involve
a journal of mathematics

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msp

2015 vol. 8, no. 2

INVOLVE 8:2 (2015)
dx.doi.org/10.2140/involve.2015.8.211

Convergence of the maximum zeros of a class of Fibonacci-type polynomials

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(Communicated by Kenneth S. Berenhaut)

Let a be a positive integer and let k be an arbitrary, fixed positive integer. We define a generalized Fibonacci-type polynomial sequence by $G_{k,0}(x) = -a$, $G_{k,1}(x) = x - a$, and $G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x)$ for $n \geq 2$. Let $g_{k,n}$ represent the maximum real zero of $G_{k,n}$. We prove that the sequence $\{g_{k,2n}\}$ is decreasing and converges to a real number β_k. Moreover, we prove that the sequence $\{g_{k,2n+1}\}$ is increasing and converges to β_k as well. We conclude by proving that $\{\beta_k\}$ is decreasing and converges to a.

1. Introduction

Let α, β, and k be integers, with $\alpha \neq 0$. Consider a Fibonacci-type polynomial sequence given by the recurrence relation $G_{k,0} = -\alpha$, $G_{k,1} = x - \beta$, and for $n \geq 2$,

$$G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x). \tag{1}$$

We should point out that the classical Fibonacci polynomial sequence F_n is obtained when $\alpha = -1$, $\beta = 0$, and $k = 1$. Moreover, the Lucas polynomial sequence L_n is obtained when $\alpha = -2$, $\beta = 0$, and $k = 1$. Hoggatt and Bicknell [1973] give explicit forms for the zeros of F_n and L_n. Even though finding explicit formulas for other Fibonacci-type polynomial sequences has been a challenge, several results about the properties of the zeros of some specific cases are known. For example, G. Moore [1994] and H. Prodinger [1996] studied the asymptotic behavior of the maximal zeros of $G_{1,n}$ when $\alpha = \beta = k = 1$, and Yu, Wang and He [Yu et al. 1996] generalized Moore's result for $\alpha = \beta = a$, where a is any positive integer. F. Mátyás [1998] studied the same problem for $\alpha = a$, $a \neq 0$ and $\beta = \pm a$. More recently, Wang and He [2004] generalized their previous result for any two integers α and β with $\alpha \neq 0$. We also mention the works of P. E. Ricci [1995] and Mátyás [1998] for boundedness results of the zeros of $G_{1,n}$. In addition, Molina and Zeleke [2007; 2009] studied the asymptotic behavior of the zeros of $G_{k,n}$ when $\alpha = \beta = 1$ and k is an arbitrary integer.

MSC2010: primary 11B39; secondary 11B37, 30C15.
Keywords: Fibonacci polynomial, convergence, zeros, roots.

Moore [1994] proved that when $\alpha = \beta = k = 1$, the maximum zeros of the odd-indexed polynomials converge to $\frac{3}{2}$ from below and the maximum roots of the even-indexed polynomials converge to $\frac{3}{2}$ from above. In that article, a remark was made about the possibilities of investigating asymptotic behaviors of maximum zeros of other Fibonacci-type polynomial sequences. In [Miller and Zeleke 2013], the first author and Zeleke studied the maximum real zeros of the Fibonacci-type polynomial sequence where $\alpha = \beta = a$, a is a positive integer, and $k = 2$. They provided asymptotic results for the maximum real zeros numerically as well as analytically. We extend those results by allowing k to be an arbitrary, fixed positive integer. The proof techniques expand those used in [Miller and Zeleke 2013] and [Molina and Zeleke 2009].

Before delving into the technical results, we provide a numerical example to motivate our work.

Example. Consider the Fibonacci-type polynomial sequence given by the recurrence relation $G_{k,0} = -2,\ G_{k,1} = x - 2$, and for $n \geq 2$,

$$G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x).$$

In the context of the generalized Fibonacci-type polynomial sequences we study in this paper, this example corresponds to the case when $a = 2$. For a fixed positive integer k and a natural number n, let $g_{k,n}$ represent the maximum real root of the polynomial $G_{k,n}$. The first six terms in the sequences of the maximum real roots for $k = 2,\ k = 3$, and $k = 4$ are shown in the following three columns, respectively.

