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Bochner (p, Y)-operator frames

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Using the concepts of Bochner measurability and Bochner space, we introduce a continuous version of (p, Y)-operator frames for a Banach space. We also define independent Bochner (p, Y)-operator frames for a Banach space and discuss some properties of Bochner (p, Y)-operator frames.

1. Introduction and preliminaries

The concept of frames was first introduced in the context of nonharmonic Fourier series [Duffin and Schaeffer 1952], and after the publication of [Daubechies et al. 1986] it has found broad application in signal processing, image processing, data compression and sampling theory. In this paper we introduce *Bochner (p, Y)-operator frames*, which are the continuous version of (p, Y)-operator frames for a Banach space, introduced in [Cao et al. 2008]. The new frames also generalize the *continuous p-frames* introduced in [Faroughi and Osgooei 2011].

Throughout this paper H will be a Hilbert space and X will be a Banach space.

Definition 1.1. Let $\{f_i\}_{i\in I}$ be a sequence of elements of H. We say that $\{f_i\}_{i\in I}$ is a *frame* for H if there exist constants $0 < A \le B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \le \sum_{i\in I} |\langle f_i, h\rangle|^2 \le B\|h\|^2. \tag{1-1}$$

The constants A and B are called frame bounds. If A, B can be chosen so that $A = B$, we call this frame an A-tight frame and if $A = B = 1$ it is called a Parseval frame. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a Bessel sequence. If $\{f_i\}_{i\in I}$ is a Bessel sequence then the following operators are bounded:

$$T : l^2(I) \to H, \quad T(c_i) = \sum_{i\in I} c_i f_i, \tag{1-2}$$

$$T^* : H \to l^2(I), \quad T^*(f) = \{\langle f, f_i\rangle\}_{i\in I}, \tag{1-3}$$

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called the *synthesis* and *analysis* operators, respectively. Hence the *frame operator* S, given by

$$Sf = TT^*f = \sum_{i\in I} \langle f, f_i\rangle f_i, \tag{1-4}$$

is also bounded.

The theory of frames has a continuous version, as follows.

Definition 1.2 [Rahimi et al. 2006]. Let (Ω, μ) be a measure space. Let $f : \Omega \to H$ be weakly measurable (i.e., for each $h \in H$, the mapping $\omega \to \langle f(\omega), h\rangle$ is measurable). Then f is called a *continuous frame* or *c-frame* for H if there exist constants $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_\Omega |\langle f(\omega), h\rangle|^2 \, d\mu \leq B\|h\|^2. \tag{1-5}$$

In this context the synthesis operator $T_f : L^2(X, \mu) \to H$ is defined by

$$\langle T_f\phi, h\rangle = \int_X \phi(x)\langle f(x), h\rangle \, d\mu(x); \tag{1-6}$$

the analysis operator $T_f^* : H \to L^2(X, \mu)$ by

$$(T_f^*h)(x) = \langle h, f(x)\rangle, \quad x \in X; \tag{1-7}$$

and the frame operator by

$$S_f = T_f T_f^*. \tag{1-8}$$

By Theorem 2.5 in [Rahimi et al. 2006], S_f is positive, self-adjoint and invertible.

Suppose (Ω, Σ, μ) is a measure space, where μ is a positive measure.

Definition 1.3. A function $f : \Omega \to X$ is called *simple* if there exist $x_1, \dots, x_n \in X$ and $E_1, \dots, E_n \in \Sigma$ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$, where $\chi_{E_i}(\omega) = 1$ if $\omega \in E_i$ and $\chi_{E_i}(\omega) = 0$ if $\omega \in E_i^c$. If $\mu(E_i)$ is finite whenever $x_i \neq 0$ then the simple function f is *integrable*, and the integral is then defined by

$$\int_\Omega f(\omega)\, d\mu(\omega) = \sum_{i=1}^n \mu(E_i)x_i.$$

Definition 1.4. A function $f : \Omega \to X$ is called *Bochner-measurable* if there exists a sequence of simple functions $\{f_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\| = 0, \quad \mu\text{-a.e.}$$

Definition 1.5. A Bochner-measurable function $f : \Omega \to X$ is called *Bochner-integrable* if there exists a sequence of integrable simple functions $\{f_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \int_\Omega \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) = 0.$$

