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# A note on moments in finite von Neumann algebras

Jon Bannon, Donald Hadwin and Maureen Jeffery

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By a result of the second author, the Connes embedding conjecture (CEC) is false if and only if there exists a self-adjoint noncommutative polynomial  $p(t_1, t_2)$  in the universal unital  $C^*$ -algebra  $\mathcal{A} = \langle t_1, t_2 : t_j = t_j^*, 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2 \rangle$  and positive, invertible contractions  $x_1, x_2$  in a finite von Neumann algebra  $\mathcal{M}$  with trace  $\tau$  such that  $\tau(p(x_1, x_2)) < 0$  and  $\text{Tr}_k(p(A_1, A_2)) \geq 0$  for every positive integer  $k$  and all positive definite contractions  $A_1, A_2$  in  $M_k(\mathbb{C})$ . We prove that if the real parts of all coefficients but the constant coefficient of a self-adjoint polynomial  $p \in \mathcal{A}$  have the same sign, then such a  $p$  cannot disprove CEC if the degree of  $p$  is less than 6, and that if at least two of these signs differ, the degree of  $p$  is 2, the coefficient of one of the  $t_i^2$  is nonnegative and the real part of the coefficient of  $t_1 t_2$  is zero then such a  $p$  disproves CEC only if either the coefficient of the corresponding linear term  $t_i$  is nonnegative or both of the coefficients of  $t_1$  and  $t_2$  are negative.

## 1. Introduction

The Connes embedding conjecture (CEC) is true if every separable type II<sub>1</sub> factor  $\mathcal{M}$  embeds in a tracial ultrapower  $\mathcal{R}^\omega$  of the amenable type II<sub>1</sub> factor  $\mathcal{R}$ . This question concerns the matricial approximation of the elements of a type II<sub>1</sub> factor  $\mathcal{M}$  with faithful normal trace state  $\tau$  in the sense we now recall. For an  $N$ -tuple  $(x_1, \dots, x_N)$  of self-adjoint elements in  $\mathcal{M}$ ,  $R > 0$ ,  $n, k \in \mathbb{N}$  and  $\varepsilon > 0$ , we let

$$\Gamma_R(x_1, \dots, x_N : n, k, \varepsilon)$$

denote the set of tuples  $(A_1, \dots, A_N)$  of those  $k \times k$  self-adjoint matrices over  $\mathbb{C}$  of operator norm at most  $R$  satisfying

$$\left| \tau(x_{i_1} x_{i_2} \dots x_{i_p}) - \frac{1}{k} \text{Tr}(A_{i_1} A_{i_2}, \dots, A_{i_p}) \right| < \varepsilon,$$

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whenever  $1 \leq p \leq n$  and  $(i_1, i_2, \dots, i_p) \in \{1, 2, \dots, N\}^p$ . We call the elements of  $\Gamma_R(x_1, \dots, x_N : n, k, \varepsilon)$  *approximating microstates* for  $(x_1, \dots, x_N)$  of precision  $(n, \varepsilon)$  using  $k \times k$  matrices of norm at most  $R$ . A separable type  $\text{II}_1$  factor  $\mathcal{M}$  embeds in an ultrapower  $\mathcal{R}^\omega$  if and only if for all tuples  $(x_1, \dots, x_N)$  of self-adjoint elements in  $\mathcal{M}$ , all  $n \in \mathbb{N}$  and all  $\varepsilon > 0$ , it is possible to find  $k \in \mathbb{N}$  and  $R > 0$  such that  $\Gamma_R(x_1, \dots, x_N : n, k, \varepsilon) \neq \emptyset$ . In [Rădulescu 1999] it is proved that this statement is true under the restriction that  $n \in \{2, 3\}$ , and that if the statement were true for  $n = 4$ , the CEC would follow.

Our paper concerns the following reformulation of the CEC:

**Theorem 1.1** [Hadwin 2001, Corollary 2.3]. *Let  $\mathcal{H}$  be a separable Hilbert space. The Connes embedding conjecture is false if and only if there is a positive integer  $n$ , a noncommutative polynomial  $p(t_1, t_2, \dots, t_n)$  in the universal unital  $C^*$ -algebra  $\mathcal{A}_n = \langle t_1, t_2, \dots, t_n : t_j = t_j^*, -1 < t_j \leq 1 \text{ for } 1 \leq j \leq n \rangle$  and an  $n$ -tuple  $(x_1, \dots, x_n)$  of self-adjoint contractions in  $B(H)$  such that*

- (i)  $\text{Tr}_k(p(A_1, A_2, \dots, A_n)) \geq 0$  for every positive integer  $k$  and every  $n$ -tuple  $(A_1, \dots, A_n)$  of self-adjoint contractions  $A_1, A_2, \dots, A_n$  in  $M_k(\mathbb{C})$ , and
- (ii)  $W^*(x_1, x_2, \dots, x_n)$  has a faithful tracial state  $\tau$  and  $\tau(p(x_1, x_2, \dots, x_n)) < 0$ .