$g_{2,1} = 2$	$g_{3,1} = 2$	$g_{4,1} = 2$
$g_{2,2} \doteq 2.359304086$	$g_{3,2} \doteq 2.190327947$	$g_{4,2} \doteq 2.102374082$
$g_{2,3} \doteq 2.350513611$	$g_{3,3} \doteq 2.188965777$	$g_{4,3} \doteq 2.102149889$
$g_{2,4} \doteq 2.350789278$	$g_{3,4} \doteq 2.188978002$	$g_{4,4} \doteq 2.102150474$
$g_{2,5} \doteq 2.350780807$	$g_{3,5} \doteq 2.188977893$	$g_{4,5} \doteq 2.102150473$
$g_{2,6} \doteq 2.350781067$	$g_{3,6} \doteq 2.188977894$	$g_{4,6} \doteq 2.102150473$

For each sequence, the subsequence created by the odd-indexed (i.e., n is odd) maximum real roots is increasing. And, the subsequence created by the even-indexed (i.e., n is even) maximum real roots is decreasing. In fact, each of the sequences converge to a real number which is dependent on k. We call this real number β_k. We should mention β_k is also dependent on our choice of a and for this example, $a = 2$. For the sequences above, we have

$$\beta_2 \doteq 2.350781059, \quad \beta_3 \doteq 2.188977894, \quad \beta_4 \doteq 2.102150473.$$

It is also the case that $\{\beta_k\}$ converges to 2 and it is not a coincidence that this is the value of a.

2. Formulas

At this time, we introduce a few handy formulas that were established in [Molina and Zeleke 2009]. The formulas in the following lemma allow us to write $G_{k,n}(x)$ in terms of smaller indexed functions.

Lemma 2.1. *For $n \geq 1$, the following recursive formulas are true*:

$$G_{k,2n+2}(x) = (x^{2k}+1)G_{k,2n}(x)+x^{2k}G_{k,2n-2}(x)+\cdots+x^{2k}G_{k,2}(x)+x^{k}G_{k,1}(x),$$
$$G_{k,2n+1}(x) = (x^{2k}+1)G_{k,2n-1}(x)+x^{2k}G_{k,2n-3}(x)+\cdots+x^{2k}G_{k,1}(x)+x^{k}G_{k,0}(x).$$

The formula that we present in the next lemma provides a type of shift from one indexed polynomial evaluated at $g_{k,n}$ to another indexed polynomial evaluated at $g_{k,n}$. The proof can be found in [Molina and Zeleke 2009, Lemma 4].

Lemma 2.2. *For $n \geq m$, $G_{k,n+m}(g_{k,n}) = (-1)^{m+1}G_{k,n-m}(g_{k,n})$.*

3. Preliminary results

We're now ready to study the maximum real roots, $g_{k,n}$, for the generalized Fibonacci-type polynomial sequence defined by $G_{k,0}(x) = -a$, $G_{k,1}(x) = x - a$, and $G_{k,n}(x) = x^k G_{k,n-1}(x) + G_{k,n-2}(x)$ for $n \geq 2$, where a is a positive integer and k is an arbitrary, fixed positive integer.

Proposition 3.1. *If $n \geq 2$, then $g_{k,n} \in (a, a+1)$.*

Proof. For $n \geq 2$, we will show $G_{k,n}(a) < 0$ and $G_{k,n}(x) > 0$ for $x \in [a+1, \infty)$; thus, our conclusion will follow. We'll begin by showing $G_{k,n}(a) < 0$ by induction. Since $G_{k,0}(a) = -a$ and $G_{k,1}(a) = a - a = 0$, we have $G_{k,2}(a) = a^k(0) - a = -a < 0$. Now suppose $G_{k,m}(a) < 0$ for all m such that $2 \leq m \leq n$. By (1) and the inductive hypothesis, $G_{k,n+1}(a) = a^k G_{k,n}(a) + G_{k,n-1}(a) < 0$. Hence, $G_{k,n}(a) < 0$ for $n \geq 2$.

