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EQUIVALENT CONDITIONS OF A HARDY-TYPE INTEGRAL INEQUALITY RELATED TO THE EXTENDED RIEMANN ZETA FUNCTION

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Abstract. By the use of techniques of real analysis and weight functions, we obtain two lemmas and build a few equivalent conditions of a Hardy-type integral inequality with a non-homogeneous kernel, related to a parameter where the constant factor is expressed in terms of the extended Riemann zeta function. Meanwhile, a few equivalent conditions for two kinds of Hardy-type integral inequalities with the homogeneous kernel are deduced. We also consider the operator expressions.

1. Introduction

If

$$0 < \int_0^\infty f^2(x)dx < \infty \ \text{ and } \ 0 < \int_0^\infty g^2(y)dy < \infty\,,$$

then we have the following Hilbert integral inequality (cf. [14]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \tag{1.1}$$

where, the constant factor π is the best possible.

In 1925, by introducing one pair of conjugate exponents (p, q), Hardy [2] gave an

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extension of (1.1) as follows:
For $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0,$

$$0 < \int_0^\infty f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty g^q(y)dy < \infty,$$

we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \quad (1.2)$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1.1) and (1.2) are important in Mathematical Analysis and its applications (cf. [3, 11, 19, 12, 13]).

In 1934, Hardy et al. presented the following extension of (1.2):
If $k_1(x, y)$ is a non-negative homogeneous function of degree -1,

$$k_p = \int_0^\infty k_1(u, 1)u^{\frac{-1}{p}}du \in \mathbf{R}_+ = (0, \infty),$$

then we have the following Hardy–Hilbert-type integral inequality

$$\int_0^\infty \int_0^\infty k_1(x, y)f(x)g(y)dxdy < k_p \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \quad (1.3)$$

where, the constant factor k_p is the best possible (cf. [3], Theorem 319). Additionally, the following Hilbert-type integral inequality with the non-homogeneous kernel is proved:
If $h(u) > 0$,

$$\phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbf{R}_+,$$

then

$$\begin{aligned} &\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy \\ < \ &\phi(\frac{1}{p}) \left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \end{aligned} \quad (1.4)$$

where, the constant factor $\phi(\frac{1}{p})$ is the best possible (cf. [3], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang provided an extension of (1.1) with the kernel $\frac{1}{(x+y)^\lambda}$ (cf. [20]). In 2004, Yang [21] introduced another pair conjugate exponents (r, s), and gave the following extension of (1.2):
If $\lambda > 0, r > 1, \frac{1}{r} + \frac{1}{s} = 1, f(x), g(y) \geq 0,$

$$0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin(\pi/r)} \left[\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)dy\right]^{\frac{1}{q}}, \quad (1.5)$$

where, the constant factor $\frac{\pi}{\lambda \sin(\pi/r)}$ is the best possible.
For $\lambda = 1, r = q, s = p$, (1.5) reduces to (1.2); For $\lambda = 1, r = p, s = q$, (1.5) reduces to the dual form of (1.2) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty x^{p-2} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{q-2} g^q(y)dy\right)^{\frac{1}{q}}. \quad (1.6)$$

For $p = q = 2$, both (1.2) and (1.6) reduce to (1.1).

In [22], inequalities (1.2) and (1.5) are also extended with the kernel $\frac{1}{(x+y)^\lambda}$. Moreover, Krnić et al. [6]-[16] proved some extensions and particular cases of (1.2), (1.3) and (1.4) with parameters.

In 2009, Yang (cf. [17, 19]) gave an extension of (1.3), (1.5). Namely he showed that:
If $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R} = (-\infty, \infty)$, $k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x, y) \quad (u, x, y > 0),$$

with

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1) u^{\lambda_1 - 1} du \in \mathbf{R}_+ = (0, \infty),$$

then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dxdy < k(\lambda_1) \left(\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{q(1-\lambda_2)-1} g^q(y)dy\right)^{\frac{1}{q}}, \quad (1.7)$$

where, the constant factor $k(\lambda_1)$ is the best possible.
For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (1.7) reduces to (1.3) and for $k_\lambda(x, y) = \frac{1}{x^\lambda + y^\lambda}$ $(\lambda > 0)$, (1.7) reduces to (1.5). Furthermore, the following extension of (1.4) was proved:

$$\int_0^\infty \int_0^\infty h(xy) f(x) g(y) dxdy < \phi(\sigma) \left(\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy\right)^{\frac{1}{q}}, \quad (1.8)$$

where, the constant factor $\phi(\sigma)$ is the best possible (cf. [18]).
For $\sigma = \frac{1}{p}$, (1.8) reduces to (1.4).

Some inequalities equivalent to (1.7) and (1.8) are considered in [19] . In 2013, Yang [18] also studied the equivalency between (1.7) and (1.8) by the addition of a particular condition. In 2017, Hong [5] studied an equivalent condition between (1.7) with a few parameters.

