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Rigidity of Ulam sets and sequences

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We give a number of results about families of Ulam sequences and sets, further exploring recent work on rigidity phenomena. For Ulam sequences, using elementary methods we give an upper bound on the density and prove regularity for various families of sequences. For Ulam sets, we consider extensions of classification work done by Kravitz and Steinerberger.

1. Introduction and main results

Introduction. Let $U(a, b)$ be the integer sequence that starts with two integers $0 < a < b$ and each subsequent term is the smallest integer that can be written as the sum of two distinct prior terms in exactly one way. Such sequences are known as Ulam sequences, in honor of Stanisław Ulam [1964], who first introduced the sequence $U(1, 2)$.

Considering the simplicity of the definition, surprisingly little is known about Ulam sequences, despite recent resurgence in interest — see [Gibbs 2015; Gibbs and McCranie 2017; Steinerberger 2017; Kravitz and Steinerberger 2017; Kuca 2018]. However, recent numerical evidence suggests that families of Ulam sequences have unexpected rigidity phenomena. In particular, in [Hinman et al. 2018], the authors make the following conjecture.

Conjecture 1.1. *There exist integer coefficients m_i, p_i, k_i, r_i such that for all integers $n \geq 4$,*

$$U(1, n) = \bigsqcup_{i=1}^{\infty} [m_i n + p_i, k_i n + r_i].$$

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While this conjecture is at present open, the authors did prove that it holds for all terms up to $50{,}000n$ — that is, for all $n \geq 4$,

$$\begin{aligned} U(1,n)\cap[1,50000n] = \{1\}\cup[n,2n]\cup\{2n+2\}\cup\{4n\}\cup[4n+2,5n-1] \\ \cup\{5n+1\}\cup[7n+3,8n+1]\cup\{10n+2\}\cup\{11n+2\} \\ \cup\cdots\cup\{49991n+6949\}\cup\{49993n+6950\}. \end{aligned}$$

This suggests that while individual Ulam sequences may be difficult to deal with, we may be able to get substantially better results about families of sequences. In the present paper, we investigate various results related to the rigidity conjecture above.

Summary of main results. We begin by revisiting the setting of Conjecture 1.1. By considering long runs of consecutive terms in an Ulam sequence $U(1,n)$, we prove the following elementary result.

Theorem 1.2. *Let m_i, p_i, k_i, r_i be integer coefficients as in Conjecture 1.1. Then for all i, we have $k_i - m_i = 0$ or 1, and $r_i \leq p_i$.*

Given a set of integers K, recall that its asymptotic density is defined as the constant

$$\delta(K) = \lim_{N\to\infty} \frac{\#(K\cap[1,N])}{N},$$

assuming that it exists. Using similar methodology as for Theorem 1.2, we also establish an upper bound on the asymptotic density for sequences $U(1,n)$.

Theorem 1.3. *The density of $U(1,n)$ is bounded above by $(n+1)/(3n)$.*

It should be noted that this is likely not a tight upper bound — asymptotically,

$$\frac{n+1}{3n} \approx \frac{1}{3},$$

but numerical data for $n \geq 4$ suggests that the actual density is $\approx \frac{1}{6}$. Furthermore, while our methods provide an upper bound, they do not provide any lower bound on the density — unfortunately, this is not surprising, as no positive lower bound on the density of sequences $U(1,n)$ is known at this time.

In Section 3, we turn to a question first studied by Queneau[1] [1972]: when is the Ulam sequence regular — that is, when is the sequence of differences between consecutive terms periodic? It was proved by Finch [1991; 1992a; 1992b] that if an Ulam sequence contains finitely many even terms, then it is regular. It is

[1] Raymond Queneau is better known for his work as a French poet and novelist, but he was interested in the role of mathematics in literature, which led him to cofound the Oulipo in 1960 [Motte 1998], together with chemical engineer and mathematician François Le Lionnais.

conjectured that $U(a, b)$ with $a < b$ coprime contains finitely many even terms if and only if

(1) $a = 2,\ b \geq 5$,

(2) $a = 4$,

(3) $a = 5,\ b = 6$, or

(4) $a \geq 6$ and a or b is even.

Schmerl and Spiegel [1994] proved the $a = 2,\ b \geq 5$ case; Cassaigne and Finch [1995] proved the case where $a = 4,\ b \equiv 1 \bmod 4$. It is worthwhile to note that the proofs of these results use a limited form of rigidity similar to Conjecture 1.1; furthermore, if some generalization of that conjecture holds for sequences $U(a, b)$ with $a \neq 1$, this would seem to give a means of proving that certain families of Ulam sequences are all regular — if you can show that some Ulam sequence $U(a, b)$ with b sufficiently large has only finitely many even terms, then this will be true of all subsequent Ulam sequences in that family. We prove a far more modest, but nevertheless interesting result that gives a semi-algorithm for determining whether an Ulam sequence is regular — unfortunately, it is only a semi-algorithm, as it is not ever guaranteed to halt. Using this, we were able to establish the following.

Theorem 1.4. *For integer pairs (a, b) given below, the sequence of differences between consecutive terms of $U(a, b)$ is eventually periodic*:

$$\begin{array}{ccccccc} (4,11), & (4,19), & (6,7), & (6,11), & (7,8), & (7,10), & (7,12), \\ (7,16), & (7,18), & (7,20), & (8,9), & (8,11), & (9,10), & (9,14), \\ (9,16), & (9,20), & (10,11), & (10,13), & (10,17), & (11,12), & (11,14), \\ (11,16), & (11,18), & (11,20), & (12,13), & (12,17), & (13,14). & \end{array}$$

In another direction, we also consider "Ulam-like" behavior and rigidity in higher dimensions. Using the terminology of [Kravitz and Steinerberger 2017], we define Ulam sets as follows.

Definition 1.5. Let $|\cdot|$ be a norm on $\mathbb{Z}^n$ that increases monotonically in each coordinate. A (k, n)-*Ulam set* $U(v_1, v_2, \ldots, v_k)$ is a recursively defined set that contains $v_1, v_2, \ldots, v_n \in \mathbb{Z}^n_{\geq 0}$ and each subsequent vector is the vector of smallest norm that can be written as a sum of two distinct vectors in the set in exactly one way. We shall say $U(v_1, v_2, \ldots, v_k)$ is *nondegenerate* if $v_i \notin U(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$ for every $1 \leq i \leq k$.

Two remarks are necessary here: first, it may appear that the definition of Ulam set depends on the choice of monotonically increasing norm $|\cdot|$. In fact, this is not so, as proved in [Kravitz and Steinerberger 2017]. Secondly, it may be unclear which vector is added if there is more than one of equal norm. However, by the above, this is irrelevant.