In this case, $\int_E f(\omega)\, d\mu(\omega)$ is defined by

$$\int_E f(\omega)\, d\mu(\omega) = \lim_{n\to\infty} \int_E f_n(\omega)\, d\mu(\omega), \quad E \in \Sigma.$$

Definition 1.6. A Banach space X has the *Radon–Nikodym property* if, for every finite measure space (Ω, Σ, μ) and every (finitely additive) X-valued measure γ on (Ω, Σ) that has bounded variation and is absolutely continuous with respect to μ, there is a Bochner-integrable function $g : \Omega \to X$ such that

$$\gamma(E) = \int_E g(\omega)\, d\mu(\omega)$$

for every measurable set $E \in \Sigma$.

Remark 1.7. Suppose that (Ω, Σ, μ) is a measure space and X^* has the Radon–Nikodym property. Let $1 \le p \le \infty$. The *Bochner space* $L^p(\mu, X)$ is defined to be the Banach space of (equivalence classes of) X-valued Bochner-measurable functions F on Ω whose L^p norm is finite; here the L^p norm is defined by

$$\|F\|_p = \left(\int_\Omega \|F(\omega)\|^p\, d\mu(\omega) \right)^{1/p}$$

if p is finite, and by the essential supremum of $\|F(\omega)\|$ if $p = \infty$. In [Diestel and Uhl 1977; Cengiz 1998; Fleming and Jamison 2008, p. 51] it is proved that if q is such that $\frac{1}{p} + \frac{1}{q} = 1$, then $L^q(\mu, X^*)$ is isometrically isomorphic to $(L^p(\mu, X))^*$ if and only if X^* has the Radon–Nikodym property. This isometric isomorphism

$$\psi : L^q(\mu, X^*) \to (L^p(\mu, X))^*$$

takes $g \in L^q(\mu, X^*)$ to ϕ_g, the linear map defined by

$$\phi_g(f) = \int_\Omega g(\omega)(f(\omega))\, d\mu(\omega), \quad f \in L^p(\mu, X).$$

So for all $f \in L^p(\mu, X)$ and $g \in L^q(\mu, X^*)$ we have

$$\psi(g)(f) = \langle f, \psi(g)\rangle = \int_\Omega g(\omega)(f(\omega))\, d\mu(\omega) = \int_\Omega \langle f(\omega), g(\omega)\rangle\, d\mu(\omega).$$

In the following, we use the notation $\langle f, g\rangle$ instead of $\langle f, \psi(g)\rangle$, so for all $f \in L^p(\mu, X)$ and $g \in L^q(\mu, X^*)$

$$\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle\, d\mu(\omega).$$

Hilbert spaces have the Radon–Nikodym property, so in particular, if H is a Hilbert space then $(L^p(\mu, H))^*$ is isometrically isomorphic to $L^q(\mu, H)$. So, for all $f \in L^p(\mu, H)$ and $g \in L^q(\mu, H)$, we have

$$\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega),$$

in which $\langle f(\omega), g(\omega)\rangle$ does not mean the inner product of elements $f(\omega)$, $g(\omega)$ in H, but

$$\langle f(\omega), g(\omega)\rangle = \nu(g(\omega))(f(\omega)),$$

where $\nu : H \to H^*$ is the isometric isomorphism between H and H^*.

Lemma 1.8. *Let (Ω, Σ, μ) be a measure space and suppose there exists $k > 0$ such that $\mu(E) \geq k$ for every nonempty measurable set E of Ω. For every $\omega \in \Omega$, define $P_\omega : L^p(\mu, X) \to X$, $P_\omega(G) = G(\omega)$. Then $\|P_\omega\| \leq k^{-1/p}$.*

Proof. For a fix $\omega_0 \in \Omega$, put

$$\Delta = \{\omega \in \Omega \mid \|G(\omega)\| \geq \|G(\omega_0)\|\}.$$

Then

$$\|G\|_p^p = \int_\Omega \|G(\omega)\|^p \, d\mu(\omega) \geq \int_\Delta \|G(\omega)\|^p \, d\mu(\omega) \geq \mu(\Delta)\|G(\omega_0)\|^p \geq k\|G(\omega_0)\|^p.$$

Hence

$$\|P_{\omega_0}\| = \sup_{\|G\|_p \leq 1} \|P_{\omega_0}(G)\| = \sup_{\|G\|_p \leq 1} \|G(\omega_0)\| \leq \sup_{\|G\|_p \leq 1} k^{-1/p}\|G\|_p = k^{-1/p}. \qquad \square$$

2. Bochner (p, Y)-Bessel mappings for X

Throughout this section and the next we will work with a second Banach space Y in addition to X. We denote by $B(X, Y)$ the space of bounded operators from X to Y.