For the remainder of the proof, let $x \in [a+1, \infty)$. We again use induction. Notice

$$G_{k,1}(x) = x - a \geq a + 1 - a > 0, \quad \text{and}$$
$$G_{k,2}(x) = x^k(x-a) - a \geq (a+1)^k(a+1-a) - a = (a+1)^k - a > 0.$$

Now suppose $G_{k,m}(x) > 0$ for all m such that $2 \leq m \leq n$. By (1) and the inductive hypothesis, it follows that $G_{k,n+1}(x) = x^k G_{k,n}(x) + G_{k,n-1}(x) > 0$. Hence, $G_{k,n}(x) > 0$ for $x \in [a+1, \infty)$ and $n \geq 2$.

Therefore, $g_{k,n} \in (a, a+1)$ for $n \geq 2$. □

Proposition 3.2. *Let a be a positive integer and let β_k be a positive real number that satisfies the equation $G_{k,2}(x) = -(a-x)^2/a$; that is, β_k is a zero of $T_k(x) = ax^k - a^2x^{k-1} + x - 2a$. Then*

$$G_{k,n}(\beta_k) = \frac{-(a-\beta_k)^n}{a^{n-1}} \quad \text{for all } n \geq 0.$$

Proof. We prove this proposition by induction. The result is true for $n=0$ and $n=1$ by simple computation. It is true for $n=2$ by construction. Now assume $G_{k,n}(\beta_k) = -(a-\beta_k)^n/a^{n-1}$ for all positive integers less than or equal to n. Then

$$\begin{aligned}
G_{k,n+1}(\beta_k) &= \beta_k^k G_{k,n}(\beta_k) + G_{k,n-1}(\beta_k) \\
&= \beta_k^k\left(\frac{-(a-\beta_k)^n}{a^{n-1}}\right) + \frac{-(a-\beta_k)^{n-1}}{a^{n-2}} \\
&= \frac{-(a-\beta_k)^{n-1}}{a^{n-2}}\left(\frac{\beta_k^k(a-\beta_k)}{a}+1\right) \\
&= \frac{-(a-\beta_k)^{n-1}}{a^{n-2}}\left(\frac{a\beta_k^k(a-\beta_k)+a^2}{a^2}\right) \\
&= \frac{-(a-\beta_k)^{n-1}}{a^{n}}\left(a\beta_k^k(a-\beta_k)+a^2\right) \\
&= \frac{-(a-\beta_k)^{n-1}}{a^{n}}\left(-a(\beta_k^k(\beta_k-a)-a)\right) \\
&= \frac{-(a-\beta_k)^{n-1}}{a^{n}}\left(-a\left(\frac{-(a-\beta_k)^2}{a}\right)\right) \\
&= \frac{-(a-\beta_k)^{n-1}}{a^{n}}(a-\beta_k)^2 \\
&= \frac{-(a-\beta_k)^{n+1}}{a^{n}}.
\end{aligned}$$

Therefore, our result is true for all nonnegative integers. □

We remind the reader that whenever β_k is used in this article, it will be dependent on the choice of a.

Corollary 3.3. $$\lim_{n\to\infty} G_{k,n}(\beta_k) = 0.$$

Proof. Before we begin, we kindly remind the reader that $k \geq 1$ and this assumption is continued throughout our work unless stated otherwise. Now the first fact we establish for this proof is that $\beta_k \in (a, a+1)$. To show this, we will again consider $T_k(x) = ax^k - a^2x^{k-1} + x - 2a$. It is easily verified that $T_k(a) < 0 < T_k(a+1)$. Moreover, T_k is strictly increasing on the interval $[a, \infty)$, which will be shown by examining the first derivative of T_k. Notice

$$\begin{aligned}
T_k'(x) &= kax^{k-1} - (k-1)a^2x^{k-2} + 1 \\
&= ax^{k-2}(kx - ka + a) + 1 \\
&= ax^{k-2}(k(x-a)+a) + 1 \\
&> 0
\end{aligned}$$

for all $x \in [a, \infty)$. Thus, $\beta_k \in (a, a+1)$. Therefore,

$$\lim_{n\to\infty} G_{k,n}(\beta_k) = \lim_{n\to\infty} \frac{-(a-\beta_k)^n}{a^{n-1}} = 0. \qquad \square$$

4. Analysis of $G'_{k,3}(x)$

In order to prove our main result on the convergence of the maximum zeros, we will need a lower bound on the values $G'_{k,n}(g_{k,n})$. This section will provide a lower bound of $G'_{k,3}(x)$ on the interval $[g_{k,3}, \infty)$. We begin with a couple of lemmas to help us achieve this lower bound.

