times$. Hence we have

$$\begin{aligned}
t'(\mu')kt(\mu) &= t'(\mu')n_{\mathbf{x}} \begin{pmatrix} \epsilon & & \\ & k^\dagger & \\ & & \epsilon^{-1} \end{pmatrix} \overline{n_{\mathbf{y}}}wt(\mu)\\
&= (t'(\mu')n_{\mathbf{x}}t'(\mu')^{-1})t'(\mu') \begin{pmatrix} \epsilon & & \\ & k^\dagger & \\ & & \epsilon^{-1} \end{pmatrix} t(\mu)(t(\mu)^{-1}\overline{n_{\mathbf{y}}}wt(\mu))\,.
\end{aligned}$$

Here $t'(\mu')n_{\mathbf{x}}t'(\mu')^{-1} \in U$ and $t(\mu)^{-1}\overline{n_{\mathbf{y}}}wt(\mu) \in K$, since w commutes with $t(\mu)$. On the other hand, we have $\psi_U(t'(\mu')n_{\mathbf{x}}t'(\mu')^{-1}) = 1$. (Recall the definition of the character ψ_U.) Set $\mu^\dagger = (\mu_2, \ldots, \mu_{r+n}) \in \Lambda^+_{m-2}$. Then, from the decomposition

$$t'(\mu')kt(\mu) = uk't'(\mu')g_{m,r}t(\mu)k_1\,,$$

we see that

$$t'(\mu')k^\dagger t(\mu^\dagger) = u^\dagger k_1' t'(\mu')g_{m,r}t(\mu^\dagger)k_1^\dagger$$

for some $k_1' \in K'$, $k_1^\dagger \in K^\dagger$, $u^\dagger \in U^\dagger = G^\dagger \cap U$ with $(t'(\mu')n_{\mathbf{x}}t'(\mu')^{-1})u^\dagger = u$. Note that the induction hypothesis implies that $\psi_{U^\dagger}(u^\dagger) = 1$. (Here $\psi_{U^\dagger}$ is the counterpart of ψ_U for $U^\dagger$.) Therefore we finally have

$$\psi_U(u) = \psi_U((t'(\mu')n_{\mathbf{x}}t'(\mu')^{-1})u^\dagger) = 1\,. \qquad \square$$

7.4. PROOF OF THE DISJOINTNESS OF THE DECOMPOSITION IN 5.1. The proof of 7.3 above and the decomposition $G = \bigcup UK't'(\lambda')g_{m,r}t(\lambda)K$ given in Section 5 show that $g \in UK't'(\lambda')g_{m,r}t(\lambda)K$ if and only if the minimum values

$$\min_{\substack{k' \in K' \\ k \in K}} v(f(k'gk))$$

for the relative invariants f defined in 3.14 are given by

$$\begin{array}{lll} -\langle \varpi_i, \lambda\rangle & \text{for } f = \alpha_i & (1 \le i \le r)\,, \\ -\langle \varpi_{r+j}, \lambda\rangle - \langle \varpi'_j, \lambda'\rangle & \text{for } f = \alpha_{r+j} & (1 \le j \le l')\,, \end{array}$$

and

$$-\langle \varpi_{r+j}, \lambda\rangle - \langle \varpi'_{j-1}, \lambda'\rangle \quad \text{for } f = \beta_j \quad \left(\begin{array}{l} 1 \le j \le l' \text{ in the odd case} \\ 1 \le j \le l'+1 \text{ in the even case} \end{array} \right).$$

This implies the disjointness of the decomposition. □

REMARK 7.5. The above approach using relative invariants (see also the proof of Theorem 7.3 (1)) to the study of double cosets is effective for general spherical homogeneous spaces. Details will appear elsewhere.

7.6. PROOF OF THEOREM 7.1. Now we can prove a "multiplicity one" result for Whittaker-Shintani functions.

Let us put $F(\mu, \mu') = F(t'(\mu')g_{m,r}t(\mu))$ for $\mu \in \Lambda_m^+$ and $\mu' \in \Lambda_{m'}^+$. By Sections 5 and 6 and the definition of Whittaker-Shintani functions (Section 4), we have only to show that $F(0,0) = 0$ implies that $F(\mu, \mu') = 0$ for any (μ, μ'). (Note that $F(0,0) = F(g_{m,r}) = F(1)$, since $g_{m,r} \in K$.)

Let $\mathrm{ch}_{Kt(\mu)K}$ and $\mathrm{ch}_{K't'(\mu')^{-1}K'}$ be the characteristic functions of $Kt(\mu)K$ and $K't'(\mu')^{-1}K'$, respectively. We then have

$$\begin{aligned} \int_{K't'(\mu')Kt(\mu)K} F(x)\,dx &= (L(\mathrm{ch}_{K't'(\mu')^{-1}K'})R(\mathrm{ch}_{Kt(\mu)K})F)(1) \\ &= \omega_\Xi(\mathrm{ch}_{Kt(\mu)K})\omega_\xi(\mathrm{ch}_{K't'(\mu')^{-1}K'})F(1)\,. \end{aligned}$$

Therefore, if we write

$$K't'(\mu')Kt(\mu)K = \bigsqcup_{i=1}^{a} u_{(i)}K't(\lambda'_{(i)})g_{m,r}t(\lambda_{(i)})K$$
$$(u_{(i)} \in U, \lambda_{(i)} \in \mathbf{Z}^r \times \Lambda_{m-2r}^+, \lambda'_{(i)} \in \Lambda_{m'}^+)$$

according to the decomposition in Section 5, we have a system of difference equations on $F(\lambda, \lambda')$ $((\lambda, \lambda') \in \Lambda_m \times \Lambda_{m'})$,

$$\omega_\Xi(\mathrm{ch}_{Kt(\mu)K})\omega_\xi(\mathrm{ch}_{K't'(\mu')^{-1}K'})F(0,0) = \sum_{\substack{\lambda \in \Lambda_m \\ \lambda' \in \Lambda_{m'}}} c_{\lambda,\lambda'}F(\lambda, \lambda') \tag{7.6.1}$$

for every $(\mu, \mu') \in \Lambda_m^+ \times \Lambda_{m'}^+$, where

$$c_{\lambda,\lambda'} = \mathrm{vol}(K't(\lambda')g_{m,r}t(\lambda)K) \sum_{i \text{ with } (\lambda_{(i)},\lambda'_{(i)})=(\lambda,\lambda')} \psi_U(u_{(i)})\,.$$

Now 7.3 (2) shows that $c_{\mu,\mu'}$ above is positive and hence is non-zero. On the other hand, by 7.3 (1), $F(\lambda, \lambda') \neq 0$ only when $(\mu, \mu') \geq_{WS} (\lambda, \lambda')$. Thus we can see that the solution to (7.6.1) is uniquely determined by the initial value $F(0,0)$ and that, especially, $F = 0$ if $F(0,0) = 0$. □

REMARK 7.7. The system of difference equations employed here is similar to those appeared in [Sh1], [K1] (see also [MS1], [MS3]). This argument implies that each value of the Whittaker-Shintani function $F(\mu, \mu')$ of F with $F(1) = 1$ is, if it exists, regular in (Ξ, ξ).

8. Rank 1 calculation. In this section, we shall evaluate some integrals related to simple roots in $G \times G'$. These results are essential in our later use for the determination of an explicit formula of Whittaker-Shintani functions.

8.1. Let us denote by $\{\Phi_w\ (w \in W)\}$ $(\Phi_w = \mathcal{P}_\Xi(\mathrm{ch}_{BwB}))$ and $\{\phi_{w'}\ (w' \in W')\}$ $(\phi_{w'} = \mathcal{P}_\xi(\mathrm{ch}_{B'w'B'}))$ the natural bases of $I(\Xi)^B$ and $I(\xi)^{B'}$ arising from Bruhat decompositions $K = BWB$ and $K' = B'W'B'$ (see 1.10), respectively.

We shall evaluate the values

$$I_\alpha := \mathrm{vol}(B)^{-1}\mathrm{vol}(B')^{-1}\Omega(\phi_1, R(g_{m,r}w_\ell)(\Phi_1 + \Phi_{w_\alpha})) \quad (\alpha \in \Delta)$$

and

$$J_\beta := \mathrm{vol}(B)^{-1}\mathrm{vol}(B')^{-1}\Omega(\phi_1 + \phi_{w_\beta}, R(g_{m,r}w_\ell)\Phi_1) \quad (\beta \in \Delta')\,.$$

Here $\Omega = \Omega_{\Xi,\xi} : I(\xi, \psi_U) \times I(\Xi) \to \boldsymbol{C}$ is a bilinear form introduced in Section 4, given by

$$\Omega(\mathcal{P}_\xi(f'), \mathcal{P}_\Xi(f)) = \int_{G'\times G} f'(x')f(x)Y(xx'^{-1})dx'dx$$

for $f' \in I(\xi, \psi_U)$, $f \in I(\Xi)$. We recall that $Y = Y_{\Xi,\xi}$ is a distribution on G satisfying

$$Y(tnw_\ell g_{m,r}t'n'u) = (\Xi^{-1}\delta^{1/2})(t)(\xi\delta'^{-1/2})(t')\psi_U(u)$$

for $t \in T, n \in N, t' \in T', n' \in N'$ and $u \in U$. Throughout this section, we assume that the parameter (Ξ, ξ) belongs to Z_c so that $Y_{\Xi,\xi}$ is actually a continuous function on G (see 4.9).

LEMMA 8.2. *The following inclusions hold*:

$$N_{(1)}g_{m,r} \subset T_{(0)}g_{m,r}T'_{(0)}N'_{(1)}U_{(1)}\,, \tag{8.2.1}$$

$$N^-_{(1)}w_\ell g_{m,r} \subset T_{(0)}w_\ell g_{m,r}T'_{(0)}N'_{(1)}U_{(1)}\,, \tag{8.2.2}$$

$$w_\ell g_{m,r}N'_{(1)} \subset N^-_{(1)}w_\ell g_{m,r}\,. \tag{8.2.3}$$

PROOF. By the commutation relations, there exists $\mathbf{y} \in (o^\times)^n \subset o^n$ such that

$$ng_{m,r} \in g_{m,r}(\mathbf{y})N'_{(1)}U_{(1)}$$

for $n \in N_{(1)}$. Therefore $g_{m,r}(\mathbf{y}) \in T_{(0)}g_{m,r}T'_{(0)}$ shows (8.2.1). (8.2.2) is a consequence of (8.2.1). As for (8.2.3), since $n \equiv 1 \pmod{\pi}$ for any $n \in N'_{(1)}$, $w_\ell g_{m,r}N'_{(1)}g_{m,r}^{-1}w_\ell^{-1} \subset N^-_{(1)}$. □

LEMMA 8.3. *One has*

$$\mathrm{vol}(B)^{-1}\mathrm{vol}(B')^{-1}\Omega(\phi_1, R(g_{m,r}w_\ell)\Phi_1) = 1\,.$$

PROOF. Since

$$R(g_{m,r}w_\ell)\Phi_1 = \mathcal{P}_\Xi(R(g_{m,r}w_\ell)\mathrm{ch}_B) = \mathcal{P}_\Xi(\mathrm{ch}_{B(g_{m,r}w_\ell)^{-1}})\,,$$

we have

$$\begin{aligned}
&\mathrm{vol}(B)^{-1}\mathrm{vol}(B')^{-1}\Omega(\phi_1, R(g_{m,r}w_\ell)\Phi_1)\\
&\quad = \mathrm{vol}(B)^{-1}\mathrm{vol}(B')^{-1}\int_{B'\times B} Y(x(g_{m,r}w_\ell)^{-1}x')dx'dx\\
&\quad = \mathrm{vol}(B)^{-1}\mathrm{vol}(B')^{-1}\int_{B'\times B} Y(xw_\ell g_{m,r}x')dx'dx\,.
\end{aligned}$$

Note that $g_{m,r}^{-1} \in T_{(0)}g_{m,r}T'_{(0)}N'_{(0)}$ in the above. By Lemma 8.2, we have

$$\begin{aligned}
Bw_\ell g_{m,r}B' &= Bw_\ell g_{m,r}N'^{-}_{(1)}N'_{(0)}T'_{(0)} = Bw_\ell g_{m,r}N'_{(0)}T'_{(0)}\\
&\subset T_{(0)}N_{(0)}w_\ell g_{m,r}T'_{(0)}N'_{(0)}U_{(1)} = P_{(0)}w_\ell g_{m,r}P_{H\,(0)}\,.
\end{aligned} \tag{8.3.1}$$

This implies that

$$Y(xw_\ell g_{m,r}x') = 1 \quad (x \in B, x' \in B')\,. \qquad \square$$

LEMMA 8.4. *For $\alpha \in \Delta$, with the normalized Haar measure dt of o,*

$$I_\alpha = 1 + q\int_o (\Xi\delta^{-1/2})(a_\alpha^{v(t)})Y(w_\ell x_{-w_\ell\alpha}(t^{-1})g_{m,r})dt\,.$$

PROOF. As in Lemma 8.3, we have

$$\begin{aligned}
&\mathrm{vol}(B)^{-1}\mathrm{vol}(B')^{-1}\Omega(\phi_1, R(g_{m,r}w_\ell)\Phi_{w_\alpha})\\
&\quad = \mathrm{vol}(B)^{-1}\mathrm{vol}(B')^{-1}\int_{B'\times Bw_\alpha B} Y(x(g_{m,r}w_\ell)^{-1}x')dx'dx\\
&\quad = \mathrm{vol}(B)^{-1}\mathrm{vol}(B')^{-1}\int_{Bw_\alpha B} Y(xw_\ell g_{m,r})dx\,.
\end{aligned}$$

In view of the decomposition

$$Bw_\alpha B = T_{(0)}N_{(0)}w_\alpha X_{\alpha,(0)}N^{-}_{(1)}$$

and the fact $\mathrm{vol}(Bw_\alpha B) = q\cdot\mathrm{vol}(B)$, we see that

$$\int_{Bw_\alpha B} Y(xw_\ell g_{m,r})dx = q\cdot\mathrm{vol}(B)\int_o Y(w_\alpha x_\alpha(t)w_\ell g_{m,r})dt$$

by using 8.2 again. Recall the formula

$$x_\alpha(t) = x_{-\alpha}(t^{-1})w_\alpha a_\alpha^{-v(t)}hx_{-\alpha}(t^{-1})$$

with some element h of $T_{(0)}$ ($t \neq 0$), see (3.5.1). Since

$$Y(w_\alpha x_\alpha(t)w_\ell g_{m,r}) = (\Xi^{-1}\delta^{1/2})(a_\alpha^{-v(t)})Y(w_\ell x_{-w_\ell\alpha}(t^{-1})g_{m,r})\,,$$

we have the lemma. $\square$

Similarly, we have the following lemma.

