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Numbers and the heights of their happiness

May Mei and Andrew Read-McFarland

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A generalized happy function, $S_{e,b}$ maps a positive integer to the sum of its base b digits raised to the e-th power. We say that x is a base-b, e-power, height-h, u-attracted number if h is the smallest positive integer such that $S^h_{e,b}(x) = u$. Happy numbers are then base-10, 2-power, 1-attracted numbers of any height. Let $\sigma_{h,e,b}(u)$ denote the smallest height-h, u-attracted number for a fixed base b and exponent e and let $g(e)$ denote the smallest number such that every integer can be written as $x_1^e + x_2^e + \cdots + x_{g(e)}^e$ for some nonnegative integers $x_1, x_2, \ldots, x_{g(e)}$. We prove that if $p_{e,b}$ is the smallest nonnegative integer such that $b^{p_{e,b}} > g(e)$,

$$d = \left\lceil \frac{g(e)+1}{1-\left(\frac{b-2}{b-1}\right)^e} + e + p_{e,b} \right\rceil,$$

and $\sigma_{h,e,b}(u) \geq b^d$, then $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$.

1. Introduction

Let $S_{e,b}$ be the function that maps a positive base-b integer to the sum of its digits raised to the e-th power, where e is a positive integer. That is, for $x = \sum_{i=0}^{n-1} a_i b^i$, with $0 \leq a_i \leq b-1$ for all i,

$$S_{e,b}\left(\sum_{i=0}^{n-1} a_i b^i\right) = \sum_{i=0}^{n-1} a_i^e.$$

If $S^h_{e,b}(x) = 1$ for some integer h, then x is said to be an e-power, b-happy number. Guy [2004] gave the smallest 2-power, 10-happy numbers of heights 0 through 6 and asked if 78999 is the smallest height-7 happy number. Grundman and Teeple [2003] answered Guy, giving the smallest 2-power, 10-happy numbers of heights 0 through 10, and 3-power, 10-happy numbers of heights 0 through 8. From Grundman and Teeple's work, one can extract an algorithm for finding the smallest happy number of height $h+1$ if the smallest happy number of height h is known. The main results of this paper are Theorems 3.1 and 3.3, which jointly imply that once

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the smallest height-$(h+1)$, u-attracted, base-b number is sufficiently large, applying $S_{e,b}$ to that number will yield the smallest height-h, u-attracted, base-b number. The results of this paper hold not only for happy numbers (i.e., 1-attracted), but more generally for u-attracted numbers. Moreover, our results hold for all bases and exponents.

Definition 1.1. For a fixed base b, exponent e, and positive integer u, we say that a positive integer x is u-attracted if $S^n_{e,b}(x) = u$ for some nonnegative integer n. If h is the smallest nonnegative integer so that $S^h_{e,b}(x) = u$ then x is a height-h, u-attracted number. (As a convention, $S^0_{e,b}(x) = x$.)

Definition 1.2. For a fixed base b, exponent e, positive integer u, and nonnegative integer h, let $\sigma_{h,e,b}(u)$ denote the smallest height-h, u-attracted number, that is, the smallest positive integer k with the property that $S^h_{e,b}(k) = u$ and $S^n_{e,b}(k) \neq u$ for $n < h$. Similarly, for positive h, let $\tau_{h,e,b}(u)$ denote the second smallest height-h, u-attracted number, that is, $S^h_{e,b}(l) = u,\ S^n_{e,b}(l) \neq u$ for $n < h$, and $\sigma_{h,e,b}(u) < l$.

Some of the following proofs rely upon knowing the smallest integer x such that for a given e, every integer is expressible as the sum of at most x many integers raised to the e-th power. We define $g(e)$ for this purpose.

Definition 1.3. For a fixed positive integer e, let $g(e)$ denote the smallest integer such that every nonnegative integer is expressible as $x_1^e + x_2^e + \cdots + x_{g(e)}^e$, where $x_1, x_2, \ldots, x_{g(e)}$ are all nonnegative integers.

This is the well-known Waring's problem. Many surveys about the history of this problem exist; see for instance [Vaughan and Wooley 2002].

For the entirety of this paper, we assume that the base $b \geq 2$ is an integer, the exponent $e \geq 1$ is an integer, the height h is a nonnegative integer, the attractor u is a positive integer, and that x denotes a positive integer. Additionally, when we say $\lceil x \rceil = y$ we mean that y is the smallest integer such that $y \geq x$, and similarly, if $\lfloor x \rfloor = y$, then y is the largest integer such that $y \leq x$.

2. Mapping attracted numbers

In this section, we establish in Theorem 2.2 a criterion, depending on $g(e)$ that ensures that $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$ for a fixed base b, exponent e, height h, and integer u.