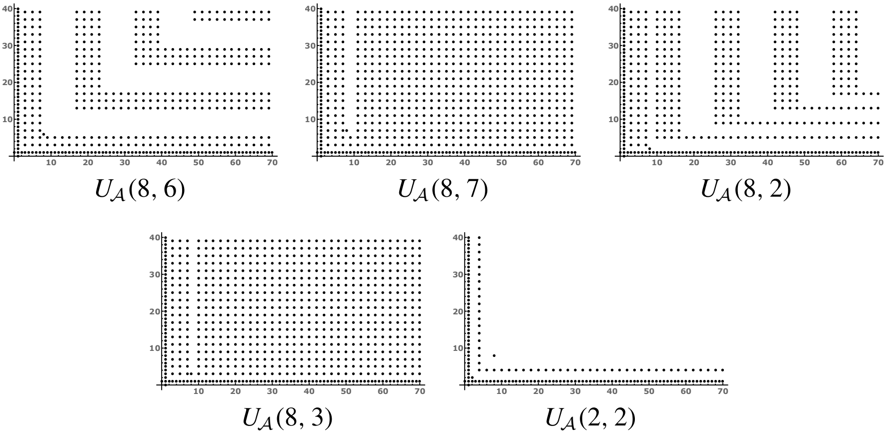

Figure 1. From left to right and top to bottom: sets $U_{\mathcal{A}}(v_1, v_2)$ of L, column-deleted, column-deleted L, shifted column-deleted, and exceptional type.

Contingent on some natural restrictions described in Section 4, we classify all $(3, 2)$-Ulam sets, showing that they necessarily belong to one of a finite number of different types, illustrated in Figure 1.

Theorem 1.6. *Let $\mathcal{U} = U((1, 0), (0, 1), (v_1, v_2))$ be a nondegenerate $(3, 2)$-Ulam set such that $v_1, v_2 \neq 0$. Then exactly one of the following is true of either $\mathcal{U}$ or its reflection about the line $y = x$:*

(1) $v_1, v_2 \in 2\mathbb{Z} \cap [4, \infty)$ *and $\mathcal{U}$ is of L type.*

(2) $v_1 \in 2\mathbb{Z},\ v_2 \in (1 + 2\mathbb{Z}) \cap [4, \infty)$, *and $\mathcal{U}$ is of column-deleted type.*

(3) $v_1 \in 2\mathbb{Z} \cap [4, \infty),\ v_2 = 2$, *and $\mathcal{U}$ is of column-deleted L type.*

(4) $v_1 \in 2\mathbb{Z},\ v_2 = 3$, *and $\mathcal{U}$ is of shifted column-deleted type.*

(5) $v_1 = v_2 = 2$ *and $\mathcal{U}$ is of exceptional type.*

See Section 4 for definitions of the various types of Ulam sets. Note that this is in a sense a higher-dimensional version of rigidity — we are varying the Ulam sets in some parameter, and outside of some odd exceptional cases when the norm of the vector is small, the parity of the coordinates of the added vector wholly determine the structure of the set. We also show that there are restrictions for more general $(k, 2)$-Ulam sets — in particular, in Section 5 we show that there is always a parity restriction.

Theorem 1.7. *Let $\mathcal{U} = U((1,0), (0,1), v_1, v_2, \dots, v_n)$ be a nondegenerate $(n+2, 2)$-Ulam set such that none of the v_i lie on the coordinate axes. Then there exists $(w_1, w_2) \in \mathbb{Z}^2_{\geq 0}$ such that for all $(m, n) \in \mathcal{U}$, if $m \geq w_1,\ n \geq w_2$, then $m = w_1 \bmod 2$, $n = w_2 \bmod 2$.*

Finally, in Section 6, we demonstrate that if the added vectors are not too small, then the corresponding Ulam set must be periodic in following sense.

Definition 1.8. A $(k, 2)$-Ulam set $\mathcal{U}$ is *eventually* (m, n)*-periodic* if there exists (m_0, n_0) such that for all $(a, b) \in \mathbb{Z}^2_{\geq 0}$ with $a \geq m_0$ and $b \geq n_0$ we have $(a, b) \in \mathcal{U}$ if and only if $(a+m, b+n) \in \mathcal{U}$. We call (m, n) a *period* of $\mathcal{U}$.

Theorem 1.9. *All* $(4, 2)$*-Ulam sets* $\mathcal{U} = U_{\mathcal{A}}(v_1, v_2)$ *with* $v_i = (x_i, y_i)$ *such that* $x_i, y_i \geq 4$ *are eventually periodic.*

2. Consecutive terms in sequences $U(1, n)$

Our main goal for this section is to find bounds on the runs of consecutive terms in the Ulam sequences $U(1, n)$. As an example, we prove the following theorem.

Theorem 2.1. *Let* $n \geq 2$ *and let* I *be a set of* $3n$ *consecutive positive integers greater than* $2n+2$*. Then* $|I \cap U(1, n)| \leq n+1$.

As an immediate corollary, this implies a bound on the density of $U(1, n)$.

Corollary 2.2.
$$\delta(U(1, n)) \leq \frac{n+1}{3n}.$$

Proof. Partition the first k integers greater than $2n+2$ into runs of $3n$ consecutive integers. Each such partition contains at most $n+1$ terms of $U(1, n)$. The proportion of Ulam numbers less than or equal to k is then no bigger than

$$\frac{(n+1)(k/(3n)+1)+2n+2}{k} = \frac{n+1}{3n}\left(1+\frac{1}{k}\right)+\frac{2n+2}{k}.$$

In the limit, we get the desired upper bound. □

We will give an improvement on this upper bound for the special case $U(1, 2)$ at the end of this section. Before we prove Theorem 2.1, we give a few useful lemmas, some of which are very useful in their own right.

Lemma 2.3. *Let* $n \geq 2$*. The first three intervals of* $U(1, n)$ *are* $\{1\}$, $[n, 2n]$, *and* $\{2n+2\}$.

Proof. Clearly, all elements of the form $n+i$ for $1 \leq i \leq n$ have the unique Ulam representation $n+i = (n+i-1)+1$. However, $2n+1 \notin U(1, n)$, because it has a second Ulam representation $n+(n+1)$. Finally, $2n+2 = n+(n+2)$, which is its only Ulam representation, and $2n+3 \notin U(1, n)$ since $2n+3 = (2n+2)+1 = n+(n+3)$. □

Lemma 2.4. *If* $a, a+k \in U(1, n)$ *for some* $1 \leq k \leq n$*, then* $[a+k+n, a+2n] \subset \mathbb{Z} \setminus U(1, n)$.

Proof. Every integer in this interval is of the form $a+k+n+i$ for $0 \leq i \leq n-k$; hence it has at least two Ulam representations: $(a+k)+(n+i)$ and $a+(n+k+i)$, where we have used the fact that $n+i, n+k+i \in [n, 2n]$, and hence are in the Ulam sequence by Lemma 2.3. □

As an immediate corollary of this lemma, we get a proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose that $m_i n + p_i, m_i n + p_i + 1 \in U(1, n)$. Then $(m_i + 1)n + (p_i + 1) \notin U(1, n)$. Therefore, $k_i - m_i = 0$ or 1 and $r_i \leq p_i$. □

Lemma 2.5. *Let* $1 \leq k \leq n$. *If* $[a, a+k] \subset U(1, n)$, *then* $[a+n+1, a+k+2n-1] \subset \mathbb{Z} \setminus U(1, n)$.