LEMMA 8.5. *For $\beta \in \Delta'$,*

$$J_\beta = 1 + q \int_o (\xi \delta'^{-1/2})(a'^{\,v(t)}_\beta) Y(w_\ell g_{m,r} x'_{-\beta}(t^{-1}))dt \,.$$

We shall give without proof the following elementary lemma which is useful in our calculation.

LEMMA 8.6. *Let χ, $\chi' \in X_{nr}(k^\times)$ be two unramified characters of $k^\times$. If $|\chi|$, $|\chi'| < q$, then*

$$1 + q \int_o \chi(t)\chi'(1+t)dt = (q-1)\frac{1 - q^{-2}\chi\chi'}{(1-q^{-1}\chi)(1-q^{-1}\chi')} \,,$$

where χ and $\chi(\pi) \in \boldsymbol{C}^\times$ (resp. χ' and $\chi'(\pi) \in \boldsymbol{C}^\times$) are identified as in 3.4.

By virtue of (4.8.1) and (4.8.2), we can apply this lemma to the calculation given below.

8.7. THE EVALUATION IN THE ODD CASE. Now we shall evaluate I_α ($\alpha \in \Delta$) and J_β ($\beta \in \Delta'$) in the odd case first. Namely, we handle the case $G' = \mathbf{SO}_{2l'}(k) \subset G = \mathbf{SO}_{2l'+2r+1}(k)$. In this case, the double coset $NTw_\ell g_{m,r}N'T'U$ is open dense in G. We note here that $g_{m,r}N' = x_{\varepsilon_{r+1}}(1)\cdots x_{\varepsilon_{r+l'}}(1)N'$. Note also that $-w_\ell\alpha = \alpha$ for any $\alpha \in \Delta$.

PROPOSITION 8.8. *For $\alpha = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le r)$,*

$$I_\alpha = q(1 - q^{-1}\Xi_i \Xi_{i+1}^{-1}) \,.$$

PROOF. We have

$$(\Xi^{-1}\delta^{1/2})(a_\alpha^{-v(t)}) = (\Xi_i\Xi_{i+1}^{-1}|\cdot|^{-1})(t)$$

for $a_\alpha = d_i(\pi)d_{i+1}(\pi)^{-1}$. Consider first the case where $1 \le i \le r-1$. We see $x_\alpha(t^{-1})g_{m,r} = g_{m,r}x_\alpha(t^{-1})$ so that

$$Y(w_\ell x_\alpha(t^{-1})g_{m,r}) = Y(w_\ell g_{m,r} x_\alpha(t^{-1})) = \psi(t^{-1}) \,.$$

On the other hand, in the case $i = r$,

$$x_{\varepsilon_r - \varepsilon_{r+1}}(t^{-1})g_{m,r} = g_{m,r}x_{\varepsilon_r-\varepsilon_{r+1}}(t^{-1})x_{\varepsilon_r}(t^{-1})x_{\varepsilon_r+\varepsilon_{r+1}}(-t^{-1}) \,.$$

Since the support of the character ψ_U is on $\varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le r-1)$ and ε_r (see Section 3), we have

$$\begin{aligned} Y(w_\ell x_{-w_\ell\alpha}(t^{-1})g_{m,r}) &= Y(w_\ell g_{m,r}x_{\varepsilon_r-\varepsilon_{r+1}}(t^{-1})x_{\varepsilon_r}(t^{-1})x_{\varepsilon_r+\varepsilon_{r+1}}(-t^{-1})) \\ &= \psi(t^{-1}) \end{aligned}$$

also in this case. By 8.4, we see that

$$\begin{aligned} I_\alpha &= 1 + q\int_o (\Xi_i\Xi_{i+1}^{-1}|\cdot|^{-1})(t)Y(w_\ell x_{-w_\ell\alpha}(t^{-1})g_{m,r})dt \\ &= 1 + q\sum_{k=0}^{\infty}(1-q^{-1})q^{-k}(\Xi_i\Xi_{i+1}^{-1})^k q^k \int_{o^\times}\psi(\pi^{-k}u)du \,, \end{aligned}$$

where du is the normalized Haar measure on $o^{\times}$. This completes the proof of our proposition, since

$$\int_{o^{\times}} \psi(\pi^{-k}u)du = \begin{cases} 1 & (k=0)\,, \\ -1/(q-1) & (k=1)\,, \\ 0 & (k>1)\,. \end{cases} \qquad \square$$

PROPOSITION 8.9. *For* $\alpha = \varepsilon_{r+i} - \varepsilon_{r+i+1}$ $(1 \le i \le l'-1)$,

$$I_\alpha = (q-1)\frac{1-q^{-1}\Xi_{r+i}\Xi_{r+i+1}^{-1}}{(1-q^{-1/2}\xi_i^{-1}\Xi_{r+i})(1-q^{-1/2}\xi_i\Xi_{r+i+1}^{-1})}\,.$$

PROOF. By 8.4,

$$I_\alpha = 1 + q\int_o (\Xi_{r+i}\Xi_{r+i+1}^{-1}|\cdot|^{-1})(t)Y(w_\ell x_\alpha(t^{-1})g_{m,r})dt\,,$$

since $a_\alpha = d_{r+i}(\pi)d_{r+i+1}(\pi)^{-1}$. The commutator relation shows that

$$\begin{aligned} x_\alpha(t^{-1})g_{m,r} &= x_{\varepsilon_{r+i}-\varepsilon_{r+i+1}}(t^{-1})g_{m,r} \\ &\in x_{\varepsilon_{r+1}}(1)\cdots x_{\varepsilon_{r+i-1}}(1)x_{\varepsilon_{r+i}}(1+t^{-1})x_{\varepsilon_{r+i+1}}(1)\cdots x_{\varepsilon_{r+n}}(1)N' \\ &= d_{r+i}(1+t^{-1})g_{m,r}d_{r+i}(1+t^{-1})^{-1}N'. \end{aligned}$$

This implies that

$$\begin{aligned} Y(w_\ell x_\alpha(t^{-1})g_{m,r}) &= (\Xi^{-1}\delta^{1/2})(d_{r+i}(1+t^{-1})^{-1})(\xi\delta'^{-1/2})(d_{r+i}(1+t^{-1})^{-1})Y(w_\ell g_{m,r}) \\ &= (\xi_i^{-1}\Xi_{r+i}|\cdot|^{-1/2})(1+t^{-1})\,. \end{aligned}$$

Thus, by 8.6, we finally have

$$\begin{aligned} I_\alpha &= 1 + q\int_o (\xi_i\Xi_{r+i+1}^{-1}|\cdot|^{-1/2})(t)(\xi_i^{-1}\Xi_{r+i}|\cdot|^{-1/2})(1+t)dt \\ &= (q-1)\frac{1-q^{-1}\Xi_{r+i}\Xi_{r+i+1}^{-1}}{(1-q^{-1/2}\xi_i^{-1}\Xi_{r+i})(1-q^{-1/2}\xi_i\Xi_{r+i+1}^{-1})}\,. \end{aligned} \qquad \square$$

PROPOSITION 8.10. *For* $\alpha = \varepsilon_{r+l'}$,

$$I_\alpha = (q-1)\frac{1-q^{-1}\Xi_{r+l'}^2}{(1-q^{-1/2}\xi_{l'}\Xi_{r+l'})(1-q^{-1/2}\xi_{l'}^{-1}\Xi_{r+l'})}\,.$$

PROOF. The evaluation is similar to that of 8.9. Since $a_\alpha = d_{r+l'}(\pi)^2$, we have

$$(\Xi^{-1}\delta^{1/2})(a_\alpha^{-v(t)}) = (\Xi_{r+l'}^2|\cdot|^{-1})(t)\,.$$

On the other hand,

$$\begin{aligned} x_\alpha(t^{-1})g_{m,r} &\in x_{\varepsilon_{r+1}}(1)\cdots x_{\varepsilon_{r+l'-1}}(1)x_{\varepsilon_{r+l'}}(1+t^{-1})N' \\ &= d_{r+l'}(1+t^{-1})g_{m,r}d_{r+l'}(1+t^{-1})^{-1}N'\,. \end{aligned}$$

Hence we have

$$\begin{aligned} Y(w_\ell x_\alpha(t^{-1})g_{m,r}) &= (\varXi^{-1}\delta^{1/2})(d_{r+l'}(1+t^{-1})^{-1})(\xi\delta'^{-1/2})(d_{r+l'}(1+t^{-1})^{-1})Y(w_\ell g_{m,r}) \\ &= (\xi_{l'}^{-1}\varXi_{r+l'}|\cdot|^{-1/2})(1+t^{-1}) \end{aligned}$$

and, by 8.6,

$$\begin{aligned} I_\alpha &= 1+q\int_o (\xi_{l'}\varXi_{r+l'}|\cdot|^{-1/2})(t)(\xi_{l'}^{-1}\varXi_{r+l'}|\cdot|^{-1/2})(1+t)dt \\ &= (q-1)\frac{1-q^{-1}\varXi_{r+l'}^2}{(1-q^{-1/2}\xi_{l'}\varXi_{r+l'})(1-q^{-1/2}\xi_{l'}^{-1}\varXi_{r+l'})}\,. \end{aligned}$$

□

PROPOSITION 8.11. *For* $\beta=\varepsilon'_i-\varepsilon'_{i+1}$ $(1\le i\le l'-1)$,

$$J_\beta=(q-1)\frac{1-q^{-1}\xi_i\xi_{i+1}^{-1}}{(1-q^{-1/2}\xi_i\varXi_{r+i+1}^{-1})(1-q^{-1/2}\xi_{i+1}^{-1}\varXi_{r+i+1})}\,.$$

PROOF. In this case, $a'_\beta=d_{r+i}(\pi)d_{r+i+1}(\pi)^{-1}$. Note that

$$g_{m,r}x_{-\beta}(t^{-1})=g_{m,r}x_{-\varepsilon_{r+i}+\varepsilon_{r+i+1}}(t^{-1})$$

is contained in

$$\begin{aligned} &x_{\varepsilon_{r+1}}(1)\cdots x_{\varepsilon_{r+l'}}(1)x_{-\varepsilon_{r+i}+\varepsilon_{r+i+1}}(t^{-1})N' \\ &\quad = x_{-\varepsilon_{r+i}+\varepsilon_{r+i+1}}(t^{-1})x_{\varepsilon_{r+1}}(1)\cdots x_{\varepsilon_{r+i-1}}(1)x_{\varepsilon_{r+i}}(1+t^{-1})x_{\varepsilon_{r+i+1}}(1)\cdots x_{\varepsilon_{r+l'}}(1)N' \\ &\quad = x_{-\beta}(t^{-1})d_{r+i}(1+t^{-1})g_{m,r}d_{r+i}(1+t^{-1})^{-1}N' \end{aligned}$$

(see 3.11). We have

$$\begin{aligned} J_\beta &= 1+q\int_o (\xi\delta'^{-1/2})(a_{\beta'}^{v(t)})(\varXi^{-1}\delta^{1/2})(d_{r+i}(1+t^{-1})^{-1})(\xi\delta'^{-1/2})(d_{r+i}(1+t^{-1})^{-1})dt \\ &= 1+q\int_o (\xi_i\varXi_{r+i+1}^{-1}|\cdot|^{-1/2})(t)(\xi_{i+1}^{-1}\varXi_{r+i+1}|\cdot|^{-1/2})(1+t)dt \\ &= (q-1)\frac{1-q^{-1}\xi_i\xi_{i+1}^{-1}}{(1-q^{-1/2}\xi_i\varXi_{r+i+1}^{-1})(1-q^{-1/2}\xi_{i+1}^{-1}\varXi_{r+i+1})}\,. \end{aligned}$$

□

PROPOSITION 8.12. *For* $\beta=\varepsilon'_{l'-1}+\varepsilon'_{l'}$,

$$J_\beta=(q-1)\frac{1-q^{-1}\xi_{l'-1}\xi_{l'}}{(1-q^{-1/2}\xi_{l'-1}\varXi_{r+l'})(1-q^{-1/2}\xi_{l'}\varXi_{r+l'}^{-1})}\,.$$