It is well known that a separable type  $\text{II}_1$  factor  $\mathcal{M}$  embeds in an  $\mathcal{R}^\omega$  if and only if  $\mathcal{M} \otimes M_k(\mathbb{C})$  does for all  $k \in \mathbb{N}$ . If  $\mathcal{M}$  is generated by  $k$  self-adjoint elements then  $\mathcal{M} \otimes M_k(\mathbb{C})$  is generated by two self-adjoint elements [Sinclair and Smith 2008, Proposition 16.1.1]. Whenever  $x \in B(H)$  is a self-adjoint contraction and  $\varepsilon > 0$ , it follows (e.g., by the continuous functional calculus for  $x$ ) that

$$\frac{(1 + \varepsilon) + x}{2 + \varepsilon}$$

is a positive invertible contraction. Therefore, if we replace  $\mathcal{A}_n$  by

$$\mathcal{A} = \langle t_1, t_2 : t_j = t_j^*, 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2 \rangle,$$

and repeat the argument in [Hadwin 2001, Section 2], we obtain the following.

**Theorem 1.2.** *Let  $\mathcal{H}$  be a separable Hilbert space. The Connes embedding conjecture is false if and only if there is a noncommutative polynomial  $p(t_1, t_2)$  in the universal unital  $C^*$ -algebra  $\mathcal{A} = \langle t_1, t_2 : t_j = t_j^*, \text{ with } 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2 \rangle$ , and positive, invertible contractions  $x_1$  and  $x_2$  in  $B(H)$  such that*

- (i)  $\text{Tr}_k(p(A_1, A_2)) \geq 0$  for every positive integer  $k$  and all positive definite contractions  $A_1$  and  $A_2$  in  $M_k(\mathbb{C})$ , and
- (ii)  $W^*(x_1, x_2)$  has a faithful tracial state  $\tau$  and  $\tau(p(x_1, x_2)) < 0$ .

Also note that if a polynomial  $p \in \mathcal{A}$  satisfies (i) and (ii) in the theorem, then so does the polynomial  $p + p^*$ . We may therefore assume that the polynomial appearing in the theorem is self-adjoint.

Note that, even if we restrict our attention in [Theorem 1.1](#) (or [Theorem 1.2](#)) to the case where the degree of  $p$  is less than or equal to 3, we cannot use [\[Rădulescu 1999\]](#) to rule out the possibility of finding such a  $p$  that will disprove the CEC, because existing methods only allow us to use, when  $R' < R$ , the existence of a microstate in  $\Gamma_R(x_1, \dots, x_N : n, k, \varepsilon)$  to guarantee the existence of a microstate in  $\Gamma_{R'}(x_1, \dots, x_N : n', k, \varepsilon')$ , where  $\varepsilon' < \varepsilon$  and  $n' > n$  — that is, decreasing  $R$  comes at the expense of increasing  $n$ . See, for example, Proposition 2.4 of [\[Voiculescu 1994\]](#) or Lemma 4 of [\[Dostál and Hadwin 2003\]](#). Even if this difficulty were overcome, there is no guarantee that the matrices in any approximating microstates found would be positive definite. It behooves us, therefore, to either look for a noncommutative polynomial that may be used to disprove the CEC as prescribed in [Theorem 1.1](#), or to proceed inductively, by degree, to show that such a polynomial cannot exist.

In [Section 2](#) of this paper we prove, in [Corollary 2.5](#) and [Theorem 2.6](#) that if the real parts of all coefficients but the constant coefficient of a self-adjoint noncommutative polynomial  $p \in \mathcal{A}$  share the same sign, then such a  $p$  cannot disprove the CEC if the degree of  $p$  is less than 6. We prove in [Section 3](#) that if the degree of a self-adjoint noncommutative polynomial  $p \in \mathcal{A}$  is 2, the real part of the coefficient of  $t_1 t_2$  is zero and the coefficient of one of the  $t_i^2$  is nonnegative, then such a  $p$  disproves the CEC only if either the coefficient of the corresponding linear term  $t_i$  is nonnegative or if both of the coefficients of  $t_1$  and  $t_2$  are negative.