Remark 1.1. (cf. [18]) If $h(xy) = 0$, for $xy > 1$, then

$$\phi(\sigma) = \int_0^1 h(u)u^{\sigma-1}du = \phi_1(\sigma) \in \mathbf{R}_+,$$

and (1.8) reduces to the following Hardy-type integral inequality with the non-homogeneous kernel:

$$\begin{aligned} &\int_0^\infty g(y)\left(\int_0^{\frac{1}{y}} h(xy)f(x)dx\right)dy \\ < \quad &\phi_1(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right)^{\frac{1}{q}}; \end{aligned} \tag{1.9}$$

if $h(xy) = 0$, for $xy < 1$, then

$$\phi(\sigma) = \int_1^\infty h(u)u^{\sigma-1}du = \phi_2(\sigma) \in \mathbf{R}_+,$$

and (1.8) reduces to another kind of Hardy-type integral inequality with the non-homogeneous kernel, namely:

$$\begin{aligned} &\int_0^\infty g(y)\left(\int_{\frac{1}{y}}^\infty h(xy)f(x)dx\right)dy \\ < \quad &\phi_2(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right)^{\frac{1}{q}}. \end{aligned} \tag{1.10}$$

In this paper, by the use of techniques of real analysis and weight functions, we obtain two lemmas and build a few equivalent conditions of a Hardy-type integral inequalities with the non-homogeneous kernel

$$\frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|}\ (\beta, \lambda > 0)\,,$$

related to a parameter where the constant factor is expressed in terms of the extended Riemann zeta function. Meanwhile, a few equivalent conditions of two kinds of Hardy-type integral inequalities with the homogeneous kernel are deduced. We also consider the operator expressions.

2. Two lemmas

For $\beta, \lambda > 0$, set

$$h(u) := \frac{|\ln u|^\beta}{|u^\lambda - 1|}\,(u > 0).$$

For $\sigma > 0$, by the Lebesgue term by term integration theorem (cf. [7]), we obtain

$$\begin{aligned} k_1(\sigma) \quad : \quad &= \int_0^1 \frac{|\ln u|^\beta}{|u^\lambda - 1|} u^{\sigma-1} du = \int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma-1} du \\ &= \int_0^1 (-\ln u)^\beta \sum_{k=0}^{\infty} u^{k\lambda+\sigma-1} du = \sum_{k=0}^{\infty} \int_0^1 (-\ln u)^\beta u^{k\lambda+\sigma-1} du. \end{aligned}$$

Setting $v = (k\lambda + \sigma)(-\ln u)$ in the above integral, we derive that

$$\begin{aligned} k_1(\sigma) &= \sum_{k=0}^{\infty} \frac{1}{(k\lambda+\sigma)^{\beta+1}} \int_0^\infty v^\beta e^{-v} dv \\ &= \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}} \zeta(\beta+1, \frac{\sigma}{\lambda}) \in \mathbf{R}_+, \end{aligned} \tag{2.1}$$

where,

$$\Gamma(\eta) := \int_0^\infty v^{\eta-1} e^{-v} dv \, (\eta > 0)$$

is the gamma function and

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \, (Re(s) > 1, a > 0)$$

is the extended Riemann zeta function ($\zeta(s,1) = \zeta(s)$ is the Riemann zeta function) (cf. [15]). In particular, for $\sigma = \lambda$, we have

$$k_1(\sigma) = \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}} \zeta(\beta+1).$$

For $0 < \sigma < \lambda$, $\mu = \lambda - \sigma > 0$, setting $v = \frac{1}{u}$, by (2.1), we deduce that

$$\begin{aligned} k_2(\sigma) \quad : \quad &= \int_1^\infty \frac{|\ln u|^\beta}{|u^\lambda - 1|} u^{\sigma-1} du \\ &= \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma-1} du = \int_0^1 \frac{(-\ln v)^\beta}{1 - v^\lambda} v^{\mu-1} dv \\ &= \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}} \zeta(\beta+1, \frac{\mu}{\lambda}) = k_1(\mu) \in \mathbf{R}_+. \end{aligned} \tag{2.2}$$

In the sequel, we will always assume that

$$p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \quad \text{and} \quad \sigma_1, \mu_1 \in \mathbf{R}.$$

Lemma 2.1. *If $\beta, \sigma, \lambda > 0$, there exists a constant M_1 such that for any non-negative measurable functions $f(x)$, $g(y)$ with $x, y \in (0, \infty)$, the inequality*

$$\begin{aligned} &\int_0^\infty g(y) \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right] dy \\ \leq \ & M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned} \tag{2.3}$$

holds. Then $\sigma_1 = \sigma$, and $M_1 \geq k_1(\sigma)$.