PROOF. In order to calculate J_β in this case, we consider $g_{m,r}x_{-\beta}(t^{-1})$ explicitly by using matrix form. It is sufficient to handle only the case $r = 0$. Set $s = t^{-1}$. Since

$$g_{m,r} = \left(\begin{array}{c|c|c} 1_{l'} & 2\mathbf{1} & -\mathbf{1}\,{}^t\mathbf{1} \\ \hline & 1 & -{}^t\mathbf{1} \\ \hline & & 1_{l'} \end{array}\right)$$

and

$$x_{-\beta}(s) = x_{-\varepsilon_{l'-1}-\varepsilon_{l'}}(s) = \left(\begin{array}{ccc|c|ccc} 1_{l'-2} & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ \hline & & & 1 & & & \\ \hline & s & & & 1 & & \\ & & -s & & & 1 & \\ & & & & & & 1_{l'-2} \end{array}\right),$$

we have

$$g_{m,r}x_{-\beta}(s) = \left(\begin{array}{ccc|cc|c|ccc} & & & -s & s & & & & \\ & 1_{l'-2} & & \vdots & \vdots & & & & \\ & & & -s & s & 2\mathbf{1} & & -\mathbf{1}\,{}^t\mathbf{1} & \\ 0 & \cdots & 0 & 1-s & s & & & & \\ 0 & \cdots & 0 & -s & 1+s & & & & \\ \hline & & & -s & s & 1 & & -{}^t\mathbf{1} & \\ \hline & & & s & 0 & & 1 & & \\ & & & 0 & -s & & & 1 & \\ & & & & & & & & 1_{l'-2} \end{array}\right)$$

$$= x_{-\beta}(s) \left(\begin{array}{ccc|c|ccc} 1_{l'-2} & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ \hline & -s & s & 1 & & & \\ \hline & s^2 & -s^2 & -2s & 1 & & \\ & -s^2 & s^2 & 2s & & 1 & \\ & & & & & & 1_{l'-2} \end{array}\right) \mathbf{d}(A)g_{m,r}\,,$$

where $\mathbf{d}(A) = \begin{pmatrix} A & & \\ & 1 & \\ & & J_{l'}{}^tA^{-1}J_{l'} \end{pmatrix}$ with

$$A = \left(\begin{array}{c|cc} & -s & s \\ 1_{l'-2} & \vdots & \vdots \\ & -s & s \\ \hline & 1-s & s \\ & -s & 1+s \end{array}\right) \in \mathbf{GL}_{l'}(k)\,.$$

Set

$$B = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & \frac{-s}{1-s} & 1 \end{pmatrix} \in \mathbf{GL}_{l'}(k)$$

and

$$C = \begin{pmatrix} 1 & & & -s & s \\ & \ddots & & \vdots & \vdots \\ & & 1 & -s & s \\ & & & 1-s & s \\ & & & & \frac{1}{1-s} \end{pmatrix} \in \mathbf{GL}_{l'}(k)\,.$$

Then $A = BC$ so that $\mathbf{d}(A) = \mathbf{d}(B)\mathbf{d}(C)$. Hence

$$\begin{aligned} \mathbf{d}(A)g_{m,r} &= \mathbf{d}(B)\mathbf{d}(C)g_{m,r} \\ &= \mathbf{d}(B)\left(\begin{array}{c|c|c} 1_{l'} & 2C\mathbf{1} & -C\mathbf{1}^t(C\mathbf{1})J_{l'} \\ \hline & 1 & -{}^t(C\mathbf{1})J_{l'} \\ \hline & & 1_{l'} \end{array}\right)\mathbf{d}(C) \\ &\in N^- x_{\varepsilon_1}(1)\cdots x_{\varepsilon_{l'-1}}(1)x_{\varepsilon_{l'}}((1-s)^{-1})d_{r+l'-1}(1-s)d_{r+l'}((1-s)^{-1})N'\,. \end{aligned}$$

Therefore we see that

$$g_{m,r}x_{-\beta}(t\ -1) \in N^- d_{r+l'}((1-t^{-1})^{-1})g_{m,r}d_{r+l'-1}(1-t^{-1})N'\,.$$

Since $a_\beta = d_{r+l'-1}(\pi)d_{r+l'}(\pi)$, we finally have

$$\begin{aligned} J_\beta &= 1 + q\int_o (\xi\delta'^{-1/2})(a_\beta'^{\,v(t)})Y(w_\ell g_{m,r}x'_{-\beta}(t^{-1}))dt \\ &= 1 + q\int_o (\xi_{l'}\Xi_{r+l'}^{-1}|\cdot|^{-1/2})(t)(\xi_{l'-1}\Xi_{r+l'}|\cdot|^{-1/2})(1-t)dt \\ &= (q-1)\frac{1-q^{-1}\xi_{l'-1}\xi_{l'}}{(1-q^{-1/2}\xi_{l'-1}\Xi_{r+l'})(1-q^{-1/2}\xi_{l'}\Xi_{r+l'}^{-1})}\,. \end{aligned}$$ □

This completes the evaluation in the odd case.

8.13. THE EVALUATION IN THE EVEN CASE. We evaluate I_α ($\alpha \in \Delta$) and J_β ($\beta \in \Delta'$) in the even case where $G' = \mathbf{SO}_{2l'+1}(k) \subset G = \mathbf{SO}_{2l'+2r+2}(k)$. In this case, $g_{m,r} = x_{\varepsilon_{r+1}-\varepsilon_{r+l'+1}}(1)\cdots x_{\varepsilon_{r+l'}-\varepsilon_{r+l'+1}}(1)$ so that $NTw_\ell g_{m,r}N'T'U$ is open dense in G.

Let γ be the outer automorphism of G which arises from the non-trivial graph automorphism of Δ . This γ is given by the conjugation by

$$\begin{pmatrix} 1_{r+l'} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1_{r+l'} \end{pmatrix},$$

which induces the substitution $\varepsilon_i \to \varepsilon_i$ $(i \neq l'+r+1)$, $\varepsilon_{r+l'+1} \leftrightarrow -\varepsilon_{r+l'+1}$ on $\mathrm{Hom}(\mathbf{T}, \mathbf{GL}_1)$. Note that $\gamma(g_{m,r}) = x_{\varepsilon_{r+1}+\varepsilon_{r+l'+1}}(1)\cdots x_{\varepsilon_{r+l'}+\varepsilon_{r+l'+1}}(1)$ and $\gamma(g_{m,r}) \in g_{m,r}N'$. The subgroups $N,\ T,\ G', N',\ T',\ U$ are invariant under γ. This implies that the open dense subset $NTw_\ell g_{m,r}N'T'U$ is also γ-invariant. Since γ naturally acts on X as $\Xi_i \leftrightarrow \Xi_i$ $(i \neq r+l'+1)$, $\Xi_{r+l'+1} \leftrightarrow \Xi_{r+l'+1}^{-1}$, we see that

$$Y_{\Xi,\xi}(\gamma(g)) = Y_{\gamma(\Xi),\xi}(g) \quad (g \in G)\,.$$

Note also that

$$-w_\ell\alpha = \begin{cases} \alpha & \text{if } r+l'+1 \text{ is even,} \\ \gamma(\alpha) & \text{if } r+l'+1 \text{ is odd} \end{cases}$$

for any $\alpha \in \Delta$. Hence we have

$$-w_\ell\varepsilon_i = \begin{cases} \varepsilon_i & (i \neq r+l'+1)\,, \\ \pm\varepsilon_{r+l'+1} & (i = r+l'+1). \end{cases}$$

PROPOSITION 8.14. *For* $\alpha = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq r)$,

$$I_\alpha = q(1 - q^{-1}\Xi_i\Xi_{i+1}^{-1})\,.$$

PROOF. The calculation of I_α in this case is similar to that of 8.8. Note that the support of the character ψ_U is on $\varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq r-1)$, $\varepsilon_r - \varepsilon_{r+l'+1}$ and $\varepsilon_r + \varepsilon_{r+l'+1}$ (see Section 3). If $1 \leq i \leq r-1$, $x_\alpha(t^{-1})g_{m,r} = g_{m,r}x_\alpha(t^{-1})$ so that

$$Y(w_\ell x_\alpha(t^{-1})g_{m,r}) = \psi(t^{-1})\,.$$

On the other hand, the equality

$$x_{\varepsilon_r-\varepsilon_{r+1}}(t^{-1})g_{m,r} = g_{m,r}x_{\varepsilon_r-\varepsilon_{r+1}}(t^{-1})x_{\varepsilon_r-\varepsilon_{r+l'+1}}(t^{-1})$$

shows that

$$Y(w_\ell x_{-w_\ell\alpha}(t^{-1})g_{m,r}) = \psi(t^{-1})$$

also in this case. Hence exactly as in 8.8, we are done. □

PROPOSITION 8.15. *For* $\alpha = \varepsilon_{r+i} - \varepsilon_{r+i+1}$ $(1 \le i \le l')$,

$$I_\alpha = (q-1)\frac{1 - q^{-1}\Xi_{r+i}\Xi_{r+i+1}^{-1}}{(1 - q^{-1/2}\xi_i^{-1}\Xi_{r+i})(1 - q^{-1/2}\xi_i\Xi_{r+i+1}^{-1})}\,.$$

PROPOSITION 8.16. *For* $\alpha = \varepsilon_{r+l'} + \varepsilon_{r+l'+1}$,

$$I_\alpha = (q-1)\frac{1 - q^{-1}\Xi_{r+l'}\Xi_{r+l'+1}}{(1 - q^{-1/2}\xi_{l'}^{-1}\Xi_{r+l'})(1 - q^{-1/2}\xi_{l'}\Xi_{r+l'+1})}\,.$$

PROOF OF 8.15 AND 8.16 First we note that 8.15 for $\alpha = \varepsilon_{r+l'} - \varepsilon_{r+l'+1}$ and 8.16 are equivalent via γ (see 8.13). Hence it suffices to calculate

$$I_\alpha = 1 + q\int_o (\Xi\delta^{-1/2})(a_\alpha^{v(t)})Y(w_\ell x_{-w_\ell\alpha}(t^{-1})g_{m,r})dt$$

for $-w_\ell\alpha = \varepsilon_{r+i} - \varepsilon_{r+i+1}$ $(1 \le i \le l')$. Note that

$$\alpha = -w_\ell(\varepsilon_{r+i} - \varepsilon_{r+i+1}) = \varepsilon_{r+i} - e\varepsilon_{r+i+1}$$

for some $e = \pm 1$. (We remark that $e = 1$ when $i < l'$.) From 3.11, we see that

$$x_{\varepsilon_{r+i}-\varepsilon_{r+i+1}}(t^{-1})g_{m,r} = \begin{cases} x_{\varepsilon_{r+1}-\varepsilon_{r+l'+1}}(1)\cdots x_{\varepsilon_{r+i-1}-\varepsilon_{r+l'+1}}(1) & \\ \quad\times x_{\varepsilon_{r+i}-\varepsilon_{r+l'+1}}(1+t^{-1})x_{\varepsilon_{r+i+1}-\varepsilon_{r+l'+1}}(1)\cdots & \text{if } i < l', \\ \quad\times x_{\varepsilon_{r+l'}-\varepsilon_{r+l'+1}}(1)x_{\varepsilon_{r+i}-\varepsilon_{r+i+1}}(t^{-1}) & \\ x_{\varepsilon_{r+1}-\varepsilon_{r+l'+1}}(1)\cdots x_{\varepsilon_{r+l'-1}-\varepsilon_{r+l'+1}}(1) & \text{if } i = l'. \\ \quad\times x_{\varepsilon_{r+l'}-\varepsilon_{r+l'+1}}(1+t^{-1}) & \end{cases}$$

Therefore we have

$$x_{\varepsilon_{r+i}-\varepsilon_{r+i+1}}(t^{-1})g_{m,r} \in d_{r+i}(1+t^{-1})g_{m,r}d_{r+i}(1+t^{-1})^{-1}N'$$

in either case. Hence we obtain

$$\begin{aligned}&(\Xi\delta^{-1/2})(a_\alpha{}^{v(t)})Y(w_\ell x_\alpha(t^{-1})g_{m,r})\\ &\quad= (\Xi_{r+i}\Xi_{r+i+1}^{-e}|\cdot|^{-1})(t)(\Xi_{r+i}|\cdot|^{-l'+1-i})(1+t^{-1})(\xi_i^{-1}|\cdot|^{l'-i-(1/2)})(1+t^{-1})\\ &\quad= (\xi_i\Xi_{r+i+1}^{-e}|\cdot|^{-1/2})(t)(\xi_i^{-1}\Xi_{r+i}|\cdot|^{-1/2})(1+t)\,,\end{aligned}$$

which yields

$$I_\alpha = (q-1)\frac{1 - q^{-1}\Xi_{r+i}\Xi_{r+i+1}^{-e}}{(1 - q^{-1/2}\xi_i^{-1}\Xi_{r+i})(1 - q^{-1/2}\xi_i\Xi_{r+i+1}^{-e})}$$

for $\alpha = \varepsilon_{r+i} - e\varepsilon_{r+i+1}$ $(1 \le i \le l')$ by 8.6. $\square$

PROPOSITION 8.17. For $\beta = \varepsilon'_i - \varepsilon'_{i+1}$ $(1 \le i \le l'-1)$,

$$J_\beta = (q-1)\frac{1 - q^{-1}\xi_i\xi_{i+1}^{-1}}{(1 - q^{-1/2}\xi_i\Xi_{r+i+1}^{-1})(1 - q^{-1/2}\xi_{i+1}^{-1}\Xi_{r+i+1})}\,.$$

PROOF. The calculation of J_β in this case is similar to that of 8.11 and is omitted. $\square$

PROPOSITION 8.18. *For* $\beta = \varepsilon'_{l'}$,

$$J_\beta = (q-1)\frac{1-q^{-1}\xi_{l'}^2}{(1-q^{-1/2}\xi_{l'}\Xi_{r+l'+1})(1-q^{-1/2}\xi_{l'}\Xi_{r+l'+1}^{-1})}\,.$$