Lemma 2.1. *Fix a base b, exponent e, and attractor u. The smallest positive integer x such that $S_{e,b}(x) = u$ has n digits, where*

$$\frac{u}{(b-1)^e} \leq n \leq \frac{u}{(b-1)^e} + g(e).$$

Proof. Since the maximum value of the image of each digit under $S_{e,b}$ is $(b-1)^e$, $u/(b-1)^e$ is a lower bound for the number of digits of x. Let q and r be the quotient and remainder of u divided by $(b-1)^e$, respectively; that is, q is a nonnegative integer, $0 \le r < (b-1)^e$, and $u = q(b-1)^e + r$. Let $x_1, \ldots, x_{g(e)}$ be integers such that $x_1^e + \cdots + x_{g(e)}^e = r$. Since $r < (b-1)^e$, we have $x_1, \ldots, x_{g(e)} < b-1$ and so they are valid digits in base b. Without loss of generality, $x_1 \le x_2 \le \cdots \le x_{g(e)}$. Let y be the positive integer formed by the digits $x_1, x_2, \ldots, x_{g(e)}$ followed by q digits, each of which is $b-1$. Since x is minimal, it follows that $x \le y$. So n, the number of digits of x, must be less than or equal to the number of digits of y, which is $\lfloor u/(b-1)^e \rfloor + g(e)$. □

Theorem 2.2. *Fix a base b, exponent e, positive height h, and attractor u. If*

$$\frac{\sigma_{h,e,b}(u)}{(b-1)^e} + g(e) \le \frac{\tau_{h,e,b}(u)}{(b-1)^e}, \tag{1}$$

then $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$.

Proof. Let x be the smallest integer such that $S_{e,b}(x) = \sigma_{h,e,b}(u)$. Let z be a height-h, u-attracted number that is greater than $\sigma_{h,e,b}(u)$ (recall that $\tau_{h,e,b}$ is the smallest such number) and y any integer such that $S_{e,b}(y) = z$. That is, y is a height-$(h+1)$, u-attracted number whose image is not $\sigma_{h,e,b}(u)$. Let n be the number of digits of x and m be the number of digits of y. We will show that $x < y$. By Lemma 2.1,

$$n \le \frac{\sigma_{h,e,b}(u)}{(b-1)^e} + g(e) \quad \text{and} \quad \frac{\tau_{h,e,b}(u)}{(b-1)^e} \le \frac{z}{(b-1)^e} \le m.$$

By the hypothesis (1), this gives $n \le m$. If $n < m$, then $x < y$, so let us suppose that $n = m$. It must then be the case that

$$\frac{\sigma_{h,e,b}(u)}{(b-1)^e} + g(e) = \frac{z}{(b-1)^e}.$$

Since $S_{e,b}(y) = z$ and y has $m = z/(b-1)^e$ digits, y is the concatenation of m digits, each of which is $b-1$. Since $x \ne y$ (as they have different images under $S_{e,b}$) and x and y have the same number of digits, at least one digit of x is not $b-1$. Thus, $x < y$. Hence x is less than every other height-$(h+1)$, u-attracted number, and so $x = \sigma_{h+1,e,b}(u)$. Since $S_{e,b}(x) = \sigma_{h,e,b}(u)$, we have $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$. □

From [Grundman and Teeple 2003], it is known that $\sigma_{7,2,10} = 78999$ and $\tau_{7,2,10}(1) = 79899$.

Question 2.3. *Under what conditions is $\tau_{h,e,b}(u)$ a permutation of the digits of $\sigma_{h,e,b}(u)$?*

3. Large u-attracted numbers

In this section, we prove Theorems 3.1 and 3.3, which imply that once $\sigma_{h,e,b}(u)$ is sufficiently large, $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$.

Theorem 3.1. *Fix a base b, exponent e, positive height h, and attractor u. Let δ be a positive integer, and let*

$$d = \frac{g(e)+1}{1-\left(\frac{b-2}{b-1}\right)^e} + \delta.$$

If $\sigma_{h,e,b}(u)$ has at least d digits, then the base-b expansion of $\sigma_{h,e,b}$ is of the form

$$\sigma_{h,e,b}(u) = \sum_{i=0}^{n-1} a_i b^i$$

with $a_0, \dots, a_\delta = b-1$. More informally, the rightmost $\delta+1$ digits of $\sigma_{h,e,b}(u)$ are all $b-1$.

Proof. In this proof, we will show that if $\sigma_{h,e,b}$ has "too many" digits which are not equal to $b-1$, we can construct a smaller number with the same image as $\sigma_{h,e,b}$. This contradicts the definition of $\sigma_{h,e,b}$.