Proof. If $\sigma_1 > \sigma$, then for $n \geq \frac{1}{\sigma_1 - \sigma}$ $(n \in \mathbf{N})$, we set:

$$f_n(x) := \begin{cases} x^{\sigma + \frac{1}{pn} - 1}, 0 < x \leq 1 \\ 0, x > 1 \end{cases}, \quad g_n(y) := \begin{cases} 0, 0 < y < 1 \\ y^{\sigma_1 - \frac{1}{qn} - 1}, y \geq 1 \end{cases},$$

and obtain

$$\begin{aligned} J_1 \quad : \quad &= \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y)dy\right]^{\frac{1}{q}} \\ &= \left(\int_0^1 x^{\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

For $u = xy$, we derive that

$$\begin{aligned} I_1 \quad : \quad &= \int_0^\infty g_n(y) \left(\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f_n(x)dx\right) dy \\ &= \int_1^\infty \left(\int_0^{\frac{1}{y}} \frac{(-\ln xy)^\beta}{1-(xy)^\lambda} x^{\sigma + \frac{1}{pn} - 1}dx\right) y^{\sigma_1 - \frac{1}{qn} - 1}dy \\ &= \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1}dy \int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma + \frac{1}{pn} - 1}du, \end{aligned}$$

and thus by (2.3), we have

$$\begin{aligned} &\int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1}dy \int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma + \frac{1}{pn} - 1}du \\ &= I_1 \leq M_1 J_1 = M_1 n < \infty. \end{aligned} \tag{2.4}$$

Since

$$(\sigma_1 - \sigma) - \frac{1}{n} \geq 0,$$

it follows that

$$\int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1}dy = \infty.$$

By (2.4), in view of

$$\int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma + \frac{1}{pn} - 1}du > 0,$$

we get that $\infty < \infty$, which is a contradiction.

If $\sigma_1 < \sigma$, then for $n \geq \frac{1}{\sigma - \sigma_1}$ $(n \in \mathbf{N})$, we set:

$$\widetilde{f}_n(x) := \begin{cases} 0, 0 < x < 1 \\ x^{\sigma - \frac{1}{pn} - 1}, x \geq 1 \end{cases}, \quad \widetilde{g}_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1}, 0 < y \leq 1 \\ 0, y > 1 \end{cases},$$

and find

$$\begin{aligned} \widetilde{J}_1 \quad : \quad &= \left[\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}} \\ &= \left(\int_1^\infty x^{-\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

For $u = xy$, we obtain

$$\begin{aligned}
\widetilde{I}_1 \quad : \quad &= \int_0^\infty \widetilde{f}_n(x) \left[\int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \widetilde{g}_n(y) dy \right] dx \\
&= \int_1^\infty \left[\int_0^{\frac{1}{x}} \frac{(-\ln xy)^\beta}{1-(xy)^\lambda} y^{\sigma_1 + \frac{1}{qn} - 1} dy \right] x^{\sigma - \frac{1}{pn} - 1} dx \\
&= \int_1^\infty x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx \int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma_1 + \frac{1}{qn} - 1} du,
\end{aligned}$$

and thus by Fubini's theorem (cf. [7]) and (2.3), we have

$$\begin{aligned}
&\int_1^\infty x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx \int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma_1 + \frac{1}{qn} - 1} du \\
= \quad &\widetilde{I}_1 = \int_0^\infty \widetilde{g}_n(y) \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta \widetilde{f}_n(x)}{|(xy)^\lambda - 1|} dx \right] dy \le M_1 \widetilde{J}_1 = M_1 n. \qquad (2.5)
\end{aligned}$$

Since

$$(\sigma - \sigma_1) - \frac{1}{n} \ge 0,$$

it follows that

$$\int_1^\infty x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx = \infty.$$

By (2.5), in view of

$$\int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma_1 + \frac{1}{qn} - 1} du > 0,$$

we deduce that $\infty < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, inequality (2.5) is reduced to the following

$$M_1 \ge \int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma + \frac{1}{qn} - 1} du. \qquad (2.6)$$

Since

$$\{\frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma + \frac{1}{qn} - 1}\}_{n=1}^\infty$$

is nonnegative and increasing in $(0, 1]$, by Levi's theorem (cf. [7]), we get

$$\begin{aligned}
M_1 \quad &\ge \quad \lim_{n\to\infty} \int_0^1 \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma + \frac{1}{qn} - 1} du \\
&= \int_0^1 \lim_{n\to\infty} \frac{(-\ln u)^\beta}{1-u^\lambda} u^{\sigma + \frac{1}{qn} - 1} du = k_1(\sigma).
\end{aligned}$$

The lemma is proved. $\square$ $\square$

Lemma 2.2. *If $\beta > 0, 0 < \sigma < \lambda$, there exists a constant M_2, such that for any non-negative measurable functions $f(x)$, $g(y)$ with $x, y \in (0, \infty)$, the inequality*

$$\begin{aligned} &\int_0^\infty g(y)\left[\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right] dy \\ \leq\ & M_2 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}} \end{aligned} \tag{2.7}$$

holds. Then $\sigma_1 = \sigma$, and $M_2 \geq k_2(\sigma)$.