Proof. We have the partition

$$[a, a+k] = \bigcup_{i=0}^{k-1} [a+i, a+i+1],$$

and so it suffices to prove the claim with $k = 1$, which is an immediate corollary of Lemma 2.4. □

Lemma 2.5 shows that if there are long runs of consecutive elements in the Ulam sequence, then there must be a longer run of consecutive elements later on that do not belong to the Ulam sequence. With this observation in hand, we proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. If $I \cap U(1, n) = \varnothing$, we are done. Otherwise, let $a > 2n+2$ be the smallest element in $I \cap U(1, n)$. There are two cases: either $[a, a+n-1]$ contains at least two consecutive elements $u, u+1 \in U(1, n)$, or it does not. We consider these cases separately.

Case 1: Since we are given that $[a, a+n-1] \cap U(1, n)$ contains at least two consecutive elements, we can partition it into disjoint intervals

$$[a, a+n-1] \cap U(1, n) = \bigsqcup_{i=1}^{m} [a+k_i, a+l_i] = \bigsqcup_{j=1}^{t} \{a+c_j\}$$

such that $k_i \leq l_i + 1 < k_{i+1}$, $c_j + 1 < c_{j+1}$, and for no i, j is $c_j \in [k_i - 1, l_i + 1]$. By Lemma 2.5, $[a+n+k_i+1, a+l_i+2n-1] \subset \mathbb{Z} \setminus U(1, n)$ for $1 \leq i \leq m$. Note that since $k_m \leq n-1$ and $l_1 \geq 1$, we have $a+n+k_m+1 \leq a+l_1+2n-1$, and hence

$$\bigcup_{i=1}^{m} [a+n+k_i+1, a+l_i+2n-1] = [a+n+k_1+1, a+2n+l_m-1] \subset \mathbb{Z} \setminus U(1, n).$$

Therefore,

$$I \cap U(1, n) \subset ([a, a+n+k_1]) \cup ([a+2n+l_m, a+3n-1]).$$

However, we claim that

$$|[a+2n+l_m, a+3n-1] \cap U(1,n)| + |[a+l_m, a+n-1] \cap U(1,n)| \leq n - l_m.$$

It suffices to prove this assuming that $[a+l_m, a+n-1] \cap U(1,n) \neq \varnothing$. Let $u_1, u_2, \ldots, u_s$ be the Ulam numbers in $[a+l_m, a+n-1]$. If $s = 1$, then

$$a+2n+l_m = (a+l_m)+2n = u_1 + (2n - (u_1 - a - l_m)),$$

and as this gives two representations, it must be that $a+2n+l_m \notin U(1,n)$. If $s > 1$, then for every $1 \leq i < j \leq s$, by Lemma 2.4,

$$[u_j + n, u_i + 2n] \subset \mathbb{Z} \setminus U(1,n).$$

Hence

$$[a+2n+l_m, u_{s-1}+2n] \subset \mathbb{Z} \setminus U(1,n).$$

Note that

$$|[a+2n+l_m, u_{s-1}+2n]| \geq s$$

unless $u_{s-1} = a+l_m+s-1$, which is to say that $[a+l_m, a+l_m+s-1] \subset U(1,n)$. But by the definition of l_m, it can only be that $a+l_m \in U(1,n)$ if $l_m = n-1$, which is not possible since we assumed that there are at least two Ulam numbers in $[a+l_m, a+n-1]$. As desired, we conclude that

$$|[a+l_m, a+n-1] \cap U(1,n)| + |[a+2n+l_m, a+3n-1] \cap U(1,n)| \leq n - l_m,$$

and therefore

$$\begin{aligned} |I \cap U(1,n)| &\leq |[a+l_m, a+n-1] \cap U(1,n)| + |[a+n, a+n+k_1] \cap U(1,n)| \\ &\qquad + |[a+2n+l_m, a+3n-1] \cap U(1,n)| \\ &\leq n - l_m + k_1 - 1 \\ &\leq n-1. \end{aligned}$$

Case 2: In this case, we are given that

$$[a, a+n-1] \cap U(1,n) = \bigsqcup_{j=1}^{t} \{a + c_j\},$$

where $c_j + 1 < c_{j+1}$. This implies that for $k > j$,

$$k - j < c_k - c_j < n.$$

By Lemma 2.4, we have

$$[a+c_k+n, a+c_j+2n] \subset \mathbb{Z} \setminus U(1,n),$$

and consequently,

$$[a+c_2+n, a+c_{t-1}+2n] = \bigcup_{1 \leq i < j \leq t} [a+c_k+n, a+c_j+2n] \subset \mathbb{Z} \setminus U(1,n).$$

Ergo,

$$\begin{aligned}|I \cap U(1,n)| &= |[a, a+n-1] \cap U(1,n)| \\ &\qquad + |[a+n, a+c_2+n-1] \cap U(1,n)| \\ &\qquad\quad + |[a+c_2+n, a+c_{t-1}+2n] \cap U(1,n)| \\ &\qquad\qquad + |[a+c_{t-1}+2n+1, a+3n-1] \cap U(1,n)| \\ &\le t+c_2+n-c_{t-1}-1 \\ &\le n+1. \end{aligned}$$

□

For $n = 2$, Corollary 2.2 gives an upper bound of $\frac{1}{2}$ on the density. Using similar techniques to the proof of Theorem 2.1, we can improve this upper bound to $\frac{6}{17} \approx 0.353$.

Theorem 2.6. *The density of $U(1,2)$ is at most $\frac{6}{17}$.*

Proof. Let $a \in U(1,n)$ and define $I = [a, a+8]$, $J = [a, a+16]$. We claim that either $|I \cap U(1,2)| \le 3$ or $|J \cap U(1,2)| \le 6$. We make use of the fact that

$$1, 2, 3, 4, 6, 8, 11, 13, 16 \in U(1,2).$$

If $|I \cap U(1,2)| > 3$, then $I = \{a, a+2, a+5, a+7\}$. Otherwise, $I \cap U(1,2)$ contains a pair of elements $u, u+1$ such that $u+1 = a+2, a+3, a+4, a+6$, or $a+8$, which gives two representations; this is a contradiction.

In this case, $J \cap U(1,2) \subset \{a, a+2, a+5, a+7, a+12, a+14\}$ — otherwise, it contains an element with two representations. Consequently, $|J \cap U(1,2)| \le 6$. This means we can now define two sequences $u_1, u_2, u_3, \ldots,$ $L_1, L_2, L_3, \ldots$ recursively — let $u_1 = 1$ and $L_1 = 17$, and then define u_{i+1} to be the smallest element of the Ulam sequence larger than $u_i + L_i$, and

$$L_{i+1} = \begin{cases} 17 & \text{if } |[u_{i+1}, u_{i+1}, +16] \cap U(1,2)| \le 6, \\ 9 & \text{otherwise.} \end{cases}$$

We can then partition the positive integers into sets of the forms $[u_{i+1}, u_{i+1} + L_i]$ and $[u_{i+1} + L_i + 1, u_{i+2} - 1]$. The density of $U(1,2)$ in any of these sets is no more than $\frac{6}{17}$, and that implies that the density of $U(1,2)$ is bounded by $\frac{6}{17}$. □

3. Regular Ulam sequences

We now consider regular sequences. Let $1_{U(a,b)}$ be the indicator function of $U(a,b)$. Given a positive integer l and a positive odd number k, define

$$S^l_{a,b}(k) = \{1_{U(a,b)}(k+2i)\}_{i=0}^{l-2}.$$

With this terminology, we can now easily state the main theorem we want to prove.