Lemma 4.1. *For $k \geq 3$, $G''_{k,3}(x)$ has exactly one zero in the interval* $(0, \infty)$.

Proof. Let $k \geq 3$ and recall $G_{k,3}(x) = x^{2k+1} - ax^{2k} - ax^k + x - a$. Thus,

$$\begin{aligned} G''_{k,3}(x) &= (2k+1)(2k)x^{2k-1} - 2ka(2k-1)x^{2k-2} - k(k-1)ax^{k-2} \\ &= kx^{k-2}\big(2(2k+1)x^{k+1} - 2a(2k-1)x^k - a(k-1)\big) \\ &= kx^{k-2} f(x), \end{aligned}$$

where $f(x) = 2(2k+1)x^{k+1} - 2a(2k-1)x^k - a(k-1)$. We can see that 0 is a zero of $G''_{k,3}$. In order to show $G''_{k,3}$ has only one zero in $(0, \infty)$, we will show that $f(x)$ has exactly one zero in $(0, \infty)$. To do so, consider

$$\begin{aligned} f'(x) &= 2(2k+1)(k+1)x^k - 2a(2k-1)kx^{k-1} \\ &= 2x^{k-1}\big((2k+1)(k+1)x - a(2k-1)k\big). \end{aligned}$$

The critical numbers of f are

$$c_1 = 0 \quad \text{and} \quad c_2 = \frac{a(2k-1)k}{(2k+1)(k+1)}.$$

Using this information, it can be verified that f is decreasing on $(0, c_2)$ and increasing on (c_2, ∞). Pairing this with $f(0) = -a(k-1) < 0$ and $\lim_{x\to\infty} f(x) = \infty$, we conclude f, and hence $G''_{k,3}$, has exactly one zero in $(0, \infty)$. Therefore, our conclusion holds. $\square$

Lemma 4.2. *For $k \geq 3$, $G'_{k,3}(x)$ has exactly two zeros in the interval* $(0, \infty)$.

Proof. Let $k \geq 3$ and recall $G_{k,3}(x) = x^{2k+1} - ax^{2k} - ax^k + x - a$. Thus,

$$G'_{k,3}(x) = (2k+1)x^{2k} - 2kax^{2k-1} - kax^{k-1} + 1.$$

Using the intermediate value theorem and the inequalities $G'_{k,3}(0) = 1 > 0$, $G'_{k,3}(1) = k(2-3a)+2 \leq -1 < 0$, and $\lim_{x\to\infty} G'_{k,3}(x) = \infty$, we can conclude $G'_{k,3}(x)$ has at least two zeros in $(0, \infty)$. To show there can be no more than two zeros in $(0, \infty)$, we will explore the possibility of $G'_{k,3}(x)$ having at least three zeros in $(0, \infty)$. If

$G'_{k,3}(x)$ has at least three zeros in $(0, \infty)$, then $G''_{k,3}$ would have at least two zeros in $(0, \infty)$ by Rolle's theorem, but, by Lemma 4.1, we know this cannot be the case. Thus, $G'_{k,3}(x)$ has exactly two zeros in $(0, \infty)$ and since $G'_{k,3}(0) \neq 0$, those two zeros are indeed in $(0, \infty)$. □

We are now ready to obtain a lower bound on $G'_{k,3}(x)$ for $x \in [g_{k,3}, \infty)$.

Proposition 4.3. *If $k \geq 1$ and $x \in [g_{k,3}, \infty)$, then $G'_{k,3}(x) > 1$.*

Proof. Let $x \in [g_{k,3}, \infty)$. We break our proof into cases.

Case 1: Consider $k = 1$. We then have

- $G_{1,3}(x) = x^3 - ax^2 - ax + x - a$,
- $G'_{1,3}(x) = 3x^2 - 2ax - a + 1$, and
- $G''_{1,3}(x) = 6x - 2a$.