PROOF. We handle $g_{m,r}x_{-\beta}(t^{-1})$ explicitly by using matrix form, as in 8.12. We may assume $r=0$. Set $s=t^{-1}$. Since

$$g_{m,r} = \left(\begin{array}{c|cc|c} 1_{l'} & \mathbf{1} & 0 & \\ \hline & 1 & & {}^t0 \\ & & 1 & -{}^t\mathbf{1} \\ \hline & & & 1_{l'} \end{array}\right)$$

and

$$x_{-\beta}(s) = x_{-\varepsilon_{l'}-\varepsilon_{l'+1}}(s)x_{-\varepsilon_{l'}-\varepsilon_{l'+1}}(s) = \left(\begin{array}{cc|cc|cc} 1_{l'-1} & & & & & \\ & 1 & & & & \\ \hline & s & 1 & & & \\ & s & & 1 & & \\ \hline & -s^2 & -s & -s & 1 & \\ & & & & & 1_{l'-1} \end{array}\right)$$

in $G = \mathbf{SO}_{2l'+2}(k)$,

$$g_{m,r}x_{-\beta}(s) = \left(\begin{array}{c|c|cc|ccc} & s & & & & & \\ 1_{l'-1} & \vdots & & & & & \\ & s & & & & & \\ \hline & 1+s & & & & & \\ \hline & s & 1 & & & & \\ & s+s^2 & s & 1+s & -1 & \cdots & -1 \\ \hline & -s^2 & -s & -s & & & \\ & & & & & 1_{l'} & \end{array}\right)$$

$$\begin{aligned} &= x_{-\varepsilon_{l'}+\varepsilon_{l'+1}}(-s^2/(1+s))x_{-\beta}(s)d_{l'+1}(1+s)^{-1}g_{m,r}d(A)^{-1} \\ &\in N^- d_{l'+1}(1+s)^{-1}g_{m,r}d_{l'}(1+s)^{-1}N', \end{aligned}$$

where $d(A) = \begin{pmatrix} A & \\ & J^t A^{-1} J \end{pmatrix}$ with

$$A = \left(\begin{array}{cc|c} & s & \\ 1_{l'-1} & \vdots & \\ & s & \\ & 1+s & \\ \hline & & 1 \end{array}\right) \in \mathbf{GL}_{l'+1}(k)\,.$$

This shows that

$$\begin{aligned}(\xi\delta'^{-1/2})&(a'^{v(t)}_{\beta})Y(w_\ell g_{m,r}x'_{-\beta}(t^{-1}))\\ &= (\xi\delta'^{-1/2})(a'^{v(t)}_{\beta})(\varXi\delta^{1/2})(d_{l'+1}(1+t^{-1})^e)(\xi\delta'^{-1/2})(d_{l'}(1+t^{-1})^{-1})\\ &= (\xi_{l'}\varXi^{-e}_{r+l'+1}|\cdot|^{-1/2})(t)(\xi_{l'}\varXi^{e}_{r+l'+1}|\cdot|^{-1/2})(1+t)\,.\end{aligned}$$

Here we put $w_\ell(\varepsilon_{l'+1}) = -e\varepsilon_{l'+1}$ for some $e = \pm 1$ as in the proof of 8.15 and 8.16. Therefore, using 8.6 as before, we have the proposition. □

9. Rationality. The purpose of this section is to show the rationality of the linear form $l_{\varXi,\xi}$ introduced in Section 4 with respect to the parameters $(\varXi, \xi)$.

We first show that Assumption 2.3 holds in our case (see 9.1, 9.2 below).

PROPOSITION 9.1. *For any* $(\varXi,\xi)$, $\dim \mathrm{Hom}_{P_H}(I(\varXi;\mathcal{O}_0), \xi^{-1}\delta'^{1/2} \otimes \psi_U) = 1$.

PROOF. This is obvious from 3.12 (3). □

PROPOSITION 9.2. *Let $\mathcal{O}$ be a $P \times P_H$-orbit in G different from $\mathcal{O}_0$. Then*

$$\dim \mathrm{Hom}_{P_H}(I(\varXi;\mathcal{O}), \xi^{-1}\delta'^{1/2} \otimes \psi_U) = 0$$

for generic $(\varXi,\xi)$.

PROOF. For $\mathcal{O} = PgP_H$, we have

$$\begin{aligned}\dim &\mathrm{Hom}_{P_H}(I(\varXi;\mathcal{O}), \xi^{-1}\delta_0^{1/2} \otimes \psi_U)\\ &= \dim \mathrm{Hom}_{P_H\cap g^{-1}Pg}(g^{-1}(\varXi\delta^{1/2}) \otimes (\xi\delta_0^{-1/2}) \otimes \psi_U^{-1}, \delta_g)\,,\end{aligned}$$

where δ_g is the modulus character of $P_H \cap g^{-1}Pg$ (see 2.2). Hence we must show that

$$g^{-1}(\varXi\delta^{1/2})|_{P_H\cap g^{-1}Pg} \cdot ((\xi\delta_0^{-1/2}) \otimes \psi_U^{-1})|_{P_H\cap g^{-1}Pg} \cdot \delta_g^{-1} \not\equiv 1$$

on $P_H \cap g^{-1}Pg$ for generic $(\varXi,\xi)$. To do this, it is sufficient to see that we can choose a representative g of the $P \times P_H$-orbit $\mathcal{O} = PgP_H$ such that

(a) $T' \cap g^{-1}Tg$ contains a non-trivial torus;

or

(b) $\psi_U|_{N_H\cap g^{-1}Ng} \not\equiv 1$.

Here $N_H = N'U$ is the unipotent radical of P_H. Let $\mathcal{O} = Pwg_{m,r}(\mathbf{y})P_H$ $(w \in W,\ \mathbf{y} \in \{0,1\}^{l'})$ be a $P \times P_H$-orbit in G. Let us put

$$g^*_{m,r}(\mathbf{y}) = \begin{cases} x_{\varepsilon_{r+1}}(y_1)\cdots x_{\varepsilon_{r+l'}}(y_{l'}) & \text{in the odd case}\,,\\ x_{\varepsilon_{r+1}-\varepsilon_{r+l'+1}}(y_1)\cdots x_{\varepsilon_{r+l'}-\varepsilon_{r+l'+1}}(y_{l'}) = g_{m,r}(\mathbf{y}) & \text{in the even case}\,. \end{cases}$$

Since $g^*_{m,r}(\mathbf{y})N_H = g_{m,r}(\mathbf{y})N_H$, we may take $g = wg^*_{m,r}(\mathbf{y})$ as a representative of $\mathcal{O}$.

Suppose that $\mathcal{O} \neq \mathcal{O}_0$. Then either $w \neq w_\ell$ or $\mathbf{y} \neq \mathbf{1} = {}^t(1,\ldots,1)$ holds. We first consider the case where $\mathbf{y} \neq \mathbf{1}$ so that the i-th component of $\mathbf{y}$ is 0 for some i with $1 \le i \le l'$. In this case, we have

$$T' \cap g^{-1}Tg \supset \text{Image of } d_{r+i}\,.$$

Hence the condition (a) holds.

We next consider the case where $\mathbf{y} = \mathbf{1}$ and $w \neq w_\ell$. We put $g = wg^*_{m,r}(\mathbf{1})$. By the assumption, there exists a simple root α satisfying $w\alpha > 0$. Assume first that $\alpha = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le r-1)$. Then, since

$$Nwg^*_{m,r}(\mathbf{1}) = Nwg^*_{m,r}(\mathbf{1})x_{\varepsilon_i-\varepsilon_{i+1}}(t)\,,$$

we have

$$N_H \cap g^{-1}Ng \ni x_{\varepsilon_i-\varepsilon_{i+1}}(t)$$

for any $t \in k$, and we see that (b) holds. Next we assume that $\alpha = \varepsilon_r - \varepsilon_{r+1}$. In the odd case,

$$\begin{aligned} Nwg^*_{m,r}(\mathbf{1}) &= Nwx_{\varepsilon_r-\varepsilon_{r+1}}(t)g^*_{m,r}(\mathbf{1})\\ &= Nwg^*_{m,r}(\mathbf{1})x_{\varepsilon_r-\varepsilon_{r+1}}(t)x_{\varepsilon_r}(t)x_{\varepsilon_r+\varepsilon_{r+1}}(-t) \end{aligned}$$

for any $t \in k$. Hence

$$N_H \cap g^{-1}Ng \ni x_{\varepsilon_r-\varepsilon_{r+1}}(t)x_{\varepsilon_r}(t)x_{\varepsilon_r+\varepsilon_{r+1}}(-t)$$

for any $t \in k$, which implies that (b) holds. Similarly, in the even case,

$$\begin{aligned} Nwg^*_{m,r}(\mathbf{1}) &= Nwx_{\varepsilon_r-\varepsilon_{r+1}}(t)g^*_{m,r}(\mathbf{1})\\ &= Nwg^*_{m,r}(\mathbf{1})x_{\varepsilon_r-\varepsilon_{r+1}}(t)x_{\varepsilon_r-\varepsilon_{r+l'+1}}(t) \end{aligned}$$

for any $t \in k$. Hence

$$N_H \cap g^{-1}Ng \ni x_{\varepsilon_r-\varepsilon_{r+1}}(t)x_{\varepsilon_r-\varepsilon_{r+l'+1}}(t)$$

for any $t \in k$ so that (b) holds. When $\alpha = \varepsilon_i - \varepsilon_{i+1}$ $(r+1 \le i \le r+l'-1)$ in the odd case,

$$\begin{aligned} Pwg^*_{m,r}(\mathbf{1}) &= Pwx_{\varepsilon_i-\varepsilon_{i+1}}(-1)g^*_{m,r}(\mathbf{1})\\ &= Pwg^*_{m,r}(\mathbf{y})x_{\varepsilon_i-\varepsilon_{i+1}}(-1)x_{\varepsilon_i+\varepsilon_{i+1}}(1) \end{aligned}$$

with $\mathbf{y} = {}^t(1,\ldots,1,\overset{i}{0},1,\ldots,1)$ so that

$$Pwg^*_{m,r}(\mathbf{1})P_H = Pwg^*_{m,r}(\mathbf{y})P_H,$$

and (a) holds. We can handle the even case where $\alpha = \varepsilon_i - \varepsilon_{i+1}$ $(r+1 \leq i \leq r+l')$ in a similar manner. Finally, if $\alpha = \varepsilon_{r+l'}$ in the odd case, an argument similar to the above shows that

$$Pwg^*_{m,r}(\mathbf{1})P_H = Pwg^*_{m,r}(\mathbf{y})P_H$$

for $\mathbf{y} = {}^t(1, \dots, 1, 0)$, hence (a) holds. On the other hand, if $\alpha = \varepsilon_{r+l'} + \varepsilon_{r+l'+1}$ in the even case, since

$$x_{\varepsilon_{r+l'}-\varepsilon_{r+l'+1}}(1)x_{\varepsilon_{r+l'}+\varepsilon_{r+l'+1}}(1) = x_{\varepsilon'_{r+l'}}(1) \in N_H\,,$$

we have

$$\begin{aligned} Pwg^*_{m,r}(\mathbf{1})P_H &= Pwx_{\varepsilon_{r+l'}+\varepsilon_{r+l'+1}}(1)g^*_{m,r}(\mathbf{1})P_H \\ &= Pw(x_{\varepsilon_1-\varepsilon_{r+l'+1}}(1)x_{\varepsilon_1+\varepsilon_{r+l'+1}}(-1)) \times \cdots \\ &\quad \times (x_{\varepsilon_{l'-1}-\varepsilon_{r+l'+1}}(1)x_{\varepsilon_{l'-1}+\varepsilon_{r+l'+1}}(-1))x_{\varepsilon_{l'}-\varepsilon_{r+l'+1}}(1)x_{\varepsilon_{l'}+\varepsilon_{r+l'+1}}(1)P_H \\ &= Pwg^*_{m,r}(\mathbf{y})P_H \end{aligned}$$

for $\mathbf{y} = {}^t(1, \dots, 1, 0)$. Therefore (a) also holds in this case. □

Together with 2.4, this proposition shows the following generic multiplicity one result.

COROLLARY 9.3. *Let $\mathcal{V}$ be a $P \times P_H$-stable open subset of G. Then*

$$\dim \operatorname{Hom}_{P_H}(I(\Xi; \mathcal{V}), \xi^{-1}\delta'^{1/2} \otimes \psi_U) \leq 1$$

for generic (Ξ, ξ). In particular,

$$\dim \operatorname{Hom}_{P_H}(I(\Xi), \xi^{-1}\delta'^{1/2} \otimes \psi_U) \leq 1.$$

9.4. As in Section 4, we define $l_{\Xi,\xi} \in \operatorname{Hom}_{P_H}(I(\Xi), \xi^{-1}\delta'^{1/2} \otimes \psi_U)$ by

$$l_{\Xi,\xi}(\mathcal{P}_\Xi(f)) = \int_G f(g)Y(g)dg \quad (f \in C_c^\infty(G))$$

for $(\Xi, \xi) \in Z_c$. Here $Y = Y_{\Xi,\xi}$ is a continuous function on G defined to be

$$Y(g) = Y(pw_\ell g_{m,r}p_H) = (\Xi^{-1}\delta^{1/2})(p)(\xi\delta'^{-1/2} \otimes \psi_U)(p_H) \quad (p \in P,\ p_H \in P_H)$$

for $g = pw_\ell g_{m,r}p_H \in \mathcal{O}_0 \simeq P \times P_H$ and $Y(g) = 0$ for $g \notin \mathcal{O}_0$. Obviously, $l_{\Xi,\xi}|_{I(\Xi,\mathcal{O}_0)}$ is defined (and rational) for any (Ξ, ξ).

Now we proceed to the rationality argument. We shall show that the equivariant linear form $l_{\Xi,\xi}$ on $I(\Xi)$ defined above is rational in (Ξ, ξ). First we shall see that the assumption 2.7 holds.