Theorem 3.1. *Let $0 < a < b$ be coprime integers. Let l, p, q be positive integers such that $p < q$, p, q are odd, $q \geq 2l$, $a < b < 2l - 2$, $S^l_{a,b}(p) = S^l_{a,b}(q)$, and*

$$U(a, b) \cap 2\mathbb{Z} \cap [2l, 3q - p] = \varnothing.$$

Then

$$U(a, b) \cap 2\mathbb{Z} \cap [2l, \infty) = \varnothing.$$

Theorem 3.1 provides a semi-algorithm for determining whether a sequence is regular — simply do a brute force search for triples (l, p, q) satisfying the conditions of the theorem. If such a triple is found, then we conclude that $U(a, b)$ is regular. This gives us the following corollary.

Corollary 3.2. *For integer pairs (a, b) given below, $U(a, b)$ is regular*:

$$\begin{array}{ccccccc}
(4, 11), & (4, 19), & (6, 7), & (6, 11), & (7, 8), & (7, 10), & (7, 12), \\
(7, 16), & (7, 18), & (7, 20), & (8, 9), & (8, 11), & (9, 10), & (9, 14), \\
(9, 16), & (9, 20), & (10, 11), & (10, 13), & (10, 17), & (11, 12), & (11, 14), \\
(11, 16), & (11, 18), & (11, 20), & (12, 13), & (12, 17), & (13, 14). &
\end{array}$$

Proof. By direct computation, we find triples (l, p, q) satisfying the conditions of Theorem 3.1:

(a, b)	(l, p, q)
(4, 11)	(25, 107, 1425)
(4, 19)	(41, 14745, 17305)
(6, 7)	(57, 8537, 70987)
(6, 11)	(89, 1032425, 1033833)
(7, 8)	(71, 14331, 57089)

(a, b)	(l, p, q)
(7, 10)	(85, 95587, 102181)
(7, 12)	(99, 79423, 80991)
(7, 16)	(127, 46957, 47965)
(7, 18)	(141, 196513, 198753)
(7, 20)	(155, 50893, 52125)

(a, b)	(l, p, q)
(8, 9)	(91, 1037093, 1038533)
(8, 11)	(111, 2125501, 4308725)
(9, 10)	(109, 117117, 747935)
(9, 14)	(145, 558073, 560377)
(9, 16)	(163, 60093, 65277)
(9, 20)	(199, 219761, 222929)
(10, 11)	(133, 470303, 485615)
(10, 13)	(157, 5804601, 5807097)
(10, 17)	(205, 3919981, 3933037)

(a, b)	(l, p, q)
(11, 12)	(155, 140511, 142975)
(11, 14)	(177, 507965, 509373)
(11, 16)	(199, 394379, 400715)
(11, 18)	(221, 29995, 37035)
(11, 20)	(243, 46291, 54035)
(12, 13)	(183, 3329465, 3330921)
(12, 17)	(239, 3204117, 3211733)
(13, 14)	(209, 1421023, 1427679)

□

To prove Theorem 3.1, we start with a useful lemma that establishes that if it is false, then there is a bijective correspondence between odd Ulam numbers in different intervals.

Lemma 3.3. *Let l, a, b be positive integers, and $p < q$ be positive odd integers such that $q \geq 2l$, $a < b < 2l - 2$, $S^l_{a,b}(p) = S^l_{a,b}(q)$,*

$$U(a,b) \cap 2\mathbb{Z} \cap [2l, 3q - p] = \varnothing,$$
$$U(a,b) \cap 2\mathbb{Z} \cap [2l, \infty) \neq \varnothing.$$

Let $\tilde{u}$ be the smallest even number in $U(a,b)$ greater than $3q - p$. Then there is a well-defined bijection

$$U(a,b) \cap (1 + 2\mathbb{Z}) \cap [p, \tilde{u} + p - q - 1] \to U(a,b) \cap (1 + 2\mathbb{Z}) \cap [q, \tilde{u} - 1],$$
$$u \mapsto u + q - p.$$

Proof. We will show that there is a well-defined bijection

$$\phi_m : U(a,b) \cap (1 + 2\mathbb{Z}) \cap [p, p + 2m] \to U(a,b) \cap (1 + 2\mathbb{Z}) \cap [q, q + 2m],$$
$$u \mapsto u + q - p,$$

for all integers $0 \leq m \leq \frac{1}{2}(\tilde{u} - q - 1)$. We know that $S^l_{a,b}(p) = S^l_{a,b}(q)$; hence $p + 2m' \in U(a,b)$ if and only if $q + 2m' \in U(a,b)$ for all $0 \leq m' \leq l - 2$, which proves the claim for $m \leq l - 2$.

For all other m, we apply induction — that is, let $l - 2 < h \leq \frac{1}{2}(\tilde{u} - q - 1)$ such that ϕ_{h-1} is a bijection. We need to show that ϕ_h is bijection. This is equivalent to proving that $p + 2h \in U(a,b)$ if and only if $q + 2h \in U(a,b)$. Define sets

$$P = \{(u, v) \in U(a,b)^2 \mid u \equiv 0 \bmod 2,\ v \equiv 1 \bmod 2,\ u + v = p + 2h\},$$
$$Q = \{(u, v) \in U(a,b)^2 \mid u \equiv 0 \bmod 2,\ v \equiv 1 \bmod 2,\ u + v = q + 2h\},$$

which enumerate the number of representations of $p + 2h$ and $q + 2h$, respectively. If we can show that $|P| = |Q|$, then this will imply that $p + 2h \in U(a,b)$ if and only if $q + 2h \in U(a,b)$. However, we can construct a bijection between these two sets by

$$\psi : P \to Q,$$
$$(u, v) \mapsto (u, \phi_{h-1}(v)) = (u, v + q - p).$$

This is well-defined since $u + v = p + 2h$ implies $v \leq p + 2h - 1$. □

Proof of Theorem 3.1. We argue by contradiction. That is, suppose that there exist even Ulam numbers larger than $3q - p$. Let $\tilde{u}$ be the smallest such element. We know $\tilde{u} = u_1 + u_2$ for some $u_1 < u_2 \in U(a,b)$. Every even Ulam number less than $\tilde{u}$ is smaller than $2l$; hence one of u_1, u_2 is odd — otherwise, we have

$$u_1 + u_2 < 4l \leq 3q - p,$$

which is a contradiction. Since $\tilde{u}$ is even, we conclude that u_1, u_2 are both odd. Next, we show that $\tilde{u} - q + p$ has at least two representations as the sum of two distinct elements of $U(a, b)$. Note that

$$\tilde{u} - q + p \geq (3q - p) - q + p = 2q > 2l,$$

and since $\tilde{u} - q + p$ is even, this implies it is not in $U(a, b)$. Consequently, it will suffice to prove that it has at least one representation. Note that

$$\begin{aligned} u_2 &> \tfrac{1}{2}\tilde{u} > \tfrac{1}{2}(3q - p) > q, \\ u_2 &\leq \tilde{u} - 1, \end{aligned}$$

so by Lemma 3.3, since $u_2 \in U(a, b)$ it follows $u_2 + q - p \in U(a, b)$. Therefore, $\tilde{u} + q - p = u_1 + (u_2 + q - p)$ is a representation.