Proof. If $\sigma_1 < \sigma$, then for $n \geq \frac{1}{\sigma - \sigma_1}$ $(n \in \mathbf{N})$, we consider two functions $\widetilde{f}_n(x)$ and $\widetilde{g}_n(y)$ as in Lemma 1, and obtain that

$$\widetilde{J}_1 = \left[\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}} = n.$$

Setting $u = xy$, we obtain

$$\begin{aligned} \widetilde{I}_2 \ &: = \int_0^\infty \widetilde{g}_n(y)\left[\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \widetilde{f}_n(x)dx\right] dy \\ &= \int_0^1 \left[\int_{\frac{1}{y}}^\infty \frac{(\ln xy)^\beta}{(xy)^\lambda - 1} x^{\sigma - \frac{1}{pn} - 1} dx\right] y^{\sigma_1 + \frac{1}{qn} - 1} dy \\ &= \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{pn} - 1} du, \end{aligned}$$

and then by (2.7), we obtain

$$\begin{aligned} &\int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{pn} - 1} du \\ =\ & \widetilde{I}_2 \leq M_2 \widetilde{J}_1 = M_2 n < \infty. \end{aligned} \tag{2.8}$$

Since

$$(\sigma_1 - \sigma) + \frac{1}{n} \leq 0,$$

it follows that

$$\int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy = \infty.$$

By (2.8), in view of

$$\int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{pn} - 1} du > 0,$$

we deduce that $\infty < \infty$, which is a contradiction.

If $\sigma_1 > \sigma$, then for $n \geq \frac{1}{\sigma_1 - \sigma}$ $(n \in \mathbf{N})$, we consider two functions $f_n(x)$ and $g_n(y)$ as in Lemma 1, and find

$$J_1 = \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y)dy\right]^{\frac{1}{q}} = n.$$

Setting $u = xy$, we obtain

$$\begin{aligned} I_2 \quad : \quad &= \int_0^\infty f_n(x)\left[\int_{\frac{1}{x}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} g_n(y)dy\right]dx \\ &= \int_0^1 \left[\int_{\frac{1}{x}}^\infty \frac{(\ln xy)^\beta}{(xy)^\lambda - 1} y^{\sigma_1 - \frac{1}{qn} - 1}dy\right] x^{\sigma + \frac{1}{pn} - 1}dx \\ &= \int_0^1 x^{(\sigma-\sigma_1)+\frac{1}{n}-1}dx \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma_1 - \frac{1}{qn} - 1}du, \end{aligned}$$

and then by Fubini's theorem (cf. [7]) and (2.7), we obtain

$$\begin{aligned} &\int_0^1 x^{(\sigma-\sigma_1)+\frac{1}{n}-1}dx \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma_1 - \frac{1}{qn} - 1}du \\ = \quad & I_2 = \int_0^\infty g_n(y)\left[\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta f_n(x)}{|(xy)^\lambda - 1|}dx\right]dy \le M_2 J_1 = M_2 n. \end{aligned} \tag{2.9}$$

Since

$$(\sigma - \sigma_1) + \frac{1}{n} \le 0,$$

it follows that

$$\int_0^1 x^{(\sigma-\sigma_1)+\frac{1}{n}-1}dx = \infty.$$

By (2.9), in view of

$$\int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma_1 - \frac{1}{qn} - 1}du > 0,$$

we deduce that $\infty < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, inequality (2.9) reduces to the following

$$M_2 \ge \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{qn} - 1}du. \tag{2.10}$$

Since

$$\{\frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{qn} - 1}\}_{n=1}^\infty$$

is nonnegative and increasing in $[1, \infty)$, again by the application of Levi's theorem (cf. [7]), we obtain that

$$\begin{aligned} M_2 \quad &\ge \quad \lim_{n\to\infty} \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{qn} - 1}du \\ &= \int_1^\infty \lim_{n\to\infty} \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{qn} - 1}du = k_2(\sigma). \end{aligned}$$

This completes the proof of the lemma. □ □

3. Main results and corollaries

Theorem 3.1. *If $\beta, \sigma, \lambda > 0$, then the following conditions are equivalent:*

(i) There exists a constant M_1, such that for any $f(x) \geq 0\,(x \in (0,\infty))$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following Hardy-type integral inequality of the first kind with the non-homogeneous kernel:

$$\begin{aligned} J \quad : \quad &= \left\{\int_0^\infty y^{p\sigma_1-1}\left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right]^p dy\right\}^{\frac{1}{p}} \\ &< \quad M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}; \end{aligned} \tag{3.1}$$

(ii) There exists a constant M_1, such that for any $f(x), g(y) \geq 0\,(x, y \in (0,\infty))$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy < \infty,$$

we have the following inequality:

$$\begin{aligned} I \quad : \quad &= \int_0^\infty g(y)\left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right] dy \\ &< \quad M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}; \end{aligned} \tag{3.2}$$

(iii) $\sigma_1 = \sigma$.

If Condition (iii) holds true, then $M_1 \geq k_1(\sigma)$ and the constant factor

$$M_1 = k_1(\sigma) = \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}} \zeta(\beta+1, \frac{\sigma}{\lambda})$$

in (3.1) and (3.2) is the best possible.

Proof.
"$(i) \Rightarrow (ii)$". By Hölder's inequality (cf. [8]), we have

$$\begin{aligned} I \quad &= \quad \int_0^\infty \left[y^{\sigma_1 - \frac{1}{p}} \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right] \left(y^{\frac{1}{p}-\sigma_1} g(y)\right) dy \\ &\leq \quad J \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{3.3}$$

Thus by (3.1), we deduce (3.2).
"$(ii) \Rightarrow (iii)$". By Lemma 1, we have $\sigma_1 = \sigma$.