PROPOSITION 9.5. *The restriction of $l_{\Xi,\xi}$ on $I(\Xi; Pw_\ell P)$ is rational in (Ξ, ξ).*

PROOF. For $\chi \in X_{nr}(k^\times)$, we first note that the function $\chi^\sim$ on k defined by

$$\chi^\sim(x) = \begin{cases} \chi(x) & (x \in k^\times)\,, \\ 0 & (x = 0) \end{cases}$$

can be viewed as a distribution on k with rational parameter $\chi = \chi(\pi) \in \boldsymbol{C}^\times$. Actually, the integral $I(\chi, f) = \int_k f(x)\chi^\sim(x)dx$ converges for any $f \in C_c^\infty(k)$ when $|\chi| < q$ and $(1 - q^{-1}\chi)I(\chi, f)$ is regular in χ . If $\mathbf{y} = {}^t(y_1, \ldots, y_{l'})$ with $y_i \neq 0$ for any i, we have $g_{m,r}(\mathbf{y}) = d(\mathbf{y})g_{m,r}d(\mathbf{y})^{-1}$ for some $d(\mathbf{y}) \in T' \subset T$. See (3.12.1) for an explicit form of $d(\mathbf{y})$. This shows that

$$\begin{aligned} Y(pw_\ell g_{m,r}(\mathbf{y})n_H) &= (\varXi^{-1}\delta^{1/2})(pw_\ell(d(\mathbf{y})))(\xi\delta'^{-1/2} \otimes \psi_U)(d(\mathbf{y})^{-1}n_H) \\ &= (\varXi^{-1}\delta^{1/2})(p)\psi_U(n_H)(w_\ell(\varXi^{-1}\delta^{1/2})\xi^{-1}\delta'^{1/2})(d(\mathbf{y})) \\ &= (\varXi^{-1}\delta^{1/2})(p)\psi_U(n_H)\prod_{i=1}^{l'}(\varXi_{r+i}\xi_i^{-1}|\cdot|^{-1/2})(y_i) \end{aligned}$$

for $p \in P,\ n_H \in N_H$. Since

$$\begin{aligned} \{pw_\ell n \in Pw_\ell N \mid n = g_{m,r}(\mathbf{y})n_H\ \ (y_1 \cdots y_{l'} \neq 0)\}\ (= \mathcal{O}_0) \\ \simeq P \times (k^\times)^{l'} \times N_H \end{aligned}$$

is an open dense subset of $Pw_\ell P \simeq P \times k^{l'} \times N_H$, the function $Y^\sim$ on $Pw_\ell P$ defined by

$$Y^\sim(pw_\ell g_{m,r}(\mathbf{y})n_H) = (\varXi^{-1}\delta^{1/2})(p)\psi_U(n_H)\prod_{i=1}^{l'}(\varXi_{r+i}\xi_i^{-1}|\cdot|^{-1/2})^\sim(y_i) \qquad (p \in P, \mathbf{y} \in k^{l'}, n_H \in N_H)$$

gives a linear form $l_{\varXi,\xi}$ on $I(\varXi;\, Pw_\ell P)$ if $(\varXi, \xi) \in Z_c$. Therefore $l_{\varXi,\xi}$ on $I(\varXi;\, Pw_\ell P)$ is rational in $(\varXi, \xi)$. □

REMARK 9.6. The above proof shows that $\prod_{i=1}^{l'}(1 - q^{1/2}\varXi_{r+1}\xi_i^{-1}) \cdot l_{\varXi,\xi}$ is regular in $(\varXi, \xi)$. Hence, together with the argument given below, we can evaluate the "denominator" of the linear form $l_{\varXi,\xi}$.

9.7. Proposition 9.5 above (see also Section 2) implies that we can extend $l_{\varXi,\xi}|_{I(\varXi;\, Pw_\ell P)}$ for generic $(\varXi, \xi)$. Then 9.3 shows that, for generic $(\varXi, \xi)$,

$$\operatorname{Hom}_{P_H}(I(\varXi;\, Pw_\ell P), \xi^{-1}\delta'^{1/2} \otimes \psi_U) = \boldsymbol{C} \cdot l_{\varXi,\xi}|_{I(\varXi;\, Pw_\ell P)}\,. \tag{9.7.1}$$

Let $T_{w_\alpha} = T_{w_\alpha, w_\alpha\varXi} : I(w_\alpha\varXi) \to I(\varXi)$ be the standard intertwining operator for $w_\alpha \in W,\ \alpha \in \varDelta$ (see Section 1). Consider generic $(\varXi, \xi) \in Z_c$. We know from 9.3 that the equivariant linear forms $T_{w_\alpha}^* l_{\varXi,\xi}|_{I(w_\alpha\varXi;\, Pw_\ell P)} = (l_{\varXi,\xi} \circ T_{w_\alpha})|_{I(w_\alpha\varXi;\, Pw_\ell P)}$ and $l_{w_\alpha\varXi,\xi}|_{I(w_\alpha\varXi;\, Pw_\ell P)}$ in $\operatorname{Hom}_{P_H}(I(w_\alpha\varXi;\, Pw_\ell P), \xi\delta'^{1/2} \otimes \psi_U)$ are proportional. Note that $l_{w_\alpha\varXi,\xi}|_{I(w_\alpha\varXi;\, Pw_\ell P)}$ is rational in the parameters thanks to 9.5.

The following result gives the explicit form of proportional constants which is crucial for our discussion on the rationality of $l_{\varXi,\xi}$ (Assumption 2.9) and the explicit formula of Whittaker-Shintani functions.

PROPOSITION 9.8. *Let $w_\alpha \in W$ be the simple reflection associated with $\alpha \in \Delta$. Then for generic $(\Xi, \xi) \in Z_c$, the constant $a(w_\alpha, \Xi, \xi)$ defined by*

$$T^*_{w_\alpha} l_{\Xi,\xi}|_{I(w_\alpha \Xi; P w_\ell P)} = a(w_\alpha, \Xi, \xi) l_{w_\alpha \Xi, \xi}|_{I(w_\alpha \Xi; P w_\ell P)} \tag{9.8.1}$$

is given as follows:

$$\begin{aligned}
a(w_\alpha, \Xi, \xi) &= \frac{1 - q^{-1}\Xi_i \Xi_{i+1}^{-1}}{1 - \Xi_i^{-1}\Xi_{i+1}} \\
&\qquad (\alpha = \varepsilon_i - \varepsilon_{i+1}, 1 \le i \le r)\,, \\
a(w_\alpha, \Xi, \xi) &= \frac{(1 - q^{-1}\Xi_i \Xi_{i+1}^{-1})(1 - q^{-1/2}\xi_i^{-1}\Xi_{i+1})(1 - q^{-1/2}\xi_i \Xi_i^{-1})}{(1 - \Xi_i^{-1}\Xi_{i+1})(1 - q^{-1/2}\xi_i^{-1}\Xi_i)(1 - q^{-1/2}\xi_i \Xi_{i+1}^{-1})} \\
&\qquad (\alpha = \varepsilon_i - \varepsilon_{i+1}, r+1 \le i \le r + l' - 1)\,, \\
a(w_\alpha, \Xi, \xi) &= \frac{(1 - q^{-1}\Xi_{l'}^2)(1 - q^{-1/2}\xi_{l'}^{-1}\Xi_{l'}^{-1})(1 - q^{-1/2}\xi_{l'}\Xi_{l'}^{-1})}{(1 - \Xi_{l'}^{-2})(1 - q^{-1/2}\xi_{l'}^{-1}\Xi_{l'})(1 - q^{-1/2}\xi_{l'}\Xi_{l'})} \\
&\qquad (\alpha = \varepsilon_{r+l'} \textit{ in the odd case})\,, \\
a(w_\alpha, \Xi, \xi) &= \frac{(1 - q^{-1}\Xi_{l'-1}\Xi_{l'})(1 - q^{-1/2}\xi_{l'-1}^{-1}\Xi_{l'}^{-1})(1 - q^{-1/2}\xi_{l'-1}\Xi_{l'-1}^{-1})}{(1 - \Xi_{l'-1}^{-1}\Xi_{l'}^{-1})(1 - q^{-1/2}\xi_{l'-1}^{-1}\Xi_{l'-1})(1 - q^{-1/2}\xi_{l'-1}\Xi_{l'})} \\
&\qquad (\alpha = \varepsilon_{l'-1} + \varepsilon_{l'} \textit{ in the even case})\,.
\end{aligned}$$

PROOF. For $\alpha \in \Delta$, let us define the elements $\Psi_1,\ \Psi_{w_\alpha} \in I(\Xi)$ by putting

$$\Psi_1 = \Psi_{1,\Xi} := R(\mathrm{ch}_{B' g_{m,r} w_\ell B})\Phi_{1,\Xi} = \mathcal{P}_\Xi(R(\mathrm{ch}_{B' g_{m,r} w_\ell B})\mathrm{ch}_B)$$

and

$$\Psi_{w_\alpha} = \Psi_{w_\alpha,\Xi} := R(\mathrm{ch}_{B' g_{m,r} w_\ell B})\Phi_{w_\alpha,\Xi} = \mathcal{P}_\Xi(R(\mathrm{ch}_{B' g_{m,r} w_\ell B})\mathrm{ch}_{B w_\alpha B}).$$

(Note that $\Phi_1 = \mathcal{P}_\Xi(\mathrm{ch}_B)$ and $\Phi_{w_\alpha} = \mathcal{P}_\Xi(\mathrm{ch}_{B w_\alpha B})$.) In particular, $\Psi_1 \in I(\Xi; P w_\ell P)$ because the support of $R(\mathrm{ch}_{B' g_{m,r} w_\ell B})\mathrm{ch}_B$ is $B(g_{mr} w_\ell)^{-1} B' \subset P w_\ell P$ (see the proof of 8.2). From Sections 4 and 8, we have

$$\begin{aligned}
l_{\Xi,\xi}(\Psi_{1,\Xi}) &= l_{\Xi,\xi}(\mathcal{P}_\Xi(R(\mathrm{ch}_{B' g_{m,r} w_\ell B})\mathrm{ch}_B)) \\
&= \int_{B \times B' g_{m,r} w_\ell B} Y(x g^{-1}) dx dg \\
&= \mathrm{vol}(B) \int_{B' g_{m,r} w_\ell B} Y(g^{-1}) dg \\
&= \mathrm{vol}(B' g_{m,r} w_\ell B)\mathrm{vol}(B')^{-1} \int_{B \times B'} Y(g (g_{m,r} w_\ell)^{-1} g') dg dg' \\
&= \mathrm{vol}(B' g_{m,r} w_\ell B)\mathrm{vol}(B')^{-1} \Omega(\phi_1, R(g_{m,r} w_\ell)\Phi_1) \\
&= \mathrm{vol}(B' g_{m,r} w_\ell B)\mathrm{vol}(B)\,.
\end{aligned}$$

Similarly,

$$\begin{aligned} l_{\Xi,\xi}(\Psi_{w_\alpha,\Xi}) &= \mathrm{vol}(B'g_{m,r}w_\ell B)\mathrm{vol}(B')^{-1}\int_{Bw_\alpha B\times B'} Y(g(g_{m,r}w_\ell)^{-1}g')dg\,dg' \\ &= \mathrm{vol}(B'g_{m,r}w_\ell B)\mathrm{vol}(B)\times \Omega(\phi_1, R(g_{m,r}w_\ell)\Phi_{w_\alpha})\,. \end{aligned}$$

On the other hand, we have

$$T_{w_\alpha}\Phi_{1,w_\alpha\Xi} = (\mathbf{c}_\alpha(\Xi^{-1})-1)\Phi_{1,\Xi} + q^{-1}\Phi_{w_\alpha,\Xi}$$

from [C2, 3.4]. Hence we get

$$T_{w_\alpha}\Psi_{1,w_\alpha\Xi} = (\mathbf{c}_\alpha(\Xi^{-1})-1)\Psi_{1,\Xi} + q^{-1}\Psi_{w_\alpha,\Xi}\,.$$

Therefore we finally have

$$\begin{aligned} &T^*_{w_\alpha}l_{\Xi,\xi}(\Psi_{1,w_\alpha\Xi}) \\ &\quad = l_{\Xi,\xi}(T_{w_\alpha}\Psi_{1,w_\alpha\Xi}) \\ &\quad = (\mathbf{c}_\alpha(\Xi^{-1})-1)l_{\Xi,\xi}(\Psi_{1,\Xi}) + q^{-1}l_{\Xi,\xi}(\Psi_{w_\alpha,\Xi}) \\ &\quad = \mathrm{vol}(B'g_{m,r}w_\ell B)\mathrm{vol}(B) \\ &\qquad \times\{(\mathbf{c}_\alpha(\Xi^{-1})-1)\Omega(\phi_1, R(g_{m,r}w_\ell)\Phi_1) + q^{-1}\Omega(\phi_1, R(g_{m,r}w_\ell)\Phi_{w_\alpha})\} \\ &\quad = \mathrm{vol}(B'g_{m,r}w_\ell B)\mathrm{vol}(B) \\ &\qquad \times\{(\mathbf{c}_\alpha(\Xi^{-1})-1-q^{-1}) + q^{-1}\Omega(\phi_1, R(g_{m,r}w_\ell)(\Phi_1+\Phi_{w_\alpha}))\} \end{aligned}$$

for generic $(\Xi, \xi) \in Z_c$. This shows that

$$a(w_\alpha, \Xi, \xi) = (\mathbf{c}_\alpha(\Xi^{-1})-1-q^{-1}) + q^{-1}\Omega(\phi_1, R(g_{m,r}w_\ell)(\Phi_1+\Phi_{w_\alpha}))$$

for any $\alpha \in \Delta$. Now substituting the values of $\Omega(\phi_1, R(g_{m,r}w_\ell)(\Phi_1+\Phi_{w_\alpha}))$ calculated in Section 8, we get the explicit form of $a(w_\alpha, \Xi, \xi)$ from case-by-case consideration. This completes the proof of the proposition. □

We have verified that all the assumptions in Section 2 are satisfied (9.1, 9.2, 9.5 and 9.8). Thus we obtain the following theorem from 2.10.