Definition 2.1. Let $1 < p < \infty$, and let $F : \Omega \to B(X, Y)$ be a map; we write F_ω for $F(\omega)$. We say that F is a *Bochner (p, Y)-Bessel mapping for X* if the following conditions are met:

(i) For each $x \in X$, the mapping $\omega \mapsto F_\omega(x)$ from Ω into Y is Bochner-measurable.

(ii) There exists a positive constant B such that

$$\|F_.(x)\|_p \leq B\|x\| \quad \text{for all } x \in X, \tag{2-1}$$

where

$$\|F_.(x)\|_p = \left(\int_\Omega \|F_\omega(x)\|^p \, d\mu\right)^{1/p}. \tag{2-2}$$

We denote by $B_X^p(Y)$ the set of all Bochner (p, Y)-Bessel mappings for X. It is easy to see that this set is closed under addition (defined in the obvious way: for $F, K \in B_X^p(Y)$, the sum $F + K$ satisfies $(F + K)_\omega(x) = F_\omega(x) + K_\omega(x)$ for all $x \in X$ and $\omega \in \Omega$) and under multiplication by scalars. Thus $B_X^p(Y)$ is a vector space. We give it a norm as follows. The *Bessel bound of* $F \in B_X^p(Y)$ is the number

$$B_F = \inf\{B > 0 : B \text{ satisfies (2-1)}\}.$$

For every $F \in B_X^p(Y)$, define $R_F : X \to L^p(\mu, Y)$ by $x \mapsto F_.(x)$. This is clearly a linear map; we should that it is also bounded. For every $F \in B_X^p(Y)$,

$$\|R_F(x)\|_p = \|F_.(x)\|_p \le B\|x\|, \tag{2-3}$$

for any B satisfying (2-1). Together with the linearity of R_F this implies that

$$\|R_F\| \le B_F; \tag{2-4}$$

that is, $R_F \in B(X, L^p(\mu, Y))$. Now set

$$\|F\|_p = \|R_F\|. \tag{2-5}$$

By (2-4), $\|F\|_p \le B_F$. It is easy to show that this gives a norm on $B_X^p(Y)$.

Theorem 2.2. *Let* (Ω, Σ, μ) *be a measure space and suppose there exists* $k > 0$ *such that* $\mu(E) \ge k$ *for every nonempty measurable set* E *of* Ω. *For every* $1 < p < \infty$, *the mapping*

$$\Lambda : B_X^p(Y) \to B(X, L^p(\mu, Y))$$

given by $\Lambda(F) = R_F$ *is a linear isometric isomorphism, and* $B_X^p(Y)$ *is a Banach space over* $\mathbb{C}$.

Proof. Clearly, the mapping Λ is a linear isometry from $B_X^p(Y)$ into $B(X, L^p(\mu, Y))$. Next we prove that Λ is surjective.

Choose $\omega \in \Omega$. For every $A \in B(X, L^p(\mu, Y))$, define $F_\omega^A : X \to Y$ by

$$F_\omega^A(x) = P_\omega(A(x)) = A(x)(\omega), \quad x \in X.$$

By Lemma 1.8, we have $\|P_\omega\| \le k^{-1/p}$; hence $F_\omega^A \in B(X, Y)$ for all $\omega \in \Omega$. Now, consider the mapping

$$F^A : \Omega \to B(X, Y)$$

given by $\omega \mapsto F_\omega^A$. Since $F_.^A(x) = A(x)(\,\cdot\,) : \Omega \to Y$ for each $x \in X$, the mapping $\omega \mapsto F_\omega^A(x)$ from Ω into Y is Bochner-measurable and

$$\|A(x)\|_p = \int_\Omega \|A(x)(\omega)\|^p \, d\mu(\omega) = \int_\Omega \|F_\omega^A(x)\|^p \, d\mu(\omega) = \|F_.^A(x)\|_p.$$