Write

$$\tilde{u} - q + p = v_1 + v_2 = v_1' + v_2',$$

where $v_1 < v_2$, $v_1' < v_2' \in U(a, b)$. Note that $v_2 > q$, since

$$v_2 > \tfrac{1}{2}(\tilde{u} - q + p) > \tfrac{1}{2}((3q - p) - q + p) > q.$$

Similarly, $v_2' > q$. From this it follows that $v_2, v_2' > 2l$, and we conclude that v_2, v_2' must be odd. Finally, note that

$$p < q < v_2, v_2' \leq \tilde{u} + p - q - 1,$$

and therefore by Lemma 3.3, $v_2 + q - p$, $v_2' + q - p \in U(a, b)$, which is a contradiction since

$$\tilde{u} = v_1 + (v_2 + q - p) = v_1' + (v_2' + q - p). \qquad \square$$

4. Classification of (3, 2)-Ulam sets

Up until this point, we have only considered (2, 1)-Ulam sets; we now turn to the problem of classifying higher-dimensional Ulam sets. The classification problem for nondegenerate (2, 2)-Ulam sets was solved by Kravitz and Steinerberger [2017]. In particular, they showed that after a linear transformation, the Ulam set becomes $U((1, 0), (0, 1))$, illustrated in Figure 2. We shall denote this set by $\mathcal{A}$.

We shall consider (3, 2)-Ulam sets that are extensions of such Ulam sets — that is, we shall assume that two of the basis vectors are $(1, 0)$ and $(0, 1)$. For convenience, we define

$$\begin{aligned} U_{\mathcal{A}}(v_1, v_2) &= U((1, 0), (0, 1), (v_1, v_2)), \\ W_{(v_1, v_2)} &= \{(m, n) \in \mathbb{Z}^2_{\geq 0} \mid m < v_1 \text{ or } n < v_2\}, \\ L_{(v_1, v_2)} &= \{(m, n) \in \mathbb{Z}_{\geq v_1} \times \mathbb{Z}_{\geq v_2}\}. \end{aligned}$$

Note that if $(a, b) \in L_{(v_1, v_2)}$, then any representations it has have to lie in the set $W_{(v_1, v_2)}$. We use this fact to our advantage to prove the following lemma.

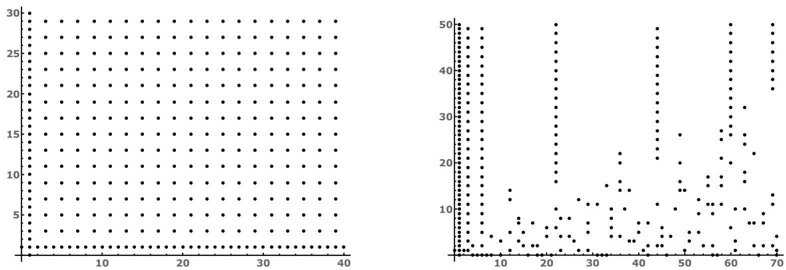

Figure 2. The (2, 2)-Ulam set $\mathcal{A}$ and the (3, 2)-Ulam set $U_{\mathcal{A}}(4, 0)$.

Lemma 4.1. *Let* $\mathcal{U} = U_{\mathcal{A}}(v_1, v_2)$ *be a nondegenerate* (3, 2)*-Ulam set with* $v_1, v_2 \neq 0$. *Then the following statements hold*:

(1) $v_1, v_2 > 1$ *and at least one of* v_1, v_2 *is even.*

(2) $\mathcal{A} \cap W_{(v_1,v_2)} = \mathcal{U} \cap W_{(v_1,v_2)}$.

(3) *Every point* $(m, n) \in \mathbb{Z}^2_{\geq 0}$ *has at least one representation.*

Proof. It was shown in [Kravitz and Steinerberger 2017] that

$$\mathcal{A} = \{(m, 1) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(1, m) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(2m+1, 2n+1) \mid m, n \in \mathbb{Z}_{\geq 0}\}.$$

For $\mathcal{U}$ to be nondegenerate, it must be that $(v_1, v_2) \notin \mathcal{A}$, and since $v_1, v_2 \neq 0$, this implies $v_1, v_2 > 1$ and at least one of v_1, v_2 is even.

All representations of points in $W_{(v_1,v_2)}$ are representations by elements in $\mathcal{U}$. It follows $\mathcal{A} \cap W_{(v_1,v_2)} = \mathcal{U} \cap W_{(v_1,v_2)}$. However, this implies

$$(m, n) = (m-1, 1) + (1, n-1)$$

is a representation of (m, n). □

We shall call $(m, n) = (m-1, 1) + (1, n-1)$ the *standard representation* of (m, n). By Lemma 4.1, proving that $(m, n) \notin U_{\mathcal{A}}(v_1, v_2)$ for $v_1, v_2 \neq 0$ is equivalent to proving that it has a nonstandard representation. This makes working with Ulam sets of this form much simpler. On the other hand, if one of v_1, v_2 is 0, then the set $U_{\mathcal{A}}(v_1, v_2)$ has a copy of a (2, 1)-Ulam set on either the x- or y-axis. An example of such a set is given in Figure 2. Some partial results about such sets are given in [Kravitz and Steinerberger 2017], but in general describing their structure is an open problem.

We now give five examples of possible structures of sets $U_{\mathcal{A}}(v_1, v_2)$ with $v_1, v_2 \neq 0$, which are derived from numerical observations. An illustration of each of these five types is provided in Figure 1.