THEOREM 9.9. *The equivariant linear form $l_{\Xi,\xi}$ is rational in (Ξ, ξ). In particular, for generic (Ξ, ξ), $l_{\Xi,\xi}$ is defined and satisfies*

$$\mathrm{Hom}_{P_H}(I(\Xi), \xi\delta'^{1/2}\otimes\psi_U) = \boldsymbol{C}\cdot l_{\Xi,\xi}\,.$$

COROLLARY 9.10. *Up to a constant factor, there uniquely exists an H-invariant bilinear form $\Omega_{\Xi,\xi} : I(\xi, \psi_U)\times I(\Xi) \to \boldsymbol{C}$ for generic (Ξ, ξ). This $\Omega_{\Xi,\xi}$ is rational in (Ξ, ξ).*

PROOF. Recall that H-invariant bilinear forms $\Omega : I(\xi, \psi_U)\times I(\Xi) \to \boldsymbol{C}$ and P_H-equivariant linear forms $l \in \mathrm{Hom}_{P_H}(I(\Xi), \xi\delta'^{1/2}\otimes\psi_U)$ are in one-to-one correspondence (see Section 4). Hence the existence and the uniqueness follow from 9.9. On the other hand, the rationality of $l_{\Xi,\xi}$ implies that of $\Omega_{\Xi,\xi}$. Actually, we have for $f \in C_c^\infty(G)$, $f_0 \in C_c^\infty(G')$,

$$\begin{aligned}\Omega_{\Xi,\xi}(\mathcal{P}_\xi(f_0), \mathcal{P}_\Xi(f)) &= \int_{G'\times G} f_0(g')f(g)Y(g{g'}^{-1})dg'dg \\ &= l_{\Xi,\xi}(\mathcal{P}f^*)\,,\end{aligned}$$

where $f^* \in C^\infty(G)$ is defined as

$$f^*(x) = \int_{G'} f_0(g')f(xg')dg'\,. \qquad \square$$

REMARK 9.11. By the rationality of $\Omega_{\Xi,\xi}$, the formulas on the values of I_α and J_β calculated in Section 8 hold for generic (Ξ, ξ).

10. An explicit formula.

10.1. In this section, we shall give an explicit formula for the Whittaker-Shintani function $S_{\Xi,\xi}$ given by

$$\begin{aligned}S_{\Xi,\xi}(g) &= \Omega_{\Xi,\xi}(\phi_{K',\xi}, R(g)\Phi_{K,\Xi}) \\ &= \int_{K'\times K} Y_{\Xi,\xi}(kg^{-1}k')dk'dk\end{aligned}$$

introduced in Section 4. Recall that the integral above defines a rational function in (Ξ, ξ) by "analytic continuation" (see Section 9).

10.2. Let $\Xi \in X$ and $\xi \in X'$. We shall identify Ξ and ξ with $(\Xi_1, \ldots, \Xi_l) \in (\boldsymbol{C}^\times)^l$ and $(\xi_1, \ldots, \xi_{l'}) \in (\boldsymbol{C}^\times)^{l'}$ respectively, as before.

For $\alpha \in \Sigma$ (resp. $\beta \in \Sigma'$), we let $\mathbf{e}_\alpha(\Xi)$ (resp. $\mathbf{e}'_\beta(\xi)$) be the numerator of the c-function $\mathbf{c}_\alpha(\Xi)$ (resp. $\mathbf{c}'_\beta(\xi)$); namely $\mathbf{e}_\alpha(\Xi) = 1 - q^{-1}\Xi(a_\alpha)$ and $\mathbf{e}'_\beta(\xi) = 1 - q^{-1}\xi(a'_\beta)$. We set $\mathbf{e}(\Xi) = \prod_{\alpha\in\Sigma^+}\mathbf{e}_\alpha(\Xi)$ and $\mathbf{e}'(\xi) = \prod_{\beta\in\Sigma'^+}\mathbf{e}'_\beta(\xi)$. We also let $\mathbf{d}_\alpha(\Xi)$ be the denominator of $\mathbf{c}_\alpha(\Xi)$ so that $\mathbf{d}_\alpha(\Xi) = 1 - \Xi(a_\alpha)$. We set $\mathbf{d}(\Xi) = \prod_{\alpha\in\Sigma^+}\mathbf{d}_\alpha(\Xi)$. Similarly we define $\mathbf{d}'_\beta(\xi)$ and $\mathbf{d}'(\xi)$.

10.3. We let

$$\mathbf{b}(\Xi, \xi) = \prod_{\substack{1\le i\le l'\\ 1\le j\le l}} (1 - q^{-1/2}(\xi_i^{-1}\Xi_j)^{\eta_{ij}})(1 - q^{-1/2}\xi_i\Xi_j)\,,$$

where

$$\eta_{ij} = \begin{cases} 1, & \text{if } j \le r+i\,, \\ -1, & \text{if } j > r+i\,. \end{cases}$$

Let us put

$$\zeta(\Xi, \xi) = \frac{\mathbf{e}(\Xi)\mathbf{e}'(\xi)}{\mathbf{b}(\Xi, \xi)}\,.$$

LEMMA 10.4. (1) *For any* $\alpha \in \Delta$,

$$\frac{\zeta(w_\alpha \Xi, \xi)}{\zeta(\Xi, \xi)} = \frac{\Omega_{w_\alpha \Xi, \xi}(\phi_1, R(g_{m,r} w_\ell)(\Phi_1 + \Phi_{w_\alpha}))}{\Omega_{\Xi, \xi}(\phi_1, R(g_{m,r} w_\ell)(\Phi_1 + \Phi_{w_\alpha}))}. \tag{10.4.1}$$

(2) *For any* $\beta \in \Delta'$,

$$\frac{\zeta(\Xi, w_\beta \xi)}{\zeta(\Xi, \xi)} = \frac{\Omega_{\Xi, w_\beta \xi}(\phi_1 + \phi_{w_\beta}, R(g_{m,r} w_\ell)\Phi_1)}{\Omega_{\Xi, \xi}(\phi_1 + \phi_{w_\beta}, R(g_{m,r} w_\ell)\Phi_1)}. \tag{10.4.2}$$

PROOF. We can verify these equalities from case-by-case considerations. For example, if $\alpha = \varepsilon_{r+i} - \varepsilon_{r+i+1}$ $(1 \le i \le l' - 1)$, it is easily seen that the left hand side of (10.4.1) is equal to

$$\frac{(1 - q^{-1/2}\xi_i^{-1}\Xi_{r+i})(1 - q^{-1/2}\xi_i \Xi_{r+i+1}^{-1})(1 - q^{-1}\Xi_{r+i}^{-1}\Xi_{r+i+1})}{(1 - q^{-1/2}\xi_i^{-1}\Xi_{r+i+1})(1 - q^{-1/2}\xi_i \Xi_{r+i}^{-1})(1 - q^{-1}\Xi_{r+i}\Xi_{r+i+1}^{-1})}.$$

On the other hand, the results of Section 8 (8.9 and 8.15) show that the right hand side of (10.4.1) is identical to the above. We can check the other cases in similar ways. □

THEOREM 10.5. *For generic* (Ξ, ξ), *the value* $S_{\Xi,\xi}(g)/\zeta(\Xi, \xi)$ $(g \in G)$ *is* $W \times W'$-*invariant as a function of* (Ξ, ξ).

PROOF. We first recall that, by the uniqueness argument in Section 7, any H-invariant bilinear form on $I(\xi, \psi_U) \times I(\Xi)$ is a scalar multiple of $\Omega_{\Xi,\xi}$ for generic (Ξ, ξ). Since a bilinear form on $I(\xi, \psi_U) \times I(\Xi)$ given by

$$(T_{w',\xi} \times T_{w,\Xi})^* \Omega_{w\Xi, w'\xi} = \Omega_{w\Xi, w'\xi} \circ (T_{w',\xi} \times T_{w,\Xi})$$

is also H-invariant, there exists a scalar factor $b_{w,w'}(\Xi, \xi)$ such that

$$(T_{w',\xi} \times T_{w,\Xi})^* \Omega_{w\Xi, w'\xi} = \mathbf{c}_w(\Xi)\mathbf{c}'_{w'}(\xi) b_{w,w'}(\Xi, \xi) \Omega_{\Xi,\xi}$$

for generic (Ξ, ξ). Consider the case where $w = w_\alpha$ $(\alpha \in \Delta)$ and $w' = 1$. Since

$$T_{w_\alpha}(\Phi_1 + \Phi_{w_\alpha}) = \mathbf{c}_\alpha(\Xi)(\Phi_1 + \Phi_{w_\alpha}),$$

we have

$$\begin{aligned}
\mathbf{c}_\alpha(\Xi) b_{w_\alpha,1}(\Xi, \xi) &\Omega_{\Xi,\xi}(\phi_1, R(g_{m,r} w_\ell)(\Phi_1 + \Phi_{w_\alpha})) \\
&= (T_{w_\alpha \Xi} \times 1)^* \Omega_{w_\alpha \Xi, \xi}(\phi_1, R(g_{m,r} w_\ell)(\Phi_1 + \Phi_{w_\alpha})) \\
&= \mathbf{c}_\alpha(\Xi) \Omega_{w_\alpha \Xi, \xi}(\phi_1, R(g_{m,r} w_\ell)(\Phi_1 + \Phi_{w_\alpha}))
\end{aligned}$$

and hence

$$\begin{aligned}
b_{w_\alpha,1}(\Xi, \xi) &= \frac{\Omega_{w_\alpha \Xi, \xi}(\phi_1, R(g_{m,r} w_\ell)(\Phi_1 + \Phi_{w_\alpha}))}{\Omega_{\Xi, \xi}(\phi_1, R(g_{m,r} w_\ell)(\Phi_1 + \Phi_{w_\alpha}))} \\
&= \frac{\zeta(w_\alpha \Xi, \xi)}{\zeta(\Xi, \xi)}.
\end{aligned}$$

Therefore we have

$$\begin{aligned} S_{w_\alpha \Xi,\xi}(g)/\zeta(w_\alpha \Xi,\xi) &= \Omega_{w_\alpha \Xi,\xi}(\phi_{K'}, R(g)\Phi_K)/\zeta(w_\alpha \Xi,\xi) \\ &= \mathbf{c}_\alpha(\Xi)^{-1}\Omega_{w_\alpha \Xi,\xi}(\phi_{K'}, T_{w_\alpha}(R(g)\Phi_K))/\zeta(w_\alpha \Xi,\xi) \\ &= b_{w_\alpha,1}(\Xi,\xi)\Omega_{\Xi,\xi}(\phi_{K'}, R(g)\Phi_K)/\zeta(w_\alpha \Xi,\xi) \\ &= \Omega_{\Xi,\xi}(\phi_{K'}, R(g)\Phi_K)/\zeta(\Xi,\xi) \\ &= S_{\Xi,\xi}(g)/\zeta(\Xi,\xi)\,. \end{aligned}$$

This implies that the function of Ξ given by $S_{\Xi,\xi}(g)/\zeta(\Xi,\xi)$ is invariant under W. The W'-invariance follows exactly in the same manner. $\square$

10.6. We are now in a position to give an explicit formula of Whittaker-Shintani function $S_{\Xi,\xi}$ in a form analogous to the case of zonal spherical functions or Whittaker functions ([Mac], [CS], [K1]).

Recall 6.1. It suffices to know the value $S_{\Xi,\xi}(g)$ with $g = t'(\lambda')g_{m,r}w_\ell t(\lambda)^{-1}$ for $\lambda' \in \Lambda^+_{m'}$, $\lambda \in \Lambda^+_m$, since $-w_\ell(\Lambda^+_m) = \Lambda^+_m$ and

$$t'(\lambda')g_{m,r}w_\ell t(\lambda)^{-1} = t'(\lambda')g_{m,r}t(-w_\ell(\lambda))w_\ell \quad (\lambda \in \Lambda^+_m,\ \lambda' \in \Lambda^+_{m'})\,.$$

Let us put

$$\mathbf{c}_{\mathrm{WS}}(\Xi,\xi) = \frac{\mathbf{c}(\Xi)\mathbf{c}'(\xi)}{\zeta(\Xi,\xi)} = \frac{\mathbf{b}(\Xi,\xi)}{\mathbf{d}(\Xi)\mathbf{d}'(\xi)}\,.$$

Then we can give the following theorem by using an argument similar to that in [CS].

THEOREM 10.7. *For $\lambda' \in \Lambda^+_{m'}$ and $\lambda \in \Lambda^+_m$,*

$$\begin{aligned} S_{\Xi,\xi}(t'(\lambda')g_{m,r}w_\ell t(\lambda)^{-1})/\zeta(\Xi,\xi) &= q^{l(w_\ell)+l(w'_\ell)}\mathrm{vol}(B)\mathrm{vol}(B') \\ \times \sum_{\substack{w\in W\\ w'\in W'}} &\mathbf{c}_{\mathrm{WS}}(w\Xi, w\xi)((w\Xi)^{-1}\delta^{1/2})(t(\lambda))((w'\xi)^{-1}{\delta'}^{1/2})(t'(\lambda'))\,. \end{aligned}$$

PROOF. We fix generic parameters (Ξ,ξ). We first note that

$$\begin{aligned} S_{\Xi,\xi}(t'(\lambda')g_{m,r}w_\ell t(\lambda)^{-1}) &= \mathrm{vol}(B't'(\lambda')^{-1}B')\mathrm{vol}(Bt(\lambda)^{-1}B) \\ &\times L(\mathrm{ch}_{B't'(\lambda')^{-1}B'})R(\mathrm{ch}_{Bt(\lambda)^{-1}B})S_{\Xi,\xi}(g_{m,r}w_\ell) \end{aligned} \tag{10.7.1}$$

for $\lambda' \in \Lambda^+_{m'}$ and $\lambda \in \Lambda^+_m$. To show this, it is sufficient to prove that

$$B't'(\lambda')B'g_{m,r}w_\ell Bt(\lambda)^{-1}B \subset U_{(0)}K't'(\lambda')g_{m,r}w_\ell t(\lambda)^{-1}K\,.$$

By the Iwahori factorization $B = N^-_{(1)}T_{(0)}N_{(0)}$ and $B' = N'_{(0)}T'_{(0)}{N'}^-_{(1)}$,

$$B't'(\lambda')B'g_{m,r}w_\ell Bt(\lambda)^{-1}B \subset K't'(\lambda'){N'}^-_{(1)}g_{m,r}w_\ell N^-_{(1)}t(\lambda)^{-1}K\,.$$

(Note that $t(\lambda)N_{(0)}t(\lambda)^{-1} \subset N_{(0)}$ and $t'(\lambda')N'_{(0)}t'(\lambda')^{-1} \subset N'_{(0)}$.) Then we see exactly as in Proposition 8.3 (see also Lemma 8.2) that

$$\begin{aligned} N'^{-}_{(1)} g_{m,r} w_\ell N^{-}_{(1)} &\subset g_{m,r} w_\ell N^{-}_{(1)} N_{(1)} \\ &\subset g_{m,r} w_\ell N^{-}_{(1)} T_{(0)} N_{(1)} \subset U_{(0)} N'_{(1)} T'_{(0)} g_{m,r} w_\ell T_{(0)} N_{(1)} \,. \end{aligned}$$

This implies (10.7.1).