Since $G''_{1,3}(x) > 0$ for $x \in (a/3, \infty)$, we know $G'_{1,3}$ is increasing on $(a/3, \infty)$. Thus, $1 \leq G'_{1,3}(a) < G'_{1,3}(x)$ when $x \in [g_{1,3}, \infty)$ as $g_{1,3} > a$ by Proposition 3.1.

Case 2: Consider $k = 2$. We then have

- $G_{2,3}(x) = x^5 - ax^4 - ax^2 + x - a$,
- $G'_{2,3}(x) = 5x^4 - 4ax^3 - 2ax + 1$, and
- $G''_{2,3}(x) = 2(10x^3 - 6ax^2 - a)$.

Since $G''_{2,3}(x) > 0$ for $x \in (a, \infty)$, we know $G'_{2,3}$ is increasing on (a, ∞). Again notice $g_{2,3} > a$ by Proposition 3.1. Applying the mean value theorem, we know there exists $c \in (a, g_{2,3})$ such that

$$G'_{2,3}(c) = \frac{G_{2,3}(g_{2,3}) - G_{2,3}(a)}{g_{2,3} - a}.$$

It follows that when $x \in [g_{2,3}, \infty)$,

$$G'_{2,3}(x) > G'_{2,3}(c) = \frac{G_{2,3}(g_{2,3}) - G_{2,3}(a)}{g_{2,3} - a} = \frac{0 - G_{2,3}(a)}{g_{2,3} - a} = \frac{a^3}{g_{2,3} - a} > 1.$$

Case 3: Consider $k \geq 3$. By Lemma 4.1, we know $G''_{k,3}(x)$ has one positive root, call it r, and, by Lemma 4.2, we know $G'_{k,3}(x)$ has two positive roots, call them s and t, where $s < t$. Moreover, by Rolle's theorem, $s < r < t$. Notice that

- $G'_{k,3}(0) = 1 > 0$,
- $G'_{k,3}(1) = k(2 - 3a) + 2 \leq -1 < 0$,
- $\lim_{x\to\infty} G'_{k,3}(x) = \infty$, and
- $G''_{k,3}$ is positive on (r, ∞).

Thus, $s < 1 < t$. Moreover, $G'_{k,3}$ is negative on (s, t) and $G'_{k,3}$ is positive and increasing on (t, ∞), and, by the mean value theorem, there exists $c \in [1, g_{k,3}]$ such that

$$G'_{k,3}(c) = \frac{G_{k,3}(g_{k,3}) - G_{k,3}(1)}{g_{k,3} - 1} = \frac{0 - (2 - 3a)}{g_{k,3} - 1} = \frac{3a - 2}{g_{k,3} - 1} \geq 1.$$

Hence, $c > t$, and thus $g_{k,3} > t$. Therefore, if $x \in [g_{k,3}, \infty)$, then

$$G'_{k,3}(x) > G'_{k,3}(c) \geq 1.$$

Therefore, our conclusion holds for all cases. □

We're now ready to prove that all of the first derivatives of the polynomials are bounded below by 1 as well as explore the characteristics of the maximum zeros. We break this up into two sections, one with the odd-indexed polynomials and the other with the even-indexed polynomials.

5. Odd-indexed polynomials

We will use the following two propositions to help establish our results. The proofs are left to the reader as they are similar to those found in [Molina and Zeleke 2009, Lemmas 6 and 7].

Proposition 5.1. *The maximum zeros of the odd-indexed polynomials $G_{k,2n+1}$ form a strictly increasing sequence.*

Proposition 5.2. *If $n \geq 0$, then the derivative of $G_{k,2n+1}(x)$ is bounded below by 1 for $x \in [g_{k,2n+1}, \infty)$.*

Proposition 5.3. *If $n \geq 0$, then $g_{k,2n+1} < \beta_k$ for each $k \geq 1$.*