From here on in this paper, the symbols  $t_1$  and  $t_2$  will denote the standard generators of the universal  $C^*$ -algebra

$$\mathcal{A} = \langle t_1, t_2 : t_j = t_j^*, 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2 \rangle.$$

We refer the reader to [\[Kadison and Ringrose 1983; Sinclair and Smith 2008\]](#) for the basic theory of finite von Neumann algebras.

## 2. $\tau$ -symmetrizable monomials

We prove that if the real parts of all coefficients but the constant coefficient of self-adjoint  $p \in \mathcal{A}$  share the same sign, and the constant coefficient is positive, then  $p$  cannot disprove the CEC if its degree is less than six. Let  $\mathcal{M}$  be a finite von Neumann algebra with faithful trace state  $\tau$ , and  $0 < x_1, x_2 \leq 1$  self-adjoint contractions in  $\mathcal{M}$ .

**Definition 2.1.** A *symmetric expression* in  $x_1, x_2$  is a finite sequence

$$(w_0, w_1, \dots, w_{N-1}, w_N)$$

of elements in  $\mathcal{M}$ , where  $N \in \mathbb{N}$ ,  $w_k = x_i^s$  with  $i \in \{1, 2\}$ ,  $s \in \{1, 1/2\}$  and  $w_k = w_{N-k}$  for all  $k \in \{0, 1, \dots, N\}$ . A monic monomial  $m(x_1, x_2) = x_{i_1}x_{i_2}\dots x_{i_l} \in \mathcal{M}$  with  $i_j \in \{1, 2\}$  for  $j \in \{1, 2, \dots, l\}$  is  $\tau$ -symmetrizable if there exists a symmetric expression  $(w_0, w_1, \dots, w_{N-1}, w_N)$  in  $x_1, x_2$  such that

$$\tau(x_{i_1}x_{i_2}\dots x_{i_l}) = \tau(w_0w_1\dots w_{N-1}w_N).$$

The element  $w_0w_1\dots w_{N-1}w_N \in \mathcal{M}$  is called the element associated to the symmetric expression  $(w_0, w_1, \dots, w_{N-1}, w_N)$ .

**Lemma 2.2.** *If  $(w_0, w_1, \dots, w_{N-1}, w_N)$  is a symmetric expression in  $x_1, x_2$ , then the associated element  $w_0w_1\dots w_{N-1}w_N$  in  $\mathcal{M}$  is a nonnegative contraction.*

*Proof.* We prove this by induction on  $N + 1$ . If  $N + 1 = 1$ , then  $N = 0$  and the result is clear from the assumptions on the  $x_i$ .

Assume now that the result holds for  $N + 1 \leq l$ , that is, for all symmetric expressions  $(w_0, w_1, \dots, w_{j-1}, w_j)$  in  $x_1, x_2$  with  $j < l$ . Let  $(w_0, w_1, \dots, w_{l-1}, w_l)$  be a symmetric expression in  $x_1, x_2$ . Then so is  $(w_1, \dots, w_{l-1})$ . By the induction hypothesis,  $w_1\dots w_{l-1} \in \mathcal{M}$  is a nonnegative contraction. Since  $w_0 = w_l = x_i^s$  for some  $i \in \{1, 2\}$  and  $s \in \{1, \frac{1}{2}\}$ , we have

$$0 \leq w_0w_1\dots w_{l-1}w_l = x_i^s w_1\dots w_{l-1}x_i^s \leq x_i^{2s} \leq x_i \leq 1. \quad \square$$

**Remark 2.3.** It is a straightforward exercise to verify that every monic noncommutative monomial  $m(x_1, x_2)$  of degree less than six is  $\tau$ -symmetrizable in any finite von Neumann algebra  $\mathcal{M}$  with faithful trace state  $\tau$ . (Here, of course, it is essential that  $0 < x_1, x_2 \leq 1$ !)