One can verify $\sigma_{1,e,b}(1) = 10$ (in base b) for all e, b and that this is the only number of the form $\sigma_{h,e,b}$ with a 0 digit. However, 10 is a two-digit number and $d > 2$ for integers $e > 1$. Thus, using the base-b expansion from the statement of the theorem, $a_{i+1} \le a_i$ for $0 \le i < n-1$ (its digits must appear in increasing order from left to right) and none of its digits can be 0 since $\sigma_{h,e,b}(u)$ is the least height-h, u-attracted number.

In the case $a_i = b-1$ for all i, this theorem is trivially true. Otherwise, let us construct z, the sum of the image of the digits which are not equal to $b-1$. In the case that some digits of $\sigma_{h,e,b,}(u)$ are $b-1$ and some are not, define an integer parameter $k \ge 2$ to be such that $a_{k-1} < b-1$ and for all $i < k-1$, $a_i = b-1$. That is, the k-th place is the first (from the right) in which a digit that is not $b-1$ appears. Hence,

$$\sigma_{h,e,b}(u) = \sum_{i=k-1}^{n-1} a_i b^i + \sum_{i=0}^{k-2} (b-1) b^i.$$

Let $y = S_{e,b}(\sigma_{h,e,b}(u))$ and let $z = y - (k-1)(b-1)^e$, that is,

$$z = \sum_{i=k-1}^{n-1} a_i^e.$$

In the case that no digits of $\sigma_{h,e,b}$ are $b-1$, set $k = 1$ and let $z = \sum_{i=0}^{n-1} a_i^e$. We proceed to show that if $k \le \delta + 1$, we can construct a number smaller than $\sigma_{h,e,b}$ with the same image as $\sigma_{h,e,b}$, a contradiction. Let $n' = n - (k-1)$ and let $m = \lfloor z/(b-1)^e \rfloor$. Since z is the sum of n' many terms of the form a_i^e, where $a_i \leq b-2$ for all i, we have $n' \geq z/(b-2)^e$. Thus,

$$\frac{(b-2)^e}{(b-1)^e} n' \geq \frac{z}{(b-1)^e} \geq m.$$

So,

$$\left(\frac{b-2}{b-1}\right)^e n' + g(e) + 1 \geq m + g(e) + 1.$$

By the definition of d,

$$d - \delta = \frac{g(e)+1}{1 - \left(\frac{b-2}{b-1}\right)^e},$$

and since $k \leq \delta + 1$,

$$d - (k-1) \geq \frac{g(e)+1}{1 - \left(\frac{b-2}{b-1}\right)^e}.$$

Thus,

$$(d-(k-1))\left(1 - \left(\frac{b-2}{b-1}\right)^e\right) \geq g(e) + 1.$$

And since $n' \geq d - (k-1)$ and $1 - \left(\frac{b-2}{b-1}\right)^e > 0$, we have

$$n'\left(1 - \left(\frac{b-2}{b-1}\right)^e\right) \geq g(e) + 1$$

and hence

$$n' \geq g(e) + 1 + n'\left(\frac{b-2}{b-1}\right)^e \geq m + g(e) + 1.$$

Therefore, $n' \geq m + g(e) + 1$.

Let r be the remainder of y divided by $(b-1)^e$; that is, $y = q(b-1)^e + r$, where $q \geq 0$ and $(b-1)^e > r \geq 0$. From the definition of m, we have $q = m + (k-1)$. Let $x_1, x_2, \ldots, x_{g(e)}$ be integers less than $b-1$ so that $x_1^e + x_2^e + \cdots + x_{g(e)}^e = r$. There are such x_j since $g(e)$ is defined so that such integers exist, and all integers must be less than $b-1$ since $r < (b-1)^e$. Without loss of generality, $x_1 \leq x_2 \leq \cdots \leq x_{g(e)}$. Let x be a base-b number with digits $x_1, \ldots, x_{g(e)}$ followed by $m + (k-1)$ many $b-1$ digits.

Hence, $S_{e,b}(x) = y$, and x has at most $g(e) + m + (k-1)$ digits. Since $n' = n - (k-1)$, we know $n \geq g(e) + 1 + m + (k-1)$. However, this means that x has fewer digits than $\sigma_{h,e,b}(u)$. This contradicts the fact that $\sigma_{h,e,b}(u)$ is the smallest height-h, u-attracted integer, and hence, $k > \delta + 1$. $\square$

For ease of notation, we define a constant $p_{e,b}$.

Definition 3.2. For a fixed exponent e and base b, let $p_{e,b}$ be the smallest integer such that $b^{p_{e,b}} > g(e)$.

Theorem 3.3. *Fix a base b, exponent e, positive height h, and attractor u. If $\sigma_{h,e,b}(u) = \sum_{i=0}^{n-1} a_i b^i$, where $a_0, \dots, a_{e+p_{e,b}} = b-1$, then $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$.*

Proof. Let $\sigma_{h,e,b}(u)$ be such that $a_0, \dots, a_k = b-1$, where $k \geq e + p_{e,b}$. Define $c_j = \sigma_{h,e,b}(u) + j$ for $1 \leq j < g(e)(b-1)^e$. We will show that c_1 through $c_{g(e)(b-1)^e-1}$ are not height-h, u-attracted numbers.