Therefore

$$\|F_.^A(x)\|_p = \|A(x)\|_p \le \|A\|\|x\|.$$

Hence $F^A \in B_X^p(Y)$. Also, for all $\omega \in \Omega$ we have $R_{F^A}(x)(\omega) = F_\omega^A(x) = A(x)(\omega)$. Thus $R_{F^A}(x) = A(x)$ for all $x \in X$. This shows that $\Lambda(F^A) = R_{F^A} = A$; thus Λ is surjective and so bijective. Consequently, $B_X^p(Y)$ is isometrically isomorphic to the Banach space $B(X, L^p(\mu, Y))$. Therefore, $B_X^p(Y)$ is a Banach space over $\mathbb{C}$. $\square$

Theorem 2.3. *Let $1 < p < \infty$ and $F \in B_X^p(Y)$. Then, for every $y^* \in Y^*$, the mapping $F_\cdot^*(y^*) : \Omega \to X^*$, $F_\cdot^*(y^*)(\omega) = F_\omega^*(y^*)$ is a Bochner pg-Bessel mapping for X with respect to $\mathbb{C}$.*

Proof. Let $y^* \in Y^*$ and $x \in X$. Clearly for each $x \in X$ the map $\omega \mapsto \langle x, F_\omega^*(y^*)\rangle$ from Ω into $\mathbb{C}$ is measurable and

$$\begin{aligned}\int_\Omega |\langle x, F_\omega^*(y^*)\rangle|^p \, d\mu(\omega) &= \int_\Omega |\langle F_\omega(x), y^*\rangle|^p \, d\mu(\omega) \\ &\leq (\|y^*\|^p)\left(\int_\Omega \|F_\omega(x)\|^p \, d\mu(\omega)\right) \\ &\leq \|y^*\|^p B_F^p \|x\|^p. \qquad \square\end{aligned}$$

Theorem 2.4. *Let (Ω, μ) be a σ-finite measure space with positive measure μ and let $\Omega = \bigcup_{n\in\mathbb{N}} K_n$ with $K_n \subseteq K_{n+1}$. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $F : \Omega \to B(X, Y)$. The following assertions are equivalent*:

(i) $F \in B_X^p(Y)$.

(ii) *For each $x \in X$, $\int_\Omega \|F_\omega(x)\|^p \, d\mu(\omega) < \infty$.*

(iii) *For each $G \in L^q(Y^*)$, $\sup_{\|x\|\leq 1} |\int_\Omega \langle x, F_\omega^*(G(\omega))\rangle \, d\mu(\omega)| < \infty$.*

(iv) *The operator $S_F : L^q(Y^*) \to X^*$ defined by*

$$\langle x, S_F(G)\rangle = \int_\Omega \langle x, F_\omega^*(G(\omega))\rangle \, d\mu(\omega) \quad \text{for } x \in X$$

is well defined and bounded.

Proof. (i) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (i) Define $A_n : X \to L^p(Y)$ by $A_n(x)(\omega) = \chi_{K_n}(\omega) F_\omega(x)$. For every $n \in \mathbb{N}$, we have

$$\|A_n\| = \sup_{\|x\|\leq 1} \|A_n(x)\|_p \leq \|F_\omega\|.$$

Hence, for all $n \in \mathbb{N}$, $A_n \in B(X, L^p(Y))$. By the definition of R_F, for every $n \in \mathbb{N}$,

$$\begin{aligned}\|(R_F - A_n)(x)\|_p^p &= \int_\Omega \|R_F(x)(\omega) - A_n(x)(\omega)\|^p \, d\mu(\omega) \\ &= \int_\Omega \|F_\omega(x) - \chi_{K_n}(\omega) F_\omega(x)\|^p \, d\mu(\omega) \\ &= \int_{\Omega - K_n} \|F_\omega(x)\|^p \, d\mu(\omega).\end{aligned}$$

This converges to 0 as $n \to \infty$, proving that $\lim_{n\to\infty} A_n(x) = R_F(x)$ for all $x \in X$. By the Banach–Steinhaus theorem, $R_F \in B(X, L^p(Y))$ and $\|R_F\| = \sup \|A_n\| < \infty$. Hence $F \in B_X^p(Y)$.