"$(iii) \Rightarrow (i)$". Setting $u = xy$, we obtain the following weight function:

$$\begin{aligned}
\omega_1(\sigma, y) \quad &: \quad = y^{\sigma} \int_0^{\frac{1}{y}} \frac{|\ln xy|^{\beta}}{|(xy)^{\lambda} - 1|} x^{\sigma - 1} dx \\
&= \int_0^1 \frac{(-\ln u)^{\beta}}{1 - u^{\lambda}} u^{\sigma - 1} du = k_1(\sigma) \ (y > 0). \qquad (3.4)
\end{aligned}$$

By the weighted Hölder inequality and (3.4), for $y \in (0, \infty)$, we have

$$\begin{aligned}
&\left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^{\beta}}{|(xy)^{\lambda} - 1|} f(x) dx\right]^p \\
&= \left\{\int_0^{\frac{1}{y}} \frac{|\ln xy|^{\beta}}{|(xy)^{\lambda} - 1|} \left[\frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x)\right] \left[\frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}}\right] dx\right\}^p \\
&\leq \int_0^{\frac{1}{y}} \frac{|\ln xy|^{\beta}}{|(xy)^{\lambda} - 1|} \frac{y^{\sigma-1} f^p(x)}{x^{(\sigma-1)p/q}} dx \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^{\beta}}{|(xy)^{\lambda} - 1|} \frac{x^{\sigma-1} dx}{y^{(\sigma-1)q/p}}\right]^{p-1} \\
&= \left[\omega_1(\sigma, y) y^{q(1-\sigma)-1}\right]^{p-1} \int_0^{\frac{1}{y}} \frac{|\ln xy|^{\beta}}{|(xy)^{\lambda} - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\
&= (k_1(\sigma))^{p-1} y^{-p\sigma+1} \int_0^{\frac{1}{y}} \frac{|\ln xy|^{\beta}}{|(xy)^{\lambda} - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx. \qquad (3.5)
\end{aligned}$$

If (3.5) obtains the form of equality for a $y \in (0, \infty)$, then (cf. [8]) there exist constants A and B, such that they are not all zero and

$$A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \quad \text{a.e. in} \quad \mathbf{R}_+.$$

Let $A \neq 0$ (otherwise $B = A = 0$). It follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \quad \text{a.e. in} \quad \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx < \infty.$$

Hence, (3.5) assumes the form of strict inequality. Then for $\sigma_1 = \sigma$, by (3.5) and Fubini's theorem (cf. [7]), we obtain

$$\begin{aligned} J &< (k_1(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \right] dy \right\}^{\frac{1}{p}} \\ &= (k_1(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \left[\int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)(p-1)}} dy \right] f^p(x) dx \right\}^{\frac{1}{p}} \\ &= (k_1(\sigma))^{\frac{1}{q}} \left[\int_0^\infty \omega_1(\sigma, x) x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \\ &= k_1(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Setting $M_1 \geq k_1(\sigma)$, (3.1) follows.

Therefore, Condition (i), Condition (ii) and Condition (iii) are equivalent.

When Condition (iii) is satisfied, if there exists a constant factor $M_1 \leq k_1(\sigma)$, such that (3.2) is valid, then by Lemma 1, we have $M_1 \geq k_1(\sigma)$. Hence, the constant factor $M_1 = k_1(\sigma)$ in (3.2) is the best possible. The constant factor $M_1 = k_1(\sigma)$ in (3.1) is still the best possible. Otherwise, by (3.3) (for $\sigma_1 = \sigma$), we would conclude that the constant factor $M_1 = k_1(\sigma)$ in (3.2) is not the best possible. □ □

Setting

$$y = \frac{1}{Y},\ G(Y) = Y^{\lambda-2} g(\frac{1}{Y}),\ \mu_1 = \lambda - \sigma_1$$

in Theorem 1, then replacing Y $(G(Y))$ by y $(g(y))$, since $\mu = \lambda - \sigma$, we have

Corollary 3.2. *If $\beta, \sigma, \lambda > 0$, then the following conditions are equivalent:*

(i) There exists a constant M_1, such that for any $f(x) \geq 0$ $(x \in (0,\infty))$, for which

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

the following Hardy-type inequality of the first kind with the homogeneous kernel is satisfied:

$$\begin{aligned} &\left\{ \int_0^\infty y^{p\mu_1 - 1} \left[\int_0^y \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ &< M_1 \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned} \tag{3.6}$$

(ii) There exists a constant M_1, such that for any $f(x), g(y) \geq 0\,(x, y \in (0,\infty))$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty \ \ \text{and} \ \ 0 < \int_0^\infty y^{q(1-\mu_1)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned}
&\int_0^\infty g(y)\left[\int_0^y \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x)dx\right]dy \\
< \;& M_1\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\mu_1)-1} g^q(y)dy\right]^{\frac{1}{q}}; \qquad (3.7)
\end{aligned}$$

(iii) $\mu_1 = \mu$.