**Corollary 2.4.** *If  $m(x_1, x_2) = x_{i_1}x_{i_2}\dots x_{i_l} \in \mathcal{M}$  is a  $\tau$ -symmetrizable monic monomial, then  $1 - \tau(m(x_1, x_2)) \geq 0$ .*

*Proof.* Since  $m$  is  $\tau$ -symmetrizable, there exists a symmetric expression

$$(w_0, w_1, \dots, w_{N-1}, w_N)$$

in  $x_1, x_2$  such that

$$\tau(x_{i_1}x_{i_2}\dots x_{i_l}) = \tau(w_0w_1\dots w_{N-1}w_N).$$

By Lemma 2.2 and the fact that  $\tau$  is a state,  $\tau(w_0w_1\dots w_{N-1}w_N) \leq 1$ .  $\square$

In the following two results,  $J = J \setminus \{0\}$  denotes a finite index set, and for all  $j \in J$ ,  $c_j \in \mathbb{C}$ , and  $m_j(t_1, t_2) \neq 1$  denotes a monic monomial in  $\mathcal{A}$ .

**Corollary 2.5.** *If  $0 < x_1, x_2 \leq 1$  in  $\mathcal{M}$  and  $p(t_1, t_2) = c_01 + \sum_{j \in J} c_j m_j(t_1, t_2)$  is a self-adjoint noncommutative polynomial in  $\mathcal{A}$  such that, such that  $c_0 > 0$ ,  $\operatorname{Re}(c_j) \geq 0$  for all  $j \in J$ ,  $p(1, 1) \geq 0$  and  $m_j(x_1, x_2)$  is  $\tau$ -symmetrizable for every  $j \in J$ , then  $\tau(p(x_1, x_2)) \geq 0$ .*

*Proof.* This is trivial application of [Corollary 2.4](#). □

**Theorem 2.6.** *If  $0 < x_1, x_2 \leq 1$  in  $\mathcal{M}$  and  $p(t_1, t_2) = c_0 1 + \sum_{j \in J} c_j m_j(t_1, t_2)$  is a self-adjoint noncommutative polynomial in  $\mathcal{A}$  such that  $c_0 > 0$ ,  $\operatorname{Re}(c_j) < 0$  for all  $j \in J$ ,  $p(1, 1) \geq 0$  and  $m_j(x_1, x_2)$  is  $\tau$ -symmetrizable for every  $j \in J$ , then  $\tau(p(x_1, x_2)) \geq 0$ .*

*Proof.* Suppose  $p(t_1, t_2)$  satisfies the hypotheses. We have

$$p(1, 1) = c_0 1 + \sum_{j \in J} c_j \geq 0,$$

and therefore

$$\tau(p(x_1, x_2)) \geq \sum_{j \in J} c_j (m_j(x_1, x_2) - 1) \geq 0. \quad \square$$

### 3. Degrees 1 and 2

In degree 1 it is convenient to consider the statement of [Theorem 1.1](#) above. The next result rules out the possibility of finding a polynomial  $p$  of degree 1 that will disprove the CEC via [Theorem 1.1](#). Observe that if  $p(s, t) = c_0 + c_1 s + c_2 t = \bar{c}_0 + \bar{c}_1 s + \bar{c}_2 t$  for any real numbers  $-1 \leq s, t \leq 1$  and that  $p(s, t) \geq 0$  for any such  $s$  and  $t$ , then  $c_0 \geq |c_1 + c_2|$ .

**Theorem 3.1.** *Let  $\mathcal{H}$  be a separable Hilbert space. Let  $x_1$  and  $x_2$  be self-adjoint contraction operators in  $B(H)$  such that  $W^*(x_1, x_2)$  has a faithful trace state  $\tau$ . If  $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 = \bar{c}_0 + \bar{c}_1 t_1 + \bar{c}_2 t_2$  is a self-adjoint polynomial in  $\mathcal{A}$  with  $c_0 \geq |c_1 + c_2|$  then  $\tau(p(x_1, x_2)) \geq 0$ .*

*Proof.* Observe that  $\tau(c_0 + c_1 x_1 + c_2 x_2) = c_0 + c_1 \tau(x_1) + c_2 \tau(x_2) \geq c_0 - |c_1 + c_2|$ , since  $-1 \leq \tau(x_i) \leq 1$  for  $i \in \{1, 2\}$ . □

We now turn to degree 2. We first prove in [Theorem 3.4](#) that if

$$p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_4 t_1 t_2 + \bar{c}_4 t_2 t_1 + c_5 t_2^2$$

is a quadratic, self-adjoint noncommutative polynomial such that either  $c_4$  is the only nonzero degree 2 term with  $2 \operatorname{Re}(c_4) \neq 0$  or one of  $c_3$  or  $c_5$  is positive, then whenever  $p(s, t)$  is nonnegative for all real numbers  $0 < s, t \leq 1$ , it follows that  $\operatorname{Tr}_k(p(A, B)) \geq 0$  for all positive definite contractions  $A$  and  $B$  in  $M_k(\mathbb{C})$ , for any  $k \in \mathbb{N}$ .