Definition 4.2. Let $U \subset \mathbb{Z}_{\geq 0}^2$ and let (v_1, v_2) be a vector in U. We say U is of *L type for* (v_1, v_2) if

$$U = \{(v_1, v_2)\} \cup \{(m, 1) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(1, m) \mid m \in \mathbb{Z}_{\geq 0}\} \\ \cup \{(a+2mv_1, b+2mv_2) \mid a, b, m \geq 0,\ a, b \in 1+2\mathbb{Z},\ m \in \mathbb{Z},\ (a, b) \in W_{(v_1, v_2)}\}.$$

We say U is of *column-deleted type for* (v_1, v_2) if

$$U = \{(v_1, v_2)\} \cup \{(m, 1) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(1, m) \mid m \in \mathbb{Z}_{\geq 0}\} \\ \cup \{(2m+1, 2n+1) \mid m, n \in \mathbb{Z}_{\geq 0},\ \text{if } 2m+1 = v_1+1 \text{ then } 2n+1 < v_2\}.$$

We say U is of *column-deleted L type for* (v_1, v_2) if

$$U = \{(v_1, v_2)\} \cup \{(m, 1) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(1, m) \mid m \in \mathbb{Z}_{\geq 0}\} \\ \cup \{(a+(m+1)v_2+2, b+2m+5) \mid a, b, m \geq 0,\ a, b, m \in 2\mathbb{Z},\ a < m \text{ or } b = 0\}.$$

We say that U is of *shifted column-deleted type for* (v_1, v_2) if

$$U = \{(v_1, v_2)\} \cup \{(m, 1) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(1, m) \mid m \in \mathbb{Z}_{\geq 0}\} \\ \cup \{(m, n) \mid m, n \geq 0,\ m < v_1,\ m, n \in 1+2\mathbb{Z}\} \\ \cup \{(m, n) \mid m, n \geq 0,\ m > v_1,\ m \in 2\mathbb{Z},\ n \in 1+2\mathbb{Z}\}.$$

We say U is of *exceptional type* if

$$U = \{(v_1, v_2)\} \cup \{(8, 8)\} \cup \{(m, 1) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(1, m) \mid m \in \mathbb{Z}_{\geq 0}\} \\ \cup \{(4, 2m+4) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(2m+4, 4) \mid m \in \mathbb{Z}_{\geq 0}\}.$$

This list enumerates all the possibilities for sets $U_{\mathcal{A}}(v_1, v_2)$ if $v_1, v_2 \neq 0$.

Theorem 4.3. *Let $\mathcal{U} = U_{\mathcal{A}}(v_1, v_2)$ be a nondegenerate $(3, 2)$-Ulam set such that $v_1, v_2 \neq 0$. Then exactly one of the following is true of either $\mathcal{U}$ or its reflection about the line $y = x$:*

(1) $v_1, v_2 \in 2\mathbb{Z} \cap [4, \infty)$ *and $\mathcal{U}$ is of L type.*

(2) $v_1 \in 2\mathbb{Z},\ v_2 \in (1+2\mathbb{Z}) \cap [4, \infty)$, *and $\mathcal{U}$ is of column-deleted type.*

(3) $v_1 \in 2\mathbb{Z} \cap [4, \infty),\ v_2 = 2$, *and $\mathcal{U}$ is of column-deleted L type.*

(4) $v_1 \in 2\mathbb{Z},\ v_2 = 3$, *and $\mathcal{U}$ is of shifted column-deleted type.*

(5) $v_1 = v_2 = 2$ *and $\mathcal{U}$ is of exceptional type.*

Proof. By Lemma 4.1, the given list enumerates all possibilities for v_1, v_2, after accounting for a possible reflection around the line $y = x$. Furthermore, it is easy to check that $\mathcal{U} \cap W_{(v_1, v_2)}$ is of the specified type in each case — that is, it is equal to the intersection of a set U of the desired type with $W_{(v_1, v_2)}$.

Consider the case $v_1, v_2 \in 2\mathbb{Z} \cap [4, \infty)$. We shall show that $\mathcal{U} \cap W_{(a,b)}$ is of L type for all $a, b \geq 0$. Note that by Lemma 4.1,

$$\begin{aligned}\mathcal{A} \cap W_{(3,3)} &= \{(m, 1) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(1, m) \mid m \in \mathbb{Z}_{\geq 0}\} \\ &\qquad \cup \{(3, 2m+1) \mid m \in \mathbb{Z}_{\geq 0}\} \cup \{(2m+1, 3) \mid m \in \mathbb{Z}_{\geq 0}\} \\ &= \mathcal{U} \cap W_{(3,3)}.\end{aligned}$$

It follows that if $(m, n) \in \mathcal{U}$ and $m, n > 1$, then $m, n \in 1 + 2\mathbb{Z}$. This is evident if $(m, n) \in W_{(3,3)}$ — otherwise, either $(m, n) = (k + 3, 2l + 2)$ or $(2l + 2, k + 3)$ for some $k, l \in \mathbb{Z}_{\geq 0}$, and we have nonstandard representations

$$\begin{aligned}(k + 3, 2l + 2) &= (3, 2l + 1) + (k, 1), \\ (2l + 2, k + 3) &= (2l + 1, 3) + (1, k).\end{aligned}$$

Furthermore, it must be that $\mathcal{U} \cap W_{(2v_1, 2v_2)}$ is of L type. To see this, it suffices to show that

$$\mathcal{U} \cap W_{(2v_1, 2v_2)} \cap L_{(v_1, v_2)} = \{(v_1, v_2)\},$$

but as we know any point in this intersection must necessarily be of the form $(2m + 1, 2n + 1)$, we have a nonstandard representation

$$(2m + 1, 2n + 1) = (v_1, v_2) + (2m + 1 - v_1, 2n + 1 - v_2).$$

We now prove that $\mathcal{U} \cap W_{(2kv_1, 2kv_2)}$ is of L type by inducting on $k \in \mathbb{Z}$ — we have proved the base case $k = 1$, so it suffices to assume $\mathcal{U} \cap W_{(2mv_1, 2mv_2)}$ is L type for some $m \in \mathbb{Z}_{\geq 0}$ and prove that $\mathcal{U} \cap W_{(2(m+1)v_1, 2(m+1)v_2)}$ is L type. This amounts to proving that

$$\begin{aligned}&\mathcal{U} \cap W_{((2m+1)v_1, (2m+1)v_2)} \cap L_{(2mv_1, 2mv_2)} \\ &\quad = W_{((2m+1)v_1, (2m+1)v_2)} \cap L_{(2mv_1, 2mv_2)} \cap (1 + 2\mathbb{Z}_{\geq 0})^2 \mathcal{U} \\ &\qquad\qquad \cap W_{((2m+2)v_1, (2m+2)v_2)} \cap L_{((2m+1)v_1, (2m+1)v_2)} \\ &\quad = \varnothing.\end{aligned}$$

This is easily proven by noting that the former set cannot possibly have any nonstandard representations, whereas the latter set is nothing more than

$$(v_1, v_2) + \mathcal{U} \cap W_{((2m+1)v_1, (2m+1)v_2)} \cap L_{(2mv_1, 2mv_2)}.$$

The other cases are similar. □

5. Parity restrictions on $(k, 2)$-Ulam sets

Let us now consider the more general case where multiple vectors are added to a $(2, 2)$-Ulam set, rather than just one. As in the previous section, we consider nondegenerate Ulam sets containing $(1,0)$, $(0,1)$, and so we define

$$U_{\mathcal{A}}(v_1, v_2, \dots, v_n) = U((1,0), (0,1), v_1, \dots, v_n).$$

We shall show that the parity of any element in $U_{\mathcal{A}}(v_1, v_2, \dots, v_n)$ is eventually fixed, as long as none of the v_i lie on the coordinate axes.