If Condition (iii) holds true, then we have $M_1 \geq k_1(\sigma)$, *and the constant* $M_1 = k_1(\sigma)$ *in (3.6) and (3.7) is the best possible.*

Remark 3.3. On the other hand, setting

$$y = \frac{1}{Y},\ G(Y) = Y^{\lambda-2} g(\frac{1}{Y}),\ \mu_1 = \lambda - \sigma_1$$

in Corollary 1, and replacing Y $(G(Y))$ by y $(g(y))$, we have Theorem 1. Hence, we conclude that Theorem 1 and Corollary 1 are equivalent.

Similarly, for $0 < \sigma = \lambda - \mu < \lambda$, we obtain the following weight function:

$$\begin{aligned}
\omega_2(\sigma, y) \; : &= y^\sigma \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta x^{\sigma-1}}{|(xy)^\lambda - 1|}dx \\
&= \int_1^\infty \frac{\ln^\beta u}{u^\lambda - 1} u^{\sigma-1}du = k_2(\sigma)\ (y > 0),
\end{aligned}$$

and thus in view of Lemma 2 and similarly to the way we showed Theorem 1, we deduce the following theorem.

Theorem 3.4. *If* $\beta > 0$, $0 < \sigma = \lambda - \mu < \lambda$, *then the following conditions are equivalent:*

(i) There exists a constant M_2, *such that for any* $f(x) \geq 0\,(x \in (0,\infty))$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following Hardy-type inequality of the second kind with the non-homogeneous kernel:

$$\begin{aligned}
&\left\{\int_0^\infty y^{p\sigma_1-1}\left[\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right]^p dy\right\}^{\frac{1}{p}} \\
< \;& M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}; \qquad (3.8)
\end{aligned}$$

(ii) There exists a constant M_2, *such that for any* $f(x), g(y) \geq 0\,(x, y \in (0,\infty))$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty \quad \textit{and} \quad 0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy < \infty,$$

we have the following inequality:

$$\int_0^\infty g(y)\left[\int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right] dy$$
$$< \quad M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}; \tag{3.9}$$

(iii) $\sigma_1 = \sigma$.

If Condition (iii) holds true, then we have $M_2 \geq k_2(\sigma)$, *and the constant factor*

$$M_2 = k_2(\sigma) = \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}}\zeta(\beta+1,\frac{\mu}{\lambda}) = k_1(\mu)$$

in (3.8) and (3.9) is the best possible.

Setting

$$y = \frac{1}{Y},\ G(Y) = Y^{\lambda-2} g(\frac{1}{Y}),\ \mu_1 = \lambda - \sigma_1$$

in Theorem 2, and replacing Y $(G(Y))$ by y $(g(y))$, we get

Corollary 3.5. *If* $\beta > 0,\ 0 < \sigma = \lambda - \mu < \lambda$, *then the following conditions are equivalent:*

(i) There exists a constant M_2, *such that for any* $f(x) \geq 0\,(x \in (0,\infty))$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we obtain the following Hardy-type inequality of the second kind with the homogeneous kernel:

$$\left\{\int_0^\infty y^{p\mu_1-1}\left[\int_y^\infty \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x)dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$< \quad M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}; \tag{3.10}$$

(ii) There exists a constant M_2, *such that for any* $f(x), g(y) \geq 0\,(x, y \in (0,\infty))$, *satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty \quad \textit{and} \quad 0 < \int_0^\infty y^{q(1-\mu_1)-1} g^q(y)dy < \infty,$$

we derive the following inequality:

$$\int_0^\infty g(y)\left[\int_y^\infty \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x)dx\right] dy$$
$$< \quad M_2\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\mu_1)-1} g^q(y)dy\right]^{\frac{1}{q}}; \tag{3.11}$$

(iii) $\mu_1 = \mu$.

If Condition (iii) holds true, then we have $M_2 \geq k_2(\sigma)$, *and the constant* $M_2 = k_2(\sigma) = k_1(\mu)$ *in (3.10) and (3.11) is the best possible.*

Remark 3.6. Similarly, Theorem 2 and Corollary 2 are equivalent.

4. Operator expressions

For $\sigma, \lambda > 0, \mu = \lambda - \sigma$, we set the following functions:

$$\varphi(x) := x^{p(1-\sigma)-1},\ \psi(y) := y^{q(1-\sigma)-1},\ \phi(y) := y^{q(1-\mu)-1},$$

wherefrom,

$$\psi^{1-p}(y) = y^{p\sigma-1},\ \phi^{1-p}(y) = y^{p\mu-1} \quad (x, y \in \mathbf{R}_+).$$

Define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}_+) := \left\{ f : ||f||_{p,\varphi} := \left(\int_0^\infty \varphi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

wherefrom,

$$\begin{aligned} L_{q,\psi}(\mathbf{R}_+) &= \left\{ g : ||g||_{q,\psi} := \left(\int_0^\infty \psi(y)|g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}, \\ L_{q,\phi}(\mathbf{R}_+) &= \left\{ g : ||g||_{q,\phi} := \left(\int_0^\infty \phi(y)|g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\}, \end{aligned}$$

$$\begin{aligned} L_{p,\psi^{1-p}}(\mathbf{R}_+) &= \left\{ h : ||h||_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{q,\phi^{1-p}}(\mathbf{R}_+) &= \left\{ h : ||h||_{p,\phi^{1-p}} = \left(\int_0^\infty \phi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