To prove the result above, we shall need the fact that any positive definite square matrix has strictly positive entries on its main diagonal. This is a direct consequence of Sylvester's minorant criterion for positive definiteness.

**Lemma 3.2.** *Let  $A = (A_{ij})_{i=1}^k \in M_k(\mathbb{C})$  be positive definite. Then  $A_{ii} > 0$  for all  $i \in \{1, 2, \dots, k\}$ .*

*Proof.* We prove this by induction on  $k$ . Recognize that the case  $k = 1$  is clear. Assume the claim holds for  $k = l$ , and that  $A = (A_{ij})_{i=1}^{l+1}$  is a positive definite matrix. By Sylvester's criterion,  $A = (A_{ij})_{i=1}^l$  is also positive definite, and therefore, by the induction hypothesis,  $A_{ii} > 0$  if  $i \in \{1, 2, \dots, l\}$ . We need only show  $A_{(l+1)(l+1)} > 0$ . Let  $v \in \mathbb{C}^{l+1}$  be the vector with 1 in its  $(l+1)$ -st row and zero elsewhere. Then  $\langle Av, v \rangle = A_{(l+1)(l+1)} > 0$  by the positive definiteness of  $A$ .  $\square$

We now observe that if a polynomial is nonnegative on  $(0, 1] \times (0, 1]$ , then its constant term must be nonnegative.

**Lemma 3.3.** *If  $p(s, t) = c_0 + c_1s + c_2t + c_3s^2 + 2 \operatorname{Re}(c_4)st + c_5t^2 \geq 0$  for all real numbers  $0 < s, t \leq 1$ , then  $c_0 \geq 0$ .*

*Proof.* For any  $\varepsilon > 0$  we have

$$0 < p(\varepsilon, \varepsilon) = c_0 + (c_1 + c_2 + (c_3 + 2 \operatorname{Re}(c_4) + c_5)\varepsilon)\varepsilon;$$

hence  $c_0 \geq 0$ .  $\square$

**Theorem 3.4.** *Let  $p(t_1, t_2) = c_0 + c_1t_1 + c_2t_2 + c_3t_1^2 + c_4t_1t_2 + \bar{c}_4t_2t_1 + c_5t_2^2$  be a self-adjoint noncommutative polynomial in  $\mathcal{A}$ . Suppose*

$$p(s, t) = c_0 + c_1s + c_2t + c_3s^2 + 2 \operatorname{Re}(c_4)st + c_5t^2 \geq 0$$

*for all real numbers  $0 < s, t \leq 1$ , and either  $c_3 = 0, c_5 = 0$  and  $2 \operatorname{Re}(c_4) \neq 0$  or  $c_3 > 0$  or  $c_5 > 0$ . Then  $\operatorname{Tr}_k(p(A, B)) \geq 0$  for any positive definite contractions  $A, B$  in  $M_k(\mathbb{C})$ .*

*Proof.* For simplicity, let us assume  $c_5 \geq 0$ . Let  $A, B$  be positive definite contractions in  $M_k(\mathbb{C})$ . By the spectral theorem, we may assume  $A = \operatorname{diag}(A_i)_{i=1}^k$  is diagonal. A simple computation establishes that, for all  $i \in \{1, 2, \dots, k\}$ ,

$$(p(A, B))_{ii} = p(A_i, B_{ii}) + \sum_{j \in \{1, 2, \dots, k\} \setminus \{i\}} c_5 |B_{ij}|^2.$$