(i) $\Rightarrow$ (iii) Let $G \in L^q(\mu, Y^*)$ be arbitrary. By the Hölder inequality, we have

$$\begin{aligned}\sup_{\|x\|\le 1}\left|\int_\Omega \langle x, F_\omega^*(G(\omega))\rangle \, d\mu(\omega)\right| &= \sup_{\|x\|\le 1}\left|\int_\Omega \langle F_\omega(x), G(\omega)\rangle \, d\mu(\omega)\right| \\ &\le \sup_{\|x\|\le 1}\left(\int_\Omega \|F_\omega(x)\|^p \, d\mu(\omega)\right)^{1/p}\left(\int_\Omega \|G\omega\|^q \, d\mu(\omega)\right)^{1/q} \le B_F\|G\|_q < \infty.\end{aligned}$$

(iii) $\Rightarrow$ (iv) Clearly S_F is well defined and by the proof of (i) $\Rightarrow$ (iii) we have

$$\|S_F\| = \sup_{\|G\|_q\le 1} \|S_F(G)\| = \sup_{\|G\|_q\le 1} \sup_{\|x\|\le 1} \langle S_F(G), x\rangle \le B_F < \infty.$$

(iv) $\Rightarrow$ (i) Take $G \in L^q(\mu, Y^*)$ such that $\|G(\omega)\| = 1$ for every $\omega \in \Omega$ and

$$\|F_\omega(x)\| = \langle F_\omega(x), G(\omega)\rangle = \langle x, F_\omega^*(G(\omega))\rangle \quad \text{for all } x \in X.$$

Define $\alpha_n : \Omega \to Y^*$ by $\alpha_n(\omega) = \chi_{K_n}(\omega)\|F_\omega(x)\|^{p-1}G(\omega)$. Then

$$\begin{aligned}\|\alpha_n\|_q &= \left(\int_\Omega \|\chi_{K_n}(\omega)\|F_\omega(x)\|^{p-1}G(\omega)\|^q \, d\mu(\omega)\right)^{1/q} \\ &= \left(\int_{K_n} \|F_\omega(x)\|^{q(p-1)} \, d\mu(\omega)\right)^{1/q} = \left(\int_{K_n} \|F_\omega(x)\|^p \, d\mu(\omega)\right)^{1/q}.\end{aligned}$$

Now, we have

$$\begin{aligned}\int_{K_n} \|F_\omega(x)\|^p \, d\mu(\omega) &= \int_{K_n} \langle x, \|F_\omega(x)\|^{p-1} F_\omega^*(G(\omega))\rangle \, d\mu(\omega) \\ &= \int_\Omega \big\langle x, \chi_{K_n}(\omega)\|F_\omega(x)\|^{p-1} F_\omega^*(G(\omega))\big\rangle \, d\mu(\omega) = \langle x, S_F(\alpha_n)\rangle \\ &\le \|x\|\|S_F\|\|\alpha_n\|_q = \|x\|\|S_F\|\left(\int_{K_n} \|F_\omega(x)\|^p \, d\mu(\omega)\right)^{1/q}.\end{aligned}$$

Thus

$$\left(\int_{K_n} \|F_\omega(x)\|^p \, d\mu(\omega)\right)^{1/p} \le \|x\|\|S_F\|. \tag{2-6}$$

By letting $n \to \infty$ in (2-6), we get $F \in B_X^p(Y)$. □

3. Bochner (p, Y)-operator frames

Definition 3.1. Let $1 < p < \infty$. A mapping $F : \Omega \to B(X, Y)$ is called a *Bochner (p, Y)-operator frame* for X if the following conditions hold:

(i) For each $x \in X$, the mapping $\omega \mapsto F_\omega(x)$ from Ω into Y is Bochner-measurable.

(ii) There exist positive constants A and B such that

$$A\|x\| \le \|F_{\cdot}(x)\|_p \le B\|x\| \quad \text{for all } x \in X, \tag{3-1}$$

where $\|F_{\cdot}(x)\|_p$ is as in (2-2). The *lower* and *upper bounds* of F are then given by

$$A_F = \sup\{A > 0 : A \text{ satisfies (3-1)}\}, \qquad B_F = \inf\{B > 0 : B \text{ satisfies (3-1)}\},$$

We denote by $F_X^p(Y)$ the set of all Bochner (p, Y)-operator frames for X.