Proof. By Proposition 3.2 and for $n \geq 1$,

$$G_{k,2n+1}(\beta_k) = \frac{-(a - \beta_k)^{2n+1}}{a^{2n}} > 0$$

as $\beta_k \in (a, a + 1)$. Our goal is to show that

$$G'_{k,2n+1}(x) > G'_{k,2n-1}(x) > \cdots > G'_{k,3}(x) > G'_{k,1}(x) = 1$$

for $x \in [\beta_k, \infty)$ as it will then follow that $g_{k,2n+1} < \beta_k$. Now, since $G_{k,3}(x) \leq 0$ on $[a, g_{k,3}]$, it must be the case that $\beta_k > g_{k,3}$. Proposition 5.2 gives

$$G'_{k,3}(x) > G'_{k,1}(x) = 1$$

on $[g_{k,3}, \infty)$. Thus,

$$G'_{k,3}(x) > G'_{k,1}(x) = 1$$

on $[\beta_k, \infty)$ as $[\beta_k, \infty) \subseteq [g_{k,3}, \infty)$. We note that the rest of the proof follows a similar format to the induction argument used in Proposition 5.2 with $[\beta_k, \infty)$ replacing $[g_{k,2n+1}, \infty)$. □

6. Even-indexed polynomials

Proposition 6.1. *If $n \geq 1$, then the derivative of $G_{k,2n}(x)$ is bounded below by 1 for $x \in [g_{k,2n-1}, \infty)$.*

Proof. We will make use of induction to obtain our result. Let $x \in [g_{k,2n-1}, \infty)$. For $n = 1$, we have

$$G'_{k,2}(x) = (k+1)x^k - akx^{k-1} = x^{k-1}((k+1)x - ak) > 1.$$

By (1), we have

$$\begin{aligned} G_{k,2n}(x) &= x^k G_{k,2n-1}(x) + G_{k,2n-2}(x), \quad \text{and} \\ G'_{k,2n}(x) &= x^k G'_{k,2n-1}(x) + kx^{k-1} G_{k,2n-1}(x) + G'_{k,2n-2}(x). \end{aligned}$$

From Proposition 5.1, we know $kx^{k-1}G_{k,2n-1}(x) \geq 0$ as $x \in [g_{k,2n-1}, \infty)$. So,

$$G'_{k,2n}(x) \geq x^k G'_{k,2n-1}(x) + G'_{k,2n-2}(x).$$

Now suppose $G'_{k,2n-2}(x) \geq 1$. Then

$$\begin{aligned} G'_{k,2n}(x) &\geq x^k G'_{k,2n-1}(x) + G'_{k,2n-2}(x) \\ &> G'_{k,2n-2}(x) \quad (\text{as } x^k G'_{k,2n-1}(x) > 1 \text{ by Proposition 5.2}) \\ &\geq 1 \quad (\text{by the induction hypothesis}). \end{aligned}$$

Therefore, the derivative of the even-indexed polynomials are bounded below by 1 for $x \in [g_{k,2n-1}, \infty)$. □

Referring back to Proposition 5.3, we should note that the result in Proposition 6.1 also holds for $x \in [\beta_k, \infty)$ as $[\beta_k, \infty) \subseteq [g_{k,2n-1}, \infty)$.

Proposition 6.2. *The maximum zeros of the even-indexed polynomials form a decreasing sequence that is bounded below by β_k.*

Proof. Let $n \geq 1$. By Proposition 3.2,

$$G_{k,2n}(\beta_k) = \frac{-(a-\beta_k)^{2n}}{a^{2n-1}} < 0.$$

Thus, $\beta_k < g_{k,2n}$. We proceed by induction to show the maximum zeros of the even-indexed polynomials form a decreasing sequence. Notice that

$$G_{k,4}(x) = x^k G_{k,3}(x) + G_{k,2}(x)$$

implies

$$G_{k,4}(g_{k,2}) = g_{k,2}^k G_{k,3}(g_{k,2}) + G_{k,2}(g_{k,2}) = g_{k,2}^k G_{k,3}(g_{k,2}) > 0$$

by utilizing Proposition 5.3. Since $G_{k,4}$ is increasing on $[\beta_k, \infty)$ as well, we conclude that $g_{k,2} > g_{k,4}$. Now assume $g_{k,2} > g_{k,4} > \cdots > g_{k,2n}$. By Lemma 2.2, $G_{k,2n-2}(g_{k,2n}) = -G_{k,2n+2}(g_{k,2n})$. Since $g_{k,2n-2} > g_{k,2n}$ (induction hypothesis), $G_{k,2n-2}$ is increasing on $[\beta_k, \infty)$, and $G_{k,2n-2}(g_{k,2n-2}) = 0$, it follows that

$$G_{k,2n-2}(g_{k,2n}) < 0 \quad \text{and} \quad G_{k,2n+2}(g_{k,2n}) > 0,$$

and, since $G_{k,2n+2}(x)$ is increasing on $[\beta_k, \infty)$, we have $g_{k,2n} > g_{k,2n+2}$. Therefore, $g_{k,2} > g_{k,4} > \cdots > \beta_k$. □