Since  $A$  is a positive definite contraction, each  $A_i$  satisfies  $0 < A_i \leq 1$ . If we could establish that the matrix  $B_0 := \operatorname{diag}(B_{ii})_{i=1}^k$  is a positive definite contraction, then each  $p(A_i, B_{ii})$  would follow nonnegative by assumption and therefore  $\operatorname{Tr}_k(p(A, B)) = \sum_{i=1}^k (p(A, B))_{ii} \geq 0$ . Positivity of  $B_0$  is a simple consequence of the positive definiteness of  $B$ , since every diagonal entry of a positive definite matrix is strictly positive by Lemma 3.2. It remains to show that  $B_0$  is a contraction, which is equivalent to proving that  $I - B_0$  is positive semidefinite. We know, however, that  $I - B$  is positive semidefinite, and hence that for all  $\varepsilon > 0$  that  $(I + \varepsilon) - B$  is positive definite. Again as a consequence of Sylvester's criterion,  $((I + \varepsilon) - B)_{ii} > 0$  for all  $i \in \{1, 2, \dots, n\}$ , therefore for all such  $i$  it follows that  $1 + \varepsilon > B_{ii}$ , and hence  $1 \geq B_{ii}$ . It follows that  $I - B_0$  is positive semidefinite, hence  $B_0$  is a contraction.  $\square$

Let  $\mathcal{M}$  be a von Neumann algebra with faithful trace state  $\tau$ . Below,

$$\langle x, y \rangle_2 = \tau(y^*x) \quad \text{and} \quad \|x\|_2^2 = \tau(x^*x)^{1/2}, \quad \text{for } x, y \in \mathcal{M}.$$

Let  $n \in \mathbb{N}$  and  $x_1, x_2$  be positive invertible contractions in  $\mathcal{M}$ . For every  $k \in \mathbb{N}$ , there are spectral projections  $\{P_i^{(k)}\}_{i=1}^k$  in  $\{1, x_1\}''$  such that  $\tau(P_i^{(k)}) = 1/k$  for each  $i$  and

$$\left\| x_1 - \sum_{i=1}^k \frac{i-1}{k} P_i^{(k)} \right\| < \frac{1}{k}.$$

If  $i = j$ , let  $V_{ij}^{(k)} = P_i^{(k)}$ , and if  $i \neq j$ , let  $V_{ij}^{(k)}$  be a partial isometry in  $\mathcal{M}$  with initial projection  $P_j^{(k)}$  (meaning that  $V_{ij}^{(k)}(V_{ij}^{(k)})^* = P_j^{(k)}$ ) and final projection  $P_i^{(k)}$  (meaning that  $(V_{ij}^{(k)})^*V_{ij}^{(k)} = P_i^{(k)}$ ). We now prove that if  $x_2$  is sufficiently close (in  $\|\cdot\|_2$ ) to a positive definite element in the type I subfactor of  $\mathcal{M}$  generated by  $\{V_{ij}^{(k)}\}_{i,j=1}^k$ , then  $\tau(p(x_1, x_2)) \geq 0$  when  $p$  satisfies the hypotheses of [Theorem 3.4](#). In the statement of the theorem, we regard  $x_2$  as an operator matrix and compare it entry-wise to the element  $(b_{ij} V_{ij}^{(k)})_{i,j=1}^k$ .

**Theorem 3.5.** *Let  $\mathcal{M}$  be a finite von Neumann algebra with faithful trace state  $\tau$ , let  $x_1, x_2$  be positive, invertible elements in  $\mathcal{M}$ , and adopt the notation in the previous paragraph. Let  $p(t_1, t_2) = c_0 + c_1t_1 + c_2t_2 + c_3t_1^2 + c_4t_1t_2 + \bar{c}_4t_2t_1 + c_5t_2^2$  be a self-adjoint noncommutative polynomial in  $\mathcal{A}$ . Suppose that*

$$p(s, t) = c_0 + c_1s + c_2t + c_3s^2 + 2\operatorname{Re}(c_4)st + c_5t^2 \geq 0$$

for all real numbers  $0 < s, t \leq 1$ , that either  $c_3 = 0, c_5 = 0$  and  $2\operatorname{Re}(c_4) \neq 0$  or  $c_3 > 0$  or  $c_5 > 0$ , and that for all  $k \in \mathbb{N}$  there exists a type I subfactor of  $\mathcal{M}$  generated by  $\{V_{ij}^{(k)}\}_{i,j=1}^k$  as in the previous paragraph, and a positive definite contraction  $(b_{ij})_{i,j=1}^k \in M_k(\mathbb{C})$  such that

$$\left\| P_i^{(k)} x_2 P_j^{(k)} - b_{ij} V_{ij}^{(k)} \right\|_2 < \frac{1}{k^{100}}, \quad \text{for all } i, j \in \{1, 2, \dots, k\}.$$

Then  $\tau(p(x_1, x_2)) \geq 0$ .