By 1.10, we have a basis $\{g_w \ (w \in W)\}$ for $I(\Xi)^B$ satisfying

$$R(\mathrm{ch}_{Bt(\lambda)^{-1}B}) g_w = \mathrm{vol}(Bt(\lambda)B)(w\Xi)^{-1}\delta^{1/2}(t(\lambda)) g_w \quad (\lambda \in \Lambda_m^+)\,; \tag{10.7.2}$$

$$g_1 = \phi_1\,; \tag{10.7.3}$$

$$\phi_K = q^{\ell(w_\ell)} \sum_{w \in W} \bar{\mathbf{c}}_w(\Xi) g_w \tag{10.7.4}$$

with $\bar{\mathbf{c}}_w(\Xi) = \prod \mathbf{c}_\alpha(\Xi)$ $(\alpha > 0, w\alpha > 0)$. We also have a basis $\{g'_{w'} \ (w' \in W')\}$ for $I(\xi)^B = I(\xi, \psi_U)^B$ with the similar properties

$$\phi_{K'} = q^{\ell(w'_\ell)} \sum_{w' \in W'} \bar{\mathbf{c}}'_{w'}(\xi) g'_{w'} \tag{10.7.5}$$

and so on. Put

$$S = S_{\Xi,\xi}(t'(\lambda')g_{m,r} w_\ell t(\lambda)^{-1})/\zeta(\Xi,\xi)\,.$$

Then we have, by substituting (10.7.4) and (10.7.5) in 10.1,

$$\begin{aligned} S = q^{\ell(w_\ell)+\ell(w'_\ell)} &\frac{\mathbf{b}(\Xi,\xi)}{\mathbf{e}(\Xi)\mathbf{e}'(\xi)} \\ \times \sum_{\substack{w \in W \\ w' \in W'}} &\bar{\mathbf{c}}_w(\Xi)\bar{\mathbf{c}}'_{w'}(\xi)(w\Xi)^{-1}\delta^{1/2}(t(\lambda))(w'\xi)^{-1}\delta'^{1/2}(t'(\lambda'))\Omega_{\Xi,\xi}(g'_{w'}, R(g_{m,r}w_\ell)g_w) \end{aligned}$$

from (10.7.1), (10.7.3) (and its counterpart for $\{g'_{w'} \ (w' \in W')\}$). We know that

$$\Omega_{\Xi,\xi}(g'_1, R(g_{m,r}w_\ell)g_1) = \mathrm{vol}(B)\mathrm{vol}(B')$$

from 8.3. Thus the coefficient for $w = 1,\ w' = 1$ in S is

$$q^{\ell(w_\ell)+\ell(w'_\ell)}\mathrm{vol}(B)\mathrm{vol}(B')\frac{\mathbf{b}(\Xi,\xi)}{\mathbf{e}(\Xi)\mathbf{e}'(\xi)} = q^{\ell(w_\ell)+\ell(w'_\ell)}\mathrm{vol}(B)\mathrm{vol}(B')\mathbf{c}_{\mathrm{WS}}(\Xi,\xi).$$

Hence the $W \times W'$-invariance of S and the linear independence of characters show that

$$\begin{aligned} S = q^{l(w_\ell)+l(w'_\ell)}&\mathrm{vol}(B)\mathrm{vol}(B') \\ \times \sum_{\substack{w \in W \\ w' \in W'}} &\mathbf{c}_{\mathrm{WS}}(w\Xi, w\xi)((w\Xi)^{-1}\delta^{1/2})(t(\lambda))((w'\xi)^{-1}\delta'^{1/2})(t'(\lambda'))\,. \end{aligned}$$

□

The value $S_{\Xi,\xi}(1)/\zeta(\Xi,\xi) = S_{\Xi,\xi}(g_{m,r})/\zeta(\Xi,\xi) = S_{\Xi,\xi}(g_{m,r}w_\ell)/\zeta(\Xi,\xi)$ is given by the following theorem.

THEOREM 10.8. *The value of* $S_{\Xi,\xi}$ *at* 1, $S_{\Xi,\xi}(1)$, *is given as*

$$S_{\Xi,\xi}(1)/\zeta(\Xi,\xi) = q^{\ell(w_\ell)+\ell(w'_\ell)}\mathrm{vol}(B)\mathrm{vol}(B') \times Q_{m'}\,,$$

where $Q_{m'}$ *is the constant given by*

$$Q_{m'} = \begin{cases} (1-q^{-l'})\displaystyle\prod_{i=1}^{l'-1}(1-q^{-2i}) & \text{if } m' = 2l'\,, \\ \displaystyle\prod_{i=1}^{l'}(1-q^{-2i}) & \text{if } m' = 2l'+1\,. \end{cases}$$

We shall prove this theorem in the next section and assume this for the moment.

Now we define the Whittaker-Shintani function $F_{\Xi,\xi}$ by normalizing $S_{\Xi,\xi}$:

$$F_{\Xi,\xi}(g) = S_{\Xi,\xi}(g)/S_{\Xi,\xi}(g_{m,r}w_\ell)\,.$$

Since we already know that $S_{\Xi,\xi}/\zeta(\Xi,\xi)$ is rational in (Ξ,ξ), the explicit formula 10.7 of $S_{\Xi,\xi}/\zeta(\Xi,\xi)$ shows that the value $F_{\Xi,\xi}(g)$ is regular in (Ξ,ξ) with $F_{\Xi,\xi}(1)=1$.

Thus we finally have the following theorem from 10.7, 10.8 and the multiplicity one result in Section 7.

THEOREM 10.9. *For any* $(\Xi,\xi)\in X\times X'$, $\dim_{\mathbf{C}} WS(\Xi,\xi) = 1$. *The basis of* $WS(\Xi,\xi)$, $F_{\Xi,\xi}\in WS(\Xi,\xi)$ *with* $F_{\Xi,\xi}(1)=1$*, is given by the formula*

$$\begin{aligned} &F_{\Xi,\xi}(t'(\lambda')g_{m,r}w_\ell t(\lambda)^{-1}) \\ &= \frac{1}{Q_{m'}}\sum_{\substack{w\in W\\ w'\in W'}} \mathbf{c}_{WS}(w\Xi,w'\xi)((w\Xi)^{-1}\delta^{1/2})(t(\lambda))((w'\xi)^{-1}\delta'^{1/2})(t'(\lambda')) \end{aligned} \tag{10.9.1}$$

for $(\lambda,\lambda')\in\Lambda_m^+\times\Lambda_{m'}^+$.

11. The value at the identity: Proof of 10.8. We shall calculate the sum

$$A_{r,m'} = A_{r,m'}(\Xi,\xi) = \sum_{\substack{w\in W\\ w'\in W'}} \frac{\mathbf{b}(w\Xi,w'\xi)}{\mathbf{d}(w\Xi)\mathbf{d}'(w'\xi)}$$

for regular $(\Xi,\xi)\in X\times X'$. (Recall that $m=2r+m'+1$.) We have

$$S_{\Xi,\xi}(1) = S_{\Xi,\xi}(g_{m,r}w_\ell) = \zeta(\Xi,\xi)q^{\ell(w_\ell)+\ell(w'_\ell)}\mathrm{vol}(B)\mathrm{vol}(B')\times A_{r,m'}$$

from 10.7. Therefore we can rewrite Theorem 10.8 as follows:

THEOREM 11.1. *The sum* $A_{r,m'}$ *is a constant, and is equal to* $Q_{m'}$ *given in* 10.8.

In what follows, we shall calculate

$$A^\dagger_{r,m'} := A_{r,m'}(\Xi^{-1},\xi^{-1}) = \sum_{\substack{w\in W\\ w'\in W'}} \frac{\mathbf{b}(w\Xi^{-1},w'\xi^{-1})}{\mathbf{d}(w\Xi^{-1})\mathbf{d}'(w'\xi^{-1})}$$

instead of $A_{r,m'}$, and show that $A^{\dagger}_{r,m}$ is equal to the above constant.

11.2. From now on, we shall consider the odd case $m = 2r + 2l' + 1,\ m' = 2l'$. We can handle the even case in a similar way.

We shall regard $\Xi_i\ (1 \le i \le r + l')$ and $\xi_j\ (1 \le j \le l')$ as indeterminates. Hence $A^{\dagger}_{r,m'}$ is in the Laurent polynomial ring $\boldsymbol{C}[\Xi_i^{\pm 1}, \xi_j^{\pm 1}]$ by Weyl's character formula. We put

$$\begin{aligned}
\mathbf{b}^{\dagger}_{r,m'}(\Xi, \xi) &:= \mathbf{b}(\Xi^{-1}, \xi^{-1}) \\
&= \prod_{\substack{1 \le i \le l' \\ 1 \le j \le r+i}} (1 - q^{-1/2}\xi_i \Xi_j^{-1}) \prod_{\substack{1 \le i \le l' \\ r+i < j \le r+l'}} (1 - q^{-1/2}\xi_i^{-1}\Xi_j) \\
&\quad \times \prod_{\substack{1 \le i \le l' \\ 1 \le j \le r+l'}} (1 - q^{-1/2}\xi_i^{-1}\Xi_j^{-1}),
\end{aligned}$$

$$\mathbf{d}(\Xi)^{\dagger} := \mathbf{d}(\Xi^{-1}) = \prod_{1 \le i < j \le r+l'} (1 - \Xi_i^{-1}\Xi_j)(1 - \Xi_i^{-1}\Xi_j^{-1}) \prod_{1 \le i \le r+l'} (1 - \Xi_i^{-2})$$

and

$$\mathbf{d}'(\xi)^{\dagger} := \mathbf{d}'(\xi^{-1}) = \prod_{1 \le i < j \le l'} (1 - \xi_i^{-1}\xi_j)(1 - \xi_i^{-1}\xi_j^{-1}).$$

11.3. Set

$$\rho = \rho_m = (r + l', r + l' - 1, \dots, 2, 1) \in \Lambda_m = \mathbf{Z}^{r+l'}$$

and

$$\rho' = \rho'_{m'} = (l' - 1, l' - 2, \dots, 1, 0) \in \Lambda_{m'} = \mathbf{Z}^{l'}.$$

Then ρ (resp. ρ') is the half-sum of positive roots in C_{r+n} (resp. D_n). We put

$$\Xi^{\rho} = \Xi_1^{r+l'}\Xi_2^{r+l'-1} \cdots \Xi_{r+l'-1}^2 \Xi_{r+l'}$$

and

$$\xi^{\rho'} = \xi_1^{l'-1}\xi_2^{l'-2} \cdots \xi_{l'-2}^2 \xi_{l'-1}.$$

As in the case of Weyl's character formula, we have

$$A^{\dagger}_{r,m'} = \mathcal{D}(\Xi)^{-1}\mathcal{D}'(\xi)^{-1} \sum_{\substack{w \in W \\ w' \in W'}} \mathrm{sgn}(w)\mathrm{sgn}(w') w w' (\Xi^{\rho}\xi^{\rho'}\mathbf{b}^{\dagger}_{r,m'}(\Xi, \xi)),$$

where

$$\mathcal{D}(\Xi) = \mathcal{D}_m(\Xi) = \prod_{1 \le i < j \le r+l'} (\Xi_i - \Xi_j)(1 - \Xi_i^{-1}\Xi_j^{-1}) \prod_{1 \le i \le r+l'} (\Xi_i - \Xi_i^{-1})$$

and

$$\mathcal{D}'(\xi) = \mathcal{D}'_{m'}(\xi) = \prod_{1 \le i < j \le l'} (\xi_i - \xi_j)(1 - \xi_i^{-1}\xi_j^{-1}).$$

We say that $\lambda \in \Lambda_m = \mathbf{Z}^{r+l'}$ or the monomial $\Xi^\lambda = \Xi_1^{\lambda_1} \cdots \Xi_{r+l'}^{\lambda_{r+l'}}$ (resp. $\mu \in \Lambda_{m'} = \mathbf{Z}^{l'}$ or $\xi^\mu = \xi_1^{\mu_1} \cdots \xi_{l'}^{\mu_{l'}}$) is *regular* if the stabilizer of λ in $W = W(\mathrm{C}_{r+l'})$ (resp. the stabilizer of μ in $W' = W(\mathrm{D}_{l'})$) is trivial. We also call the monomial $\Xi^\lambda \xi^\mu$ regular if both Ξ^λ and ξ^μ are regular. Let us set $B_{r,m'} = \Xi^\rho \xi^{\rho'} \mathbf{b}^\dagger_{r,m'}(\Xi, \xi)$. Then, by expanding $B_{r,m'}$ as $B_{r,m'} = \sum c_{\lambda,\mu} \Xi^\lambda \xi^\mu$, we have

$$A^\dagger_{r,m'} = \mathcal{D}(\Xi)^{-1} \mathcal{D}'(\xi)^{-1} \sum_{\Xi^\lambda \xi^\mu \text{ regular}} c_{\lambda,\mu} \sum_{\substack{w \in W(\mathrm{C}_{r+l'}) \\ w' \in W(\mathrm{D}_{l'})}} \mathrm{sgn}(w)\mathrm{sgn}(w') w w' (\Xi^\lambda \xi^\mu) \,.$$