(a) In view of Theorem 1 (for $\sigma_1 = \sigma$), where $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_1(y) := \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx \ (y \in \mathbf{R}_+),$$

by (3.1), we obtain that

$$||h_1||_{p,\psi^{1-p}} = \left[\int_0^\infty \psi^{1-p}(y) h_1^p(y) dy \right]^{\frac{1}{p}} < M_1 ||f||_{p,\varphi} < \infty. \tag{4.1}$$

Definition 4.1. We define a Hardy-type integral operator of the first kind with the non-homogeneous kernel

$$T_1^{(1)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\psi^{1-p}}(\mathbf{R}_+)$$

as follows:
For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_1^{(1)} f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying

$$T_1^{(1)} f(y) = h_1(y),$$

for any $y \in \mathbf{R}_+$.

In view of (4.1), it follows that

$$||T_1^{(1)}f||_{p,\psi^{1-p}} = ||h_1||_{p,\psi^{1-p}} \leq M_1||f||_{p,\varphi},$$

and thus the operator $T_1^{(1)}$ is bounded satisfying

$$||T_1^{(1)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_1^{(1)}f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq M_1.$$

If we define the formal inner product of $T_1^{(1)}f$ and g as follows:

$$(T_1^{(1)}f, g) := \int_0^\infty \left[\int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx\right] g(y)dy,$$

then we can rewrite Theorem 1 in the following manner:

Theorem 4.2. *For $\beta, \sigma, \lambda > 0$, the following conditions are equivalent:*

(i) There exists a constant M_1, such that for any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$, $||f||_{p,\varphi} > 0$, the following inequality holds true:

$$||T_1^{(1)}f||_{p,\psi^{1-p}} < M_1||f||_{p,\varphi}; \tag{4.2}$$

(ii) There exists a constant M_1, such that for any

$$f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+), g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\varphi}, ||g||_{q,\psi} > 0,$$

the following inequality holds true

$$(T_1^{(1)}f, g) < M_1||f||_{p,\varphi}||g||_{q,\psi}. \tag{4.3}$$

We still have $||T_1^{(1)}|| = k_1(\sigma) \leq M_1$.

(b) In view of Corollary 1 (for $\mu_1 = \mu$), where $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_2(y) := \int_0^y \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x)dx \; (y \in \mathbf{R}_+),$$

by (3.6), we deduce that

$$||h_2||_{p,\phi^{1-p}} = \left[\int_0^\infty \phi^{1-p}(y)h_2^p(y)dy\right]^{\frac{1}{p}} < M_1||f||_{p,\varphi} < \infty. \tag{4.4}$$

Definition 4.3. We define a Hardy-type integral operator of the first kind with the homogeneous kernel

$$T_1^{(2)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\phi^{1-p}}(\mathbf{R}_+)$$

as follows:
For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation

$$T_1^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R}_+),$$

satisfying $T_1^{(2)}f(y) = h_2(y)$, for any $y \in \mathbf{R}_+$.

In view of (4.4), it follows that

$$||T_1^{(2)}f||_{p,\phi^{1-p}} = ||h_2||_{p,\phi^{1-p}} \leq M_1||f||_{p,\varphi},$$

and thus the operator $T_1^{(2)}$ is bounded satisfying

$$||T_1^{(2)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_1^{(2)}f||_{p,\phi^{1-p}}}{||f||_{p,\varphi}} \leq M_1.$$

If we define the formal inner product of $T_1^{(2)}f$ and g as

$$(T_1^{(2)}f, g) := \int_0^\infty \left[\int_0^y \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x)dx\right] g(y)dy,$$

then Corollary 1 assumes the following form:

Corollary 4.4. *For $\beta, \sigma, \lambda > 0$, the following conditions are equivalent:*

(i) There exists a constant M_1, such that for any $f(x) \geq 0$, $f \in L_{p,\varphi}(\mathbf{R}_+)$, $||f||_{p,\varphi} > 0$, the following inequality holds true:

$$||T_1^{(2)}f||_{p,\phi^{1-p}} < M_1||f||_{p,\varphi}; \tag{4.5}$$

(ii) There exists a constant M_1, such that for any $f(x), g(y) \geq 0$, $f \in L_{p,\varphi}(\mathbf{R}_+)$, $g \in L_{q,\phi}(\mathbf{R}_+)$, $||f||_{p,\varphi}$, $||g||_{q,\phi} > 0$, the following inequality holds true:

$$(T_1^{(2)}f, g) < M_1||f||_{p,\varphi}||g||_{q,\phi}. \tag{4.6}$$

We still have $||T_1^{(2)}|| = k_1(\sigma) \leq M_1$.

Remark 4.5. Similarly, Theorem 3 and Corollary 3 are equivalent.