Definition 3.2. A Bochner (p, Y)-operator frame F is called *tight* if $A_F = B_F$. If $A_F = B_F = 1$, we call F *normalized*. We denote by $TF_X^p(Y)$ and $NF_X^p(Y)$, respectively, the sets of all tight and normalized Bochner (p, Y)-operator frames for X.

Corollary 3.3. *Let $F \in B_X^p(Y)$.*

(i) *$F \in F_X^p(Y)$ if and only if R_F is bounded below if and only if R_F^* is surjective.*

(ii) *$F \in TF_X^p(Y)$ if and only if R_F is a scaled isometry.*

Lemma 3.4. (i) *If $F \in B_X^p(Y)$ then $R_F^*\psi = S_F$.*

(ii) *If Y is reflexive then $L^p(\mu, Y)$ is reflexive.*

Proof. (i) For all $g \in L^q(\mu, Y^*)$ and $x \in X$, we have

$$\begin{aligned}\langle x, R_F^*\psi(g)\rangle = \langle R_F x, \psi(g)\rangle &= \int_\Omega \langle F_\omega(x), g(\omega)\rangle \, d\mu(\omega) \\ &= \int_\Omega \langle x, F_\omega^*(g(\omega))\rangle \, d\mu(\omega) = \langle x, S_F g\rangle.\end{aligned}$$

(ii) Let $J_Y : Y \to Y^{**}$ be the canonical mapping. Suppose that Y is reflexive, that is $J_Y(Y) = Y^{**}$. For every $f \in L^p(\mu, Y)$, define $L^p(J_Y)(f(\omega)) = J_Y f(\omega)$, $\omega \in \Omega$. This gives a bijection $L^p(J_Y) : L^p(\mu, Y) \to L^p(\mu, Y^{**})$. By using Remark 1.7, we know that the mapping $\psi : L^q(\mu, Y^*) \to (L^p(\mu, Y))^*$ is a bijective bounded operator and so the adjoint $\psi^* : (L^p(\mu, Y))^{**} \to (L^q(\mu, Y^*))^*$ is bijective.
By using Remark 1.7 again, we obtain a bijective bounded operator

$$\psi' : L^p(\mu, Y^{**}) \to (L^q(\mu, Y^*))^*$$

such that for all $f \in L^p(\mu, Y^{**})$ and $g \in L^q(\mu, Y^*)$

$$\langle f, \psi' g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega).$$

For all $f \in L^p(\mu, Y)$, $g \in L^q(\mu, Y^*)$ we have

$$\big\langle g, (\psi^* \circ J_{L^p(\mu,Y)}) f\big\rangle = \langle \psi(g), J_{L^p(\mu,Y)} f\rangle = \langle f, \psi(g)\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega)$$

and

$$\begin{aligned}\big\langle g, (\psi' \circ L^p(J_Y)) f\big\rangle &= \langle g, (\psi'(J_Y f(\cdot)))\rangle \\ &= \int_\Omega \langle g(\omega), J_Y f(\omega)\rangle \, d\mu(\omega) \\ &= \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega).\end{aligned}$$

Therefore, $\psi^* \circ J_{L^p(\mu,Y)} = \psi' \circ L^p(J_Y)$ and hence $J_{L^p(\mu,Y)} = (\psi^*)^{-1} \circ \psi' \circ L^p(J_Y)$, which is a bijection. Hence $L^p(\mu, Y)$ is reflexive. □

Theorem 3.5. *Let $F \in B_X^p(Y)$, $G \in F_X^p(Y)$ and $\|F\|_p \le A_G$. Then*

$$F \pm G \in F_X^p(Y).$$

Proof. For each $x \in X$, we have

$$\|(F \pm G)_.(x)\|_p = \|F_.(x) \pm G_.(x)\|_p \ge \|G_.(x)\|_p - \|F_.(x)\|_p \ge (A_G - \|F\|_p)\|x\|$$

and

$$\|(F \pm G)_.(x)\|_p \le (\|F\|_p + \|G\|_p)\|x\|.$$

So $F \pm G \in F_X^p(Y)$. □

Theorem 3.6. *Let $F \in F_X^p(Y)$. Then for each $x^* \in X^*$, there exists an element $G \in L^p(\mu, Y^*)$ such that*

$$\langle y, x^*\rangle = \int_\Omega \langle y, F_\omega^*(G(\omega))\rangle \, d\mu(\omega), \quad y \in X.$$