*Proof.* Let  $D_k = \sum_{i=1}^k \frac{i-1}{k} P_i^{(k)}$  and  $B_k = \sum_{i,j=1}^k b_{ij} V_{ij}^{(k)}$ . Writing  $x_1 = D_k + (x_1 - D_k)$  and  $x_2 = B_k + (x_2 - B_k)$ , we have

$$\begin{aligned} & \tau(p(x_1, x_2)) \\ &= c_0 + c_1\tau(D_k + (x_1 - D_k)) + c_2\tau(B_k + (x_2 - B_k)) + c_3\tau((D_k + (x_1 - D_k))^2) \\ & \quad + 2\operatorname{Re}(c_4)\tau((D_k + (x_1 - D_k))(B_k + (x_2 - B_k))) + c_5\tau((B_k + (x_2 - B_k))^2) \\ &= p(\tau(D_k), \tau(B_k)) + c_1\tau(x_1 - D_k) + c_2\tau(x_2 - B_k) \\ & \quad + 2c_3\tau(D_k(x_1 - D_k)) + c_3\tau(x_1 - D_k)^2 + 2\operatorname{Re}(c_4)\tau(D_k(x_2 - B_k)) \\ & \quad + 2\operatorname{Re}(c_4)\tau(B_k(x_1 - D_k)) + 2\operatorname{Re}(c_4)\tau((x_1 - D_k)(x_2 - B_k)) \\ & \quad + 2c_5\tau(B_k(x_2 - B_k)) + c_5\tau((x_2 - B_k)^2). \end{aligned}$$

Therefore, by the triangle and Cauchy–Schwartz inequalities and the fact that the operator norm dominates the  $\|\cdot\|_2$ -norm,

$$|\tau(p(x_1), p(x_2)) - p(\tau(D_k), \tau(B_k))| \leq (|c_1| + |c_2| + 3|c_3| + 6\operatorname{Re}(c_4) + 3c_5) \frac{1}{k}.$$

Since  $W^*(D_k, B_k) \cong W^*(\operatorname{diag}((i-1)/k, i \in \{1, \dots, k\}), (b_{ij})_{i,j=1}^k) \subseteq M_k(\mathbb{C})$  via the obvious trace-preserving  $*$ -isomorphism, it follows that

$$\tau(p(x_1), p(x_2)) \geq 0. \quad \square$$

**Proposition 3.6.** *Let  $p(t_1, t_2) = c_0 + c_1t_1 + c_2t_2 + c_3t_1^2 + c_4t_1t_2 + \bar{c}_4t_2t_1 + c_5t_2^2$  be a self-adjoint noncommutative polynomial in  $\mathcal{A}$  satisfying the hypotheses of [Theorem 3.4](#), and let  $\mathcal{M}$  be a finite von Neumann algebra with faithful trace state  $\tau$ . If  $0 < x_1, x_2 \leq 1$  in  $\mathcal{M}$  then  $\tau(p(x_1, x_2)) < 0$  if and only if*

$$c_5\|x_2 - \tau(x_2)\|_2^2 + c_3\|x_1 - \tau(x_1)\|_2^2 + 2\operatorname{Re}(c_4)\langle x_1 - \tau(x_1), x_2 - \tau(x_2) \rangle_2 < -p(\tau(x_1), \tau(y_1)).$$

*Proof.* Writing each  $\tau(x_i x_j)$  as  $\tau((x_i - \tau(x_i)1)(x_j - \tau(x_j)1)) + \tau(x_i)\tau(x_j)$ , we see that

$$\begin{aligned} \tau(p(x_1, x_2)) &= p(\tau(x_1), \tau(y_1)) + c_5\|x_2 - \tau(x_2)\|_2^2 + c_3\|x_1 - \tau(x_1)\|_2^2 \\ &\quad + 2\operatorname{Re}(c_4)\langle x_1 - \tau(x_1), x_2 - \tau(x_2) \rangle_2. \end{aligned}$$

The result follows.  $\square$

In the rest of this section, we narrow down the possibilities for disproving the CEC using polynomials satisfying the hypotheses of [Theorem 3.4](#) in the nonrotated case, where  $\operatorname{Re}(c_4) = 0$ . We point out that if  $p(t_1, t_2) = c_0 + c_1t_1 + c_2t_2 + c_3t_1^2 + c_5t_2^2$  is a self-adjoint noncommutative polynomial in  $\mathcal{A}$  satisfying the hypotheses of [Theorem 3.4](#) with both  $c_5 \geq 0$  and  $c_3 \geq 0$ , then  $\tau(p(x_1, x_2)) \geq 0$  by the proof of [Proposition 3.6](#).