(c) In view of Theorem 2 (for $\sigma_1 = \sigma$), where $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$H_1(y) := \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx \; (y \in \mathbf{R}_+),$$

by (3.8), we derive that

$$||H_1||_{p,\psi^{1-p}} = \left[\int_0^\infty \psi^{1-p}(y)H_1^p(y)dy\right]^{\frac{1}{p}} < M_2||f||_{p,\varphi} < \infty. \tag{4.7}$$

Definition 4.6. We define a Hardy-type integral operator of the second kind with the non-homogeneous kernel

$$T_2^{(1)} : L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\psi^{1-p}}(\mathbf{R}_+)$$

as follows:
For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_2^{(1)}f = H_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying $T_2^{(1)}f(y) = H_1(y)$, for any $y \in \mathbf{R}_+$.

In view of (4.7), it follows that

$$||T_2^{(1)}f||_{p,\psi^{1-p}} = ||H_1||_{p,\psi^{1-p}} \leq M_2||f||_{p,\varphi},$$

and thus the operator $T_2^{(1)}$ is bounded satisfying

$$||T_2^{(1)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_2^{(1)}f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq M_2.$$

If we define the formal inner product of $T_2^{(1)}f$ and g as follows:

$$(T_2^{(1)}f, g) := \int_0^\infty \left[\int_{\frac{1}{y}}^\infty \frac{(\ln xy)^\beta}{(xy)^\lambda - 1} f(x)dx\right] g(y)dy,$$

then Theorem 2 obtains the following form:

Theorem 4.7. *For $\beta > 0, 0 < \sigma = \lambda - \mu < \lambda$, the following conditions are equivalent:*

(i) There exists a constant M_2, such that for any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$, $||f||_{p,\varphi} > 0$, the following inequality holds true:

$$||T_2^{(1)}f||_{p,\psi^{1-p}} < M_2||f||_{p,\varphi}; \tag{4.8}$$

(ii) There exists a constant M_2, such that for any $f(x), g(y) \geq 0$, $f \in L_{p,\varphi}(\mathbf{R}_+)$, $g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\varphi}, ||g||_{q,\psi} > 0$, we have the following inequality:

$$(T_2^{(1)}f, g) < M_2||f||_{p,\varphi}||g||_{q,\psi}. \tag{4.9}$$

We still have $||T_2^{(1)}|| = k_2(\sigma) = k_1(\mu) \leq M_2$.

(d) In view of Corollary 2 (for $\mu_1 = \mu$), where $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$H_2(y) := \int_y^\infty \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x)dx \; (y \in \mathbf{R}_+),$$

by (3.10), we derive that

$$||H_2||_{p,\phi^{1-p}} = \left[\int_0^\infty \phi^{1-p}(y)H_2^p(y)dy\right]^{\frac{1}{p}} < M_2||f||_{p,\varphi} < \infty. \tag{4.10}$$

Definition 4.8. We define a Hardy-type integral operator of the second kind with the homogeneous kernel

$$T_2^{(2)} \; : \; L_{p,\varphi}(\mathbf{R}_+) \to L_{p,\phi^{1-p}}(\mathbf{R}_+)$$

as follows:
For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T_2^{(2)}f = H_2 \in L_{p,\phi^{1-p}}(\mathbf{R}_+)$, satisfying $T_2^{(2)}f(y) = H_2(y)$, for any $y \in \mathbf{R}_+$.

In view of (4.10), it follows that

$$||T_2^{(2)}f||_{p,\phi^{1-p}} = ||H_2||_{p,\phi^{1-p}} \leq M_2||f||_{p,\varphi},$$

and thus the operator $T_2^{(2)}$ is bounded satisfying

$$||T_2^{(2)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_2^{(2)}f||_{p,\phi^{1-p}}}{||f||_{p,\varphi}} \leq M_2.$$

If we define the formal inner product of $T_1^{(2)} f$ and g as follows:

$$(T_2^{(2)} f, g) := \int_0^\infty \left[\int_y^\infty \frac{[\ln(x/y)]^\beta}{x^\lambda - y^\lambda} f(x) dx \right] g(y) dy,$$

then Corollary 2 obtains the following form:

Corollary 4.9. *For $\beta > 0, 0 < \sigma = \lambda - \mu < \lambda$, the following conditions are equivalent:*

(i) There exists a constant M_2, such that for any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$, $||f||_{p,\varphi} > 0$, the following inequality holds true:

$$||T_2^{(2)} f||_{p,\phi^{1-p}} < M_2 ||f||_{p,\varphi}; \tag{4.11}$$

(ii) There exists a constant M_2, such that for any $f(x), g(y) \geq 0$, $f \in L_{p,\varphi}(\mathbf{R}_+)$, $g \in L_{q,\phi}(\mathbf{R}_+)$, $||f||_{p,\varphi}$, $||g||_{q,\phi} > 0$, we have the following inequality:

$$(T_2^{(2)} f, g) < M_2 ||f||_{p,\varphi} ||g||_{q,\phi}. \tag{4.12}$$

We still have $||T_2^{(2)}|| = k_2(\sigma) = k_1(\mu) \leq M_2$.

Remark 4.10. Similarly, Theorem 4 and Corollary 4 are equivalent.

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