Proof. By Lemma 3.4, we have $R_F^* \psi = S_F$. Since $F \in F_X^p(Y)$, it follows from Corollary 3.3 that R_F^* is surjective. Thus the operator $S_F : L^q(\mu, Y^*) \to X^*$ is a surjection. Let $x^* \in X^*$; then there exists a $G \in L^p(\mu, Y^*)$ such that $x^* = S_F(G)$, so

$$\langle y, x^*\rangle = \int_\Omega \langle y, F_\omega^*(G(\omega))\rangle \, d\mu(\omega), \quad y \in X. \qquad \square$$

Definition 3.7. A Bochner (p, Y)-operator frame for X is called *independent* if the operator S_F is injective, i.e., if for every $f \neq 0$ there exists $x \in X$ such that

$$\int_\Omega \langle x, F_\omega^*(f(\omega))\rangle \, d\mu(\omega) \neq 0.$$

We denote by $IF_X^p(Y)$ the set of all independent Bochner (p, Y)-operator frames for X.

Theorem 3.8. *Let F be an independent Bochner (p, Y)-operator frame for X. Then R_F is invertible.*

Proof. We already know that S_F is injective. By Lemma 3.4 and Corollary 3.3, we know that R_F^* is bijective. Hence R_F is invertible. □

Theorem 3.9. *Let (Ω, Σ, μ) be a measure space and suppose there exists $k > 0$ such that $\mu(E) \geq k$ for every nonempty measurable set E of Ω. For each $F \in IF_X^p(Y)$, there exists a unique Bochner (q, Y^*)-operator frame Q for X^* such that for all $y \in X$*

$$\langle y, x^*\rangle = \int_\Omega \langle y, F_\omega^* R_Q x^*(\omega)\rangle \, d\mu(\omega).$$

Proof. Let F be an independent Bochner (p, Y)-operator frame for X. Then Theorem 3.8 yields that the operator R_F is invertible, so by Lemma 3.4, S_F is invertible. We can define $Q_\omega = P_\omega S_F^{-1}, \omega \in \Omega$, where $P_\omega : L^q(\mu, Y^*) \to Y^*$ is defined by $P_\omega(G) = G(\omega)$. By Lemma 1.8, P_ω is bounded. Therefore $Q_\omega \in B(X^*, Y^*)$, $\omega \in \Omega$. For each $x^* \in X^*$, we have $Q_.(x^*) = S_F^{-1}(x^*)$, so for each $x^* \in X^*$, the mapping $\omega \mapsto Q_\omega(x^*)$ is Bochner-measurable and

$$\frac{1}{\|S_F\|}\|x^*\| \leq \left(\int_\Omega \|Q_\omega(x^*)\|^q \, d\mu\right)^{1/q} = \|S_F^{-1}(x^*)\| \leq \|S_F^{-1}\|\|x^*\|.$$

Hence, Q is a Bochner (q, Y^*)-operator frame for X^* with bounds $\|S_F\|^{-1}$ and $\|S_F^{-1}\|$. By the definition of Q, we obtain that $R_Q = S_F^{-1}$ and so $x^* = S_F R_Q x^*$, $x^* \in X^*$. Thus

$$\langle y, x^*\rangle = \int_\Omega \langle y, F_\omega^* R_Q x^*(\omega)\rangle \, d\mu(\omega), \quad y \in X.$$

Next, we will show the uniqueness of Q. Let W be a Bochner (q, Y^*)-operator frame for X^* such that for all $y \in X$

$$\langle y, x^*\rangle = \int_\Omega \langle y, F_\omega^* R_W x^*(\omega)\rangle \, d\mu(\omega), \quad x^* \in X^*.$$

Thus $S_F R_W = I_{X^*}$, or $R_W = S_F^{-1} = R_Q$. Therefore, $W = Q$. □

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