7. Main results

Theorem 7.1. *The sequence of odd-indexed zeros is increasing and converges to β_k, and the sequence of even-indexed zeros is decreasing and converges to β_k as well.*

Proof. By Proposition 5.1 and Proposition 5.3, we have shown the maximum zeros of the odd-indexed polynomials form an increasing sequence bounded above by β_k, and, by Proposition 6.2, we know the maximum zeros of the even-indexed polynomials form a decreasing sequence bounded below by β_k. In order to show both of the sequences converge to β_k, we will show that $\lim_{n\to\infty} g_{k,n} = \beta_k$. The mean value theorem tells us there exists a real number c between $g_{k,n}$ and β_k such that

$$|G'_{k,n}(c)| = \left|\frac{G_{k,n}(\beta_k) - G_{k,n}(g_{k,n})}{\beta_k - g_{k,n}}\right| = \left|\frac{G_{k,n}(\beta_k)}{\beta_k - g_{k,n}}\right|.$$

Since $G'_{k,n}(c) \geq 1$, $|\beta_k - g_{k,n}| \leq |G_{k,n}(\beta_k)|$. By utilizing Corollary 3.3, which states $\lim_{n\to\infty} G_{k,n}(\beta_k) = 0$, we can say $\lim_{n\to\infty} g_{k,n} = \beta_k$. Therefore, the sequence of odd-indexed zeros and the sequence of even-indexed zeros converge to β_k. □

Theorem 7.2. *The sequence $\{\beta_k\}$ is decreasing and converges to a.*

Proof. We begin by referring the reader back to $T_k(x)$ as defined in Proposition 3.2. Recall that T_k is increasing on $[a, \infty)$ and $\beta_k \in (a, a+1)$ is a zero of T_k. Using the fact that β_k is a zero of T_k, we have $a\beta_k^k - a^2\beta_k^{k-1} = 2a - \beta_k$. Then

$$\begin{aligned} T_{k+1}(\beta_k) &= a\beta_k^{k+1} - a^2\beta_k^k + \beta_k - 2a = \beta_k(a\beta_k^k - a^2\beta_k^{k-1}) + \beta_k - 2a \\ &= \beta_k(2a - \beta_k) + \beta_k - 2a = (\beta_k - 1)(2a - \beta_k) \\ &> 0. \end{aligned}$$

Thus, $\beta_{k+1} < \beta_k$, which verifies that $\{\beta_k\}$ is decreasing. Now let $\varepsilon > 0$. Then

$$\begin{aligned}\lim_{k\to\infty} T_k(a+\varepsilon) &= \lim_{k\to\infty} [a(a+\varepsilon)^k - a^2(a+\varepsilon)^{k-1} + (a+\varepsilon) - 2a] \\ &= \lim_{k\to\infty} [a(a+\varepsilon)^{k-1}(a+\varepsilon-a) + a + \varepsilon - 2a] \\ &= \lim_{k\to\infty} [\varepsilon a(a+\varepsilon)^{k-1} + \varepsilon - a] \\ &= \infty.\end{aligned}$$

We then know that there exists $j \in \mathbb{Z}$ such that $T_j(a+\varepsilon) > 0$ and so $\beta_j \in (a, a+\varepsilon)$. Therefore, $\lim_{k\to\infty} \beta_k = a$. □

Acknowledgements

The authors would like to thank and acknowledge A. Zeleke for his introduction to this research topic via an REU program at Michigan State University.

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Received: 2012-10-07 Revised: 2013-06-16 Accepted: 2013-10-19

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2015 vol. 8 no. 2