11.4. Reduction to the case $r = 0$. Now we look at the expansion of $B_{r,m'}$ in the above more closely to study regular terms in it. We write down $B_{r,m'}$ as

$$B_{r,m'} = \prod_{\substack{1 \le i \le l' \\ 1 \le j \le r+i}} (\Xi_j - q^{-1/2} \xi_i) \prod_{\substack{1 \le i \le l' \\ r+i < j \le r+l'}} (\xi_i - q^{-1/2} \Xi_j)$$
$$\times \prod_{\substack{1 \le i \le l' \\ 1 \le j \le r+l'}} (1 - q^{-1/2} \xi_i^{-1} \Xi_j^{-1}) \prod_{j=1}^{r} \Xi_j^{r-j+1} \,.$$

If a monomial $\Xi^\lambda \xi^\mu$ in the expansion of $B_{r,m'}$ is regular, then we must have

$$\begin{cases} |\lambda_{\sigma(1)}| > |\lambda_{\sigma(2)}| > \cdots > |\lambda_{\sigma(r+l')}| > 0 \,, \\ |\mu_{\tau(1)}| > |\mu_{\tau(2)}| > \cdots > |\mu_{\tau(l')}| \ge 0 \end{cases}$$

for some permutations $\sigma \in S_{r+l'}$ and $\tau \in S_{l'}$. In particular, we have

$$\begin{cases} |\lambda_{\sigma(i)}| \ge r + l' + 1 - i \,, \\ |\mu_{\tau(j)}| \ge l' - j \,. \end{cases} \tag{11.4.1}$$

However we can see easily that the exponent λ_i of the power of Ξ_i in $B_{r,m'}$ must satisfy

$$\begin{cases} -l' + r - i + 1 \le \lambda_i \le r + l' - i + 1 & \text{if } i \le r, \\ -l' \le \lambda_i \le l' & \text{if } r < i \,. \end{cases}$$

This shows that

$$\lambda_{\sigma(1)} = \lambda_1 = l' + r > \lambda_{\sigma(2)} = \lambda_2 = l' + r - 1 > \cdots > \lambda_{\sigma(r)} = \lambda_r = l' \,, \tag{11.4.2}$$

and that

$$|\lambda_{\sigma(r+i)}| = l' - i \quad (1 \le i \le l') \,. \tag{11.4.3}$$

In particular, we have $\lambda = y(\rho)$ for some $y \in W(\mathrm{C}_{l'})$. Here we regard $W(\mathrm{C}_{l'})$ as the subgroup of $W = W(\mathrm{C}_{r+l'})$ which acts trivially on the first r entries. Note that

$$\prod_{1 \le i \le l'} (\Xi_j - q^{-1/2} \xi_i) = \Xi_j^{l'} + (\text{lower terms in } \Xi_j)$$

and

$$\prod_{1 \le i \le l'} (1 - q^{-1/2} \xi_i^{-1} \Xi_j^{-1}) = 1 + \Xi_j^{-1} \cdot (\text{a polynomial in } \Xi_j^{-1})$$

for $1 \le j \le r$. Therefore we have

$$A^{\dagger}_{r,m'} = \mathcal{D}(\Xi)^{-1}\mathcal{D}'(\xi)^{-1} \sum_{\substack{w\in W \\ w'\in W'}} \mathrm{sgn}(w)\mathrm{sgn}(w')ww'(B^{*}_{r,m'}) ,$$

where

$$B^{*}_{r,m'} = \prod_{\substack{1\le i\le l' \\ r+1\le j\le r+i}} (\Xi_j - q^{-1/2}\xi_i) \prod_{\substack{1\le i\le l' \\ r+i<j\le r+l'}} (\xi_i - q^{-1/2}\Xi_j) \times \prod_{\substack{1\le i\le l' \\ r+1\le j\le r+l'}} (1 - q^{-1/2}\xi_i^{-1}\Xi_j^{-1}) \prod_{j=1}^{r} \Xi_j^{r+l'-j+1} .$$

But then the equalities

$$\begin{aligned} \mathrm{sgn}(y) &= \mathcal{D}_{2r+2l'+1}(\Xi)^{-1} \sum_{w\in W(\mathrm{C}_{r+l'})} \mathrm{sgn}(w)w(\Xi^{y(\rho_{2r+2l'+1})}) \\ &= \mathcal{D}_{2l'+1}(\Xi)^{-1} \sum_{w\in W(\mathrm{C}_{l'})} \mathrm{sgn}(w)w(\Xi^{y(\rho_{2l'+1})}) \end{aligned} \tag{11.4.4}$$

for $y \in W(\mathrm{C}_{l'}) \subset W(\mathrm{C}_{r+l'})$ imply the following lemma.

LEMMA 11.5. *The sum $A^{\dagger}_{r,m'}$ is constant in Ξ and is independent of r. In particular*, $A^{\dagger}_{r,m'} = A^{\dagger}_{0,m'}$.

11.6. THE CASE $r = 0$. Now we shall study

$$A = A^{\dagger}_{0,m'} = \mathcal{D}_{m'+1}(\Xi)^{-1}\mathcal{D}'_{m'}(\xi)^{-1} \sum_{\substack{w\in W(\mathrm{C}_{l'}) \\ w'\in W(\mathrm{D}_{l'})}} \mathrm{sgn}(w)\mathrm{sgn}(w')ww'(B_{0,m'}) ,$$

where

$$\begin{aligned} B_{0,m'} = &\prod_{\substack{1\le i\le l' \\ 1\le j\le i}} (\Xi_j - q^{-1/2}\xi_i) \prod_{\substack{1\le i\le l' \\ i<j\le l'}} (\xi_i - q^{-1/2}\Xi_j) \\ &\times \prod_{1\le i,j\le l'} (1 - q^{-1/2}\xi_i^{-1}\Xi_j^{-1}) . \end{aligned} \tag{11.6.1}$$

Recall that the inequalities $|\lambda_i| \le l'$ and $|\mu_j| \le l'$ hold if the monomial $\Xi^{\lambda}\xi^{\mu}$ appears in the expansion of $B_{0,m'}$.

Suppose that a monomial $\Xi^{\lambda}\xi^{\mu}$ with $\mu_i = l'$ for some $i = i_0$ appears in the expansion of $B_{0,m'}$. Note that

$$\prod_{1\le j\le i_0} (\Xi_j - q^{-1/2}\xi_{i_0}) \prod_{i_0<j\le l'} (\xi_{i_0} - q^{-1/2}\Xi_j) = c \cdot \xi_{i_0}^{l'} + (\text{lower terms in } \xi_{i_0})$$

for some non-zero constant c, and

$$\prod_{1\le j\le l'}(1-q^{-1/2}\xi_{i_0}^{-1}\Xi_j^{-1}) = 1+\xi_{i_0}^{-1}\cdot(\text{a polynomial in } \xi_{i_0}^{-1})\,.$$

Therefore, for any j, only the product

$$\prod_{\substack{j\le k\le l'\\ k\ne i_0}}(\Xi_j-q^{-1/2}\xi_k)\prod_{\substack{1\le k<j\\ k\ne i_0}}(\xi_k-q^{-1/2}\Xi_j)$$

contributes to the power $\Xi_j^{\lambda_j}$ in $\Xi^\lambda\xi^\mu$. This implies that $0\le\lambda_j<l'$ for any j, and hence $\Xi^\lambda\xi^\mu$ is not regular. Similarly, we can see that $\Xi^\lambda\xi^\mu$ appearing in $B_{0,m'}$ is not regular if $\mu_i=-l'$ for some i.

Thus we see that $|\mu_i|<l'$ for any i if the monomial $\Xi^\lambda\xi^\mu$ that appears in $B_{0,m'}$ is regular. This and (11.4.1) show that $\mu=u(\rho')$ for some $u\in W(\mathrm{D}_{l'})$. Therefore, as in 11.5, we have:

LEMMA 11.7. *The sum $A^\dagger_{0,m'}$ is a constant.*

11.8. THE EVALUATION OF THE CONSTANT. To evaluate the constant $A=A^\dagger_{0,m'}$, we specialize (Ξ,ξ) to $(\tilde\Xi,\tilde\xi)$, where $\tilde\Xi_k=q^{l'-k+1/2}$ and $\tilde\xi_i=q^{l'-i}$ $(1\le i,k\le l')$. Namely,

$$A=\sum_{\substack{w\in W(\mathrm{C}_{l'})\\ w'\in W(\mathrm{D}_{l'})}}\frac{\mathbf{b}^\dagger_{0,l'}(w\tilde\Xi,w'\tilde\xi)}{\mathbf{d}^\dagger(w\tilde\Xi)\mathbf{d}'^\dagger(w'\tilde\xi)}\,.$$

Note that $\mathbf{d}(w\tilde\Xi)\mathbf{d}'(w'\tilde\xi)\ne 0$ for any $w\in W(\mathrm{C}_{l'})$, $w'\in W(\mathrm{D}_{l'})$.

Now, to every $w\in W(\mathrm{C}_{l'})$, $w'\in W(\mathrm{D}_{l'})$, we shall assign permutations $\sigma,\tau\in S_{l'}$ and $\varepsilon_i,\varepsilon'_j=\pm1$ with $\prod\varepsilon'_j=1$ in the following way:

$$w\Xi=(\Xi_{\sigma(1)}^{\varepsilon_1},\dots,\Xi_{\sigma(l')}^{\varepsilon_{l'}})\,,\quad w'\xi=(\xi_{\tau(1)}^{\varepsilon'_1},\dots,\xi_{\tau(l')}^{\varepsilon'_{l'}})\,.$$

LEMMA 11.9. *If the product $\mathbf{b}^\dagger_{0,m'}(w\tilde\Xi,w'\tilde\xi)\ne0$, then $w=w'=1$.*

PROOF. To show the lemma, we first rewrite $\mathbf{b}^\dagger_{0,m'}(w\tilde\Xi,w'\tilde\xi)$ as

$$\mathbf{b}^\dagger_{0,m'}(w\tilde\Xi,w'\tilde\xi)=\prod_{1\le j\le i\le l'}(1-q^{\alpha(i,j)})\prod_{1\le i<j\le l'}(1-q^{\beta(i,j)})\prod_{1\le i,j\le l'}(1-q^{\gamma(i,j)})\,,$$

where we put

$$\alpha(i,j)=-\frac12-\varepsilon_j\left(l'-\sigma(j)+\frac12\right)+\varepsilon'_i(l'-\tau(i))\,,$$
$$\beta(i,j)=-\frac12+\varepsilon_j\left(l'-\sigma(j)+\frac12\right)-\varepsilon'_i(l'-\tau(i))\,,$$

and

$$\gamma(i,j) = -\frac{1}{2} - \varepsilon_j \left(l' - \sigma(j) + \frac{1}{2} \right) - \varepsilon'_i (l' - \tau(i)) \,.$$

If $\mathbf{b}^{\dagger}_{0,m'}(w\tilde{\Xi}, w'\tilde{\xi}) \neq 0$, we must have

$$\alpha(i,j) \neq 0 \ (1 \leq j \leq i \leq l')\,, \quad \beta(i,j) \neq 0 \ (1 \leq i < j \leq l')\,, \quad \gamma(i,j) \neq 0 \ (1 \leq i, j \leq l')\,.$$

Define $i_1, \ldots, i_{l'}$ and $j_1 \ldots, j_{l'}$ by

$$\tau(i_s) = l' - s + 1 \quad (1 \leq s \leq l')$$

and

$$\sigma(j_t) = l' - t + 1 \quad (1 \leq t \leq l')$$

so that $l' - \tau(i_s) = s - 1$ and $l' - \sigma(j_t) = t - 1$. Then we can deduce Lemma 11.9 easily from the following lemma, since the conditions (11.10.1) and (11.10.2) given below occur only when $w = w' = 1$. Actually, we have $\varepsilon'_{i_1} = 1$ from $\prod \varepsilon'_i = 1$. (Note that $w' \in W(\mathrm{D}_{l'})$.)

LEMMA 11.10. *If the product* $\mathbf{b}^{\dagger}_{0,m'}(w\tilde{\Xi}, w'\tilde{\xi}) \neq 0$, *then the following hold*:

$$i_1 \geq j_1 > i_2 \geq j_2 > \cdots > i_{l'} \geq j_{l'}\,, \tag{11.10.1}$$

$$\varepsilon'_{i_2} = \cdots = \varepsilon'_{i_{l'}} = \varepsilon_{j_1} = \cdots = \varepsilon_{j_{l'}} = 1\,. \tag{11.10.2}$$

The proof of this lemma is as follows. Since $\gamma(i_1, j_1) = -1/2 - \varepsilon_{j_1}(1/2) \neq 0$, we have $\varepsilon_{j_1} = 1$. Then $\beta(i_1, j_1) = -1/2 + \varepsilon_{j_1}(1/2) = 0$ implies that $j_1 \leq i_1$. Next consider $\gamma(i_2, j_1) = -1/2 - 1/2 - \varepsilon'_{i_2}$. The assumption $\gamma(i_2, j_1) \neq 0$ shows that $\varepsilon'_{i_2} = 1$, which in turn implies that $i_2 < j_1$, since $\alpha(i_2, j_1) = -1/2 - \varepsilon_{j_1}(1/2) + \varepsilon'_{i_2} = 0$. In this way, we have

$$i_1 \geq j_1 > i_2 \geq j_2 > \cdots > i_{l'} \geq j_{l'}$$

and

$$\varepsilon'_{i_2} = \cdots = \varepsilon'_{i_{l'}} = \varepsilon_{j_1} = \cdots = \varepsilon_{j_{l'}} = 1$$

by induction. Details are left to the readers. □

As for the value of $A = A^{\dagger}_{0,m'} = A^{\dagger}_{r,m'}$, Lemma 11.9 and the direct calculation show that

$$A = \frac{\mathbf{b}^{\dagger}_{0,m'}(\tilde{\xi}, \tilde{\Xi})}{\mathbf{d}^{\dagger}(\tilde{\Xi})\mathbf{d}'^{\dagger}(\tilde{\xi})} = (1 - q^{-l'}) \prod_{i=1}^{l'-1} (1 - q^{-2i})\,.$$

Therefore we have proved Theorem 11.1, and hence Theorem 10.8, and have completed the proof of Theorem 10.9. □

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