Theorem 5.1. *Let $\mathcal{U} = U_{\mathcal{A}}(v_1, v_2, \dots, v_n)$ be a nondegenerate $(n+2, 2)$-Ulam set such that none of the v_i lie on the coordinate axes. Then there exists a v such that for all $u \in \mathcal{U} \cap L_v$, we have $u = v \mod 2$.*

To prove Theorem 5.1, we first note that if $\mathcal{U}$ contains a point (u_1, u_2) such that $(u_1, u_2 + 2k) \in \mathcal{U}$ for all $k \in \mathbb{Z}_{\geq 0}$, then for all $(u_1', u_2') \in \mathcal{U} \cap L_{(u_1,u_2)}$, we have $u_2 = u_2' \mod 2$. This is because if $u_2' \neq u_2 \mod 2$,

$$(u_1', u_2') = (u_1, u_2' - 1) + (u_1' - u_1, 1)$$

gives a nonstandard representation. It shall therefore suffice to prove the existence of such a point. Toward this end, we give a useful lemma.

Lemma 5.2. *Let $\mathcal{U} = U_{\mathcal{A}}(v_1, v_2, \dots, v_n)$ be a nondegenerate $(n+2, 2)$-Ulam set such that none of the v_i lie on the coordinate axes. If there exists $m \in \mathbb{Z}_{>1}$ such that there are infinitely many points of the form $(m, n) \in \mathcal{U}$, then there exists a point (u_1, u_2) such that $(u_1, u_2 + 2k) \in \mathcal{U}$ for all $k \in \mathbb{Z}_{\geq 0}$.*

Proof. Let $M \in \mathbb{Z}_{>1}$ be the smallest m such that there are infinitely many points of the form $(m, n) \in \mathcal{U}$. Note that in fact $M > 2$, since every element $(2, n)$ has at least two representations. Therefore, we can define N be the largest n such that $(m, n) \in \mathcal{U}$, where $1 < m < M$.

Consider any point $(M, n) \in \mathbb{Z}^2_{\geq 0}$ with $n > 2N$. For any representation of (M, n), at least one of the summands must have x-coordinate 1 or M — otherwise, the y-coordinates are too small to add up to n. If this representation is

$$(M, n) = (1, n') + (M - 1, n - n'),$$

then it is nonstandard if and only if $n - n' \neq 1$. However, if $n - n' \neq 1$, then every point (M, n'') with $n'' > n$ has a nonstandard representation, which is impossible.

On the other hand, the only other possible representation is $(M, n) = (M, n-1) + (0, 1)$, so we conclude that $(M, n) \in \mathcal{U}$ if and only if $(M, n-1) \notin \mathcal{U}$. Thus if we take

$$(u_1, u_2) = \begin{cases} (M, n) & \text{if } (M, n) \in \mathcal{U}, \\ (M, n+1) & \text{otherwise}, \end{cases}$$

it satisfies the desired conditions. $\square$

This is sufficient to prove Theorem 5.1.

Proof of Theorem 5.1. We claim that there must exist some $m \in \mathbb{Z}_{>1}$ such that there are infinitely many points of the form $(m, n) \in \mathcal{U}$. Suppose otherwise — then there must exist some strictly increasing function $\phi : \mathbb{Z}_{>1} \to \mathbb{Z}_{>1}$ such that if $(m, n) \in \mathcal{U}$ and $m, n > 1$, then $n < \phi(m)$.

Let $m > 2$ and $n > 2\phi(m)$. Then if

$$(m, n) = (m_1, n_1) + (m_2, n_2)$$

is a representation of (m, n), it must be the standard representation — otherwise, $n_1 + n_2 < 2\phi(m) < n$. But this implies $(m, n) \in \mathcal{U}$, which is a contradiction.

Consequently, we can apply Lemma 5.2. By our earlier remarks, we know there exists a point $(u_1, u_2) \in \mathcal{U}$ such that for all $(u_1', u_2') \in \mathcal{U} \cap W_{(u_1,u_2)}$, we have $u_2' \equiv u_2 \bmod 2$.

On the other hand, the reflection of $\mathcal{U}$ about the line $y = x$ is also an Ulam set, which we shall denote by $\mathcal{V}$. It is easy to check that $\mathcal{V}$ also satisfies the requirements of the theorem, and therefore must contain a point (v_1, v_2) such that for all $(v_1', v_2') \in \mathcal{V} \cap W_{(v_1,v_2)}$, we have $v_2' \equiv v_2 \bmod 2$. However, this means that if we take

$$v = (\max\{u_1, v_2\}, \max\{u_2, v_1\}),$$

then for all $u \in \mathcal{U} \cap L_v$, we have $u = v \bmod 2$, as desired. □

6. Periodicity of Ulam sets

We close by considering the periodicity of Ulam sets $U_{\mathcal{A}}(v_1, v_2, \ldots, v_n)$, under the additional constraint that the added vectors are not too small — that is, all of their components are at least 4. With this restriction, such sets become far more manageable.

Lemma 6.1. *Let $\mathcal{U} := U_{\mathcal{A}}(v_1, v_2, \ldots, v_n)$ be a $(k, 2)$-Ulam set such that all $v_i = (x_i, y_i)$ have $x_i, y_i \geq 4$. Then $\mathcal{U} \subset \mathcal{A} \cup \{v_1, v_2, \ldots, v_n\}$.*

Proof. Since all the initial vectors have components greater than or equal to 4, all elements of $\mathcal{A}$ with at least one coordinate less than 4 are also in $\mathcal{U}$. In particular, $\mathcal{U}$ contains all vectors of the forms $(2n-1, 3)$ and $(3, 2n-1)$ for $n \geq 1$. Thus for $n \geq 2$, we have $(2n, m) = (2n-1, 3) + (1, m-3)$ is a representation of $(2n, m)$ as a sum of vectors in the sets — as this representation is not the standard one, we conclude that $(2n, m) \notin \mathcal{U}$. By symmetry, we also have that $(m, 2n) \notin \mathcal{U}$ for $n \geq 2$. Thus, all vectors with at least one coordinate less than 4 in $\mathcal{U}$ are the vectors in $\mathcal{A}$ with at least one coordinate less than 4, and all other vectors are in $\mathcal{U}$ only if their coordinates are both odd; hence they are in $\mathcal{U}$. This proves the claim. □

In fact, we can be far more precise in our characterization of this set.