**Theorem 3.7.** *Let  $p(t_1, t_2) = c_0 + c_1t_1 + c_2t_2 + c_3t_1^2 + c_5t_2^2$  be a self-adjoint noncommutative polynomial in  $\mathcal{A}$  satisfying the hypotheses of [Theorem 3.4](#) with  $c_3 > 0$ ,  $c_5 < 0$  and such that  $c_1 \geq 0$  and  $c_2 \leq 0$ . Then, for any finite von Neumann algebra  $\mathcal{M}$  with faithful trace state  $\tau$ , we have*

$$\tau(p(x_1, x_2)) \geq 0,$$

for any positive definite contractions  $x_1$  and  $x_2$  in  $\mathcal{M}$ .

*Proof.* Assume that  $p(t_1, t_2)$  satisfies the hypotheses. Suppose that there exists a finite von Neumann algebra  $\mathcal{M}$  with faithful trace state  $\tau$  and positive definite

contractions  $x_1$  and  $x_2$  such that  $\tau(p(x_1, x_2)) < 0$ . If  $c_1 \geq 0$  and  $c_2 \leq 0$ , then

$$p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2,$$

so  $c_0 + (c_1 + c_3\varepsilon)\varepsilon + c_2 + c_5 \geq 0$  for every  $\varepsilon > 0$ , and hence  $c_0 \geq -c_5 - c_2$ . Thus

$$\begin{aligned} 0 &> c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2 \\ &\geq -c_5 - c_2 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2 \\ &= -c_5(1 - t_2^2) + c_3 t_1^2 + c_1 t_1 - c_2(1 - t_2), \end{aligned}$$

and

$$0 > -c_5 \tau(1 - x_2^2) + c_3 \tau(x_1^2) + c_1 \tau(x_1) - c_2 \tau(1 - x_2) \geq 0.$$

This is a contradiction. □

**Theorem 3.8.** *Let  $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$  be a self-adjoint non-commutative polynomial in  $\mathcal{A}$  satisfying the hypotheses of [Theorem 3.4](#) with  $c_3 > 0$ ,  $c_5 < 0$  and such that  $c_1 < 0$  and  $c_2 = 0$ . Then for any finite von Neumann algebra  $\mathcal{M}$  with faithful trace state  $\tau$ ,*

$$\tau(p(x_1, x_2)) \geq 0,$$

for any positive definite contractions  $x_1$  and  $x_2$  in  $\mathcal{M}$ .

*Proof.* Assume that  $p(t_1, t_2)$  satisfies the hypotheses. Let  $\mathcal{M}$  be a finite von Neumann algebra with faithful trace state  $\tau$  and let  $x_1$  and  $x_2$  be positive definite contractions. If  $c_1 < 0$  and  $c_2 = 0$ , then for every  $\varepsilon > 0$  letting  $t_1 = \varepsilon - c_1/(2c_3)$ ,

$$c_0 + c_3 \varepsilon^2 - \frac{c_1^2}{4c_3} + c_5 \geq 0,$$

and therefore  $c_0 \geq \frac{c_1^2}{4c_3} - c_5$ . Then

$$\begin{aligned} p(t_1, t_2) &= c_0 + c_3 \left( t_1 + \frac{c_1}{2c_3} \right)^2 - \frac{c_1^2}{4c_3} + c_5 t_2^2 \\ &\geq \frac{c_1^2}{4c_3} - c_5 + c_3 \left( t_1 + \frac{c_1}{2c_3} \right)^2 - \frac{c_1^2}{4c_3} + c_5 t_2^2 = -c_5(1 - t_2^2) + c_3 \left( t_1 + \frac{c_1}{2c_3} \right)^2. \end{aligned}$$

Therefore

$$\tau(p(x_1, x_2)) = -c_5 \tau(1 - x_2^2) + c_3 \tau \left( \left( x_1 + \frac{c_1}{2c_3} \right)^2 \right) \geq 0. \quad \square$$

The previous two theorems establish that any polynomial  $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$  in  $\mathcal{A}$  that has a chance to disprove the CEC must satisfy either  $c_2 > 0$  or both  $c_1 < 0$  and  $c_2 < 0$ .

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