If $b > 2$, using the definition of $p_{e,b}$ we get

$$j < g(e)(b-1)^e < b^{p_{e,b}}(b-1)^e < b^{p_{e,b}} b^e = b^{e+p_{e,b}}.$$

Since $\sigma_{h,e,b}$ has at least $e + p_{e,b} + 1$ trailing digits equal to $b-1$, we know c_1 has at least $e + p_{e,b} + 1$ trailing zeros. Since $j < b^{e+p_{e,b}}$, we know j has at most $e + p_{e,b}$ many digits. Hence c_j has at least one digit which is zero for $1 \leq j < g(e)(b-1)^e$. Let c'_j be formed by removing the all zero digits of c_j. We claim that $c'_j < \sigma_{h,e,b}(u)$. Recall that n denotes the number of digits of $\sigma_{h,e,b}(u)$. If $a_i \neq b-1$ for some i, then $n \geq e + p_{e,b} + 2$ and c_j has n digits for all j. Thus, c'_j has at most $n-1$ digits and hence $c'_j < \sigma_{h,e,b}$. If $a_i = b-1$ for all i, then $\sigma_{h,e,b}(u) = b^n - 1$ and $c_1 = b^n = b^{e+p_{e,b}+1}$, which means that $c_j < b^{e+p_{e,b}+1} + b^{e+p_{e,b}}$. Thus c'_j has at most n digits, while the leading digit of $\sigma_{h,e,b}$ is $b-1$, but the leading digit of c'_j is 1, and since $b \neq 2$, $c'_j < \sigma_{h,e,b}$.

This leaves only the case that $b = 2$. In this case,

$$j < g(e)(2-1)^e = g(e) < 2^{p_{e,2}}.$$

Since the only allowable digits are 0 and 1, and we argued in the proof of Theorem 3.1 that $\sigma_{h,e,b}$ does not have any digits that are equal to zero, $\sigma_{h,e,2} = 2^{n+1} - 1$ for some $n \geq e + p_{e,2}$, so $2^{n+1} \leq c_j < 2^{n+1} + 2^{p_{e,2}}$ for all j. Since $n \geq e + p_{e,2}$ and e is at least 1, c_j has at least two digits that are equal to 0. Again, let c'_j be formed by removing the all zero digits of c_j. Then c'_j has fewer than n digits and hence $c'_j < \sigma_{h,e,2}$.

So, if any c_j are height-h, u-attracted numbers, then c'_j is a smaller height-h, u-attracted number than $\sigma_{h,e,b}(u)$, contradicting the definition of $\sigma_{h,e,b}(u)$. Hence, $\tau_{h,e,b}(u) \geq g(e)(b-1)^e + \sigma_{h,e,b}(u)$. Therefore, by Theorem 2.2, $S_{e,b}(\sigma_{h+1,e,b}) = \sigma_{h,e,b}$. □

Corollary 3.4. *Fix a base b and exponent e. Let*

$$d = \left\lceil \frac{g(e)+1}{1 - \left(\frac{b-2}{b-1}\right)^e} + e + p_{e,b} \right\rceil.$$

If $\sigma_{h,e,b}(u) \geq b^d$, then $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$.

Proof. Since $\sigma_{h,e,b}(u) \geq b^d$, we know $\sigma_{h,e,b}(u)$ must have at least $d-1$ digits. Hence, by Theorem 3.1, $\sigma_{h,e,b}(u) = \sum_{i=0}^{n-1} a_i b^i$, where for $i \leq e + p_{e,b}$, we have $a_i = b-1$. Therefore, by Theorem 3.3, $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$. □

Corollary 3.4 gives a bound b^d for $\sigma_{h,e,b}(u)$ (in terms of e and b) so that if $\sigma_{h,e,b}(u) \geq b^d$, then $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}$. This leads to the natural question:

Question 3.5. *Is there a bound β for h (in terms of e and b) so that if $h \geq \beta$, $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}$?*

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References

[Grundman and Teeple 2003] H. G. Grundman and E. A. Teeple, "Heights of happy numbers and cubic happy numbers", *Fibonacci Quart.* **41**:4 (2003), 301–306. MR Zbl

[Guy 2004] R. K. Guy, *Unsolved problems in number theory*, 3rd ed., Springer, 2004. MR Zbl

[Vaughan and Wooley 2002] R. C. Vaughan and T. D. Wooley, "Waring's problem: a survey", pp. 301–340 in *Number theory for the millennium, III* (Urbana, IL, 2000), edited by M. A. Bennett et al., A K Peters, Natick, MA, 2002. MR Zbl

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