Lemma 6.2. *Let $\mathcal{U} = U_{\mathcal{A}}(v_1, v_2, \ldots, v_n)$ be a $(k, 2)$-Ulam set such that the vectors v_i all have both components greater than or equal to 4. Let $a, b \geq 1$ be odd integers such that $(a, b) \neq v_i$ for any i. Then $(a, b) \in \mathcal{U}$ if and only if $(a, b) - v_i \notin \mathcal{U}$ for all i, $1 \leq i \leq n$.*

Proof. If $(a, b) - v_i = u \in \mathcal{U}$, then clearly $u + v_i$ is a second representation of (a, b) outside of the standard one, and so $(a, b) \notin \mathcal{U}$. On the other hand, if $(a, b) \notin \mathcal{U}$, then we know there must be some nonstandard representation of it. We know that $(a, b) \in \mathcal{A}$; hence at least one term in this representation must come from $\mathcal{U} \backslash \mathcal{A}$. Since $\mathcal{U} \subset \mathcal{A} \cup \{v_1, v_2, \ldots, v_n\}$, that means that one of the summands must be v_i for some i, which is to say that $(a, b) - v_i \in \mathcal{U}$, as desired. $\square$

Note that if $n = 1$, Lemma 6.2 tells us precisely that $\mathcal{U}$ is eventually periodic, which is consistent with the result of Section 4. On the other hand, based on numerical evidence, it is almost certainly not true that all $(k, 2)$-Ulam sets are eventually periodic. However, we are interested in whether one can build new eventually periodic sets from existing eventually periodic sets. As an example, we know from the results of Section 4 that adding an initial vector whose coordinates are at least 4 to $\mathcal{A}$ yields another set that is eventually periodic. This leads us to conjecture that adding an initial vector whose coordinates are sufficiently large to an eventually periodic set yields an eventually periodic set. We prove two theorems in this direction.

Theorem 6.3. *Let $\mathcal{U} = U_{\mathcal{A}}(v_1, v_2, \ldots, v_n)$ be a nondegenerate $(k, 2)$-Ulam set such that all vectors $v_i = (x_i, y_i)$ have $x_i, y_i \geq 4$ and even. Furthermore, suppose that there exist integers m, n such for all i, there exists a j such that $m - v_i = v_j$. Then $\mathcal{U}$ is eventually (m, n)-periodic, and for any other vector $v_{n+1} = (x_{n+1}, y_{n+1})$ with $x_{n+1}, y_{n+1} \geq 4$ such that at least one of x_{n+1}, y_{n+1} is odd, $\mathcal{U}' := U_{\mathcal{A}}(v_1, v_2, \ldots, v_{n+1})$ is eventually (m, n)-periodic.*

Proof. Let (a, b) be a vector such that $a, b > 0$ are both odd. We shall show that $(a, b) \in \mathcal{U}$ if and only if $(a, b) + (m, n) \in U$. Indeed, for all i, we have $(a, b) + (m, n) - v_i = (a, b) + v_j$ for some j. If $(a, b) \in \mathcal{U}$, this gives a nonstandard representation of $(a, b) + (m, n) - v_i$; hence it is not in $\mathcal{U}$, and so by Lemma 6.2, it follows that $(a, b) + (m, n) \in U$. On the other hand, if $(a, b) \notin \mathcal{U}$, then again by Lemma 6.2, we know that $(a, b) - v_i \in \mathcal{U}$ for some i, and it follows that $(a, b) + (m, n) - v_i \in \mathcal{U}$. But this implies $(a, b) + (m, n) \notin \mathcal{U}$. Since by Lemma 6.1 we know that all sufficiently large vectors in $\mathcal{U}$ have odd coordinates, we conclude that $\mathcal{U}$ is eventually (m, n)-periodic.

It remains to prove that $\mathcal{U}'$ is eventually periodic. Let (a, b) be a vector such that a, b are both odd, and $a > x_i$, $b > y_i$ for every $1 \leq i \leq n+1$. If $(a, b) \in \mathcal{U}'$, then $(a, b) + (m, n) - v_i = (a, b) + v_j \notin \mathcal{U}'$ for every $1 \leq i \leq n$, so it suffices to prove that $(a, b) + (m, n) - v_{n+1} \notin \mathcal{U}'$ to conclude that $(a, b) + (m, n) \in \mathcal{U}'$. However, the coordinates of $(a, b) + (m, n) - v_{n+1}$ are both integers greater than 1, and at least one of them is even. By Lemma 6.1, this implies $(a, b) + (m, n) - v_{n+1} \notin U'$, and so $(a, b) + (m, n) \in U'$. In the other direction, we know that if $(a, b) \notin \mathcal{U}'$, then $(a, b) - v_i \in U'$ for some i. If $i \neq n + 1$, the proof is the same as before. If $(a, b) - v_{n+1} \in U'$, then we note that $(a, b) + (m, n) \notin U'$ by parity considerations. We thus conclude that $\mathcal{U}'$ is eventually (m, n)-periodic. □

Theorem 6.4. *Let $\mathcal{U} = U_{\mathcal{A}}(v_1, v_2, \ldots, v_n)$ be a nondegenerate $(k, 2)$-Ulam set such that all vectors $v_i = (x_i, y_i)$ have $x_i, y_i \geq 4$ and at least one of x_i, y_i is odd. Then $\mathcal{U}$ is eventually periodic, with period $(2, 2)$.*

Proof. Note that if x_i, y_i are both odd, then $(x_i, y_i) \in \mathcal{A}$, which would contradict the fact that $\mathcal{U}$ is nondegenerate. Thus all vectors v_i have one even component. We claim that if x_n is even, then

$$\mathcal{U} = \{v_n\} \cup U_{\mathcal{A}}(v_1, \ldots, v_{n-1}) \backslash (\{(x_n+1, y_n+2l) \mid l \in \mathbb{Z}_{\geq 0}\} \cup \{v_i + v_j \mid 1 \leq i < j \leq n\}),$$

and if y_n is even, then

$$\mathcal{U} = \{v_n\} \cup U_{\mathcal{A}}(v_1, \ldots, v_{n-1}) \backslash (\{(x_n+2l, y_n+1) \mid l \in \mathbb{Z}_{\geq 0}\} \cup \{v_i + v_j \mid 1 \leq i < j \leq n\}).$$

The base case follows from the results of Section 4. Now, note that if $v_n \in \mathcal{U}$, then certainly either $v_n + (1, 2l)$ or $v_n + (2l, 1)$ is a nonstandard representation, so correspondingly $(x_n + 1, y_n + 2l)$ or $(x_n + 2l, y_n + 1)$ is not in $\mathcal{U}$. Similarly, all vectors $v_i + v_j$ have at least two representations. It remains to prove that removing these vectors doesn't lead to removing representations of other points. This cannot be — all removed vectors have both coordinates odd, and $\mathcal{U}$ contains all vectors with positive odd coordinates, all of which have one standard representation that we know has not been removed. That this is true for all the sets $U_{\mathcal{A}}(v_1, \ldots, v_k)$ follows by induction. This concludes the proof, since it is clear that each of the sets $U_{\mathcal{A}}(v_1, \ldots, v_n)$ is eventually periodic, with period $(2, 2)$, by induction. □

These two results immediately imply Theorem 1.9.

Proof of Theorem 1.9. If both v_1 and v_2 have at least one odd coordinate, the result follows from Theorem 6.4. Otherwise, let v_i, v_j be the vectors that have both coordinates even — here, i, j need not be distinct. Then by Theorem 6.3, $\mathcal{U}$ is eventually $(v_i + v_j)$-periodic. □

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