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A family of 2-arc transitive pentagraphs with unbounded valency

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Abstract

We construct polygonal graphs on the points of a generalized polygon in general position with respect to a polarity.

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1 Polygonal graphs

Let (X, L, I) be a generalized n-gon with polarity σ. Let Z be the set of points in general position with respect to σ, i.e., $Z = \{x \in X \mid d(x, x^\sigma) \geq n-1\}$, with distances measured in the point-line incidence graph Σ of (X, L, I). (Thus, if n is even then $d(x, x^\sigma) = n-1$ and if n is odd then $d(x, x^\sigma) = n$ for $x \in Z$.) Define a graph Γ with vertex set Z by letting distinct vertices $x, y \in Z$ be adjacent (notation $x \sim y$) when $x \, I \, y^\sigma$.

Theorem 1.1. *If n is odd, then Γ has girth $g \geq n$ and each edge is contained in a unique n-gon. If n is even, then Γ has girth $g \geq n+1$ and each 2-path is contained in a unique $(n+1)$-gon.*

Proof. Let us first collect information about the vertex set Z.

Step 1. *If $x_0 \, I \, x_1^\sigma \, I \, x_2 \, I \, \ldots \, I \, x_{l-1} \, I \, x_0^\sigma \, I \, x_1 \, I \, \ldots \, I \, x_{l-1}^\sigma \, I \, x_0$ is a self-polar $2l$-circuit in Σ, and $l \leq n+1$, then $x_i \in Z$ $(0 \leq i \leq l-1)$.*

(Indeed, if $d_\Sigma(x_i, x_i^\sigma) = m$, then we find an $(m+l)$-circuit in Σ, so that $m + l \geq 2n$.)

Step 2. *If n is even, and $x \in Z$, and $x\,I\,x_1^\sigma\,I\,\ldots\,I\,x_{n-2}\,I\,x^\sigma$ is the unique path of length $n-1$ joining x to x^σ in Σ, then $x_i \notin Z$ $(1 \le i \le n-2)$.*

(Indeed, applying σ to this path, we find another path that must coincide with this path, so that $x_i^\sigma = x_{n-1-i}$ $(1 \le i \le n-2)$.)

Now look at the graph Γ. Note that if $x \sim y \sim z$ in Γ, then $x\,I\,y^\sigma\,I\,z$ in Σ.

Step 3. *Γ does not have even circuits of length less than $2n$ and no odd circuits of length less than n. In particular, if two vertices have distance less than n in Γ, then there is a unique shortest path in Γ joining them.*

(Indeed, if $x_0 \sim x_1 \sim \ldots \sim x_{l-1} \sim x_0$ is an l-circuit in Γ, and l is even, then $x_0\,I\,x_1^\sigma\,I\,x_2\,I\,\ldots\,I\,x_{l-1}^\sigma\,I\,x_0$ is an l-circuit in Σ, and it follows that $l \ge 2n$. If l is odd, then $x_0\,I\,x_1^\sigma\,I\,x_2\,I\,\ldots\,I\,x_{l-1}\,I\,x_0^\sigma$ is an l-path in Σ, and by Step 2 we have $l \ge n$.)

Step 4. *If n is odd, then each edge is contained in a unique n-gon.*

(Indeed, if n is odd, and xy is an edge in Γ, then $d_\Sigma(x,y) = n-1$ and in Σ there is a unique geodesic $x = x_0\,I\,x_1^\sigma\,I\,x_2\,I\,\ldots\,I\,x_{n-1} = y$ joining x and y. This geodesic is part of the self-polar $2n$-circuit

$$x_0\,I\,x_1^\sigma\,I\,x_2\,I\,\ldots\,I\,x_{n-1}\,I\,x_0^\sigma\,I\,x_1\,I\,x_2^\sigma\,I\,\ldots\,I\,x_{n-1}^\sigma\,I\,x_0$$

in Σ. Thus, by Step 1, $x_0 \sim x_1 \sim \ldots \sim x_{n-1} \sim x_0$ is the unique n-gon on the edge xy in Γ.)

Step 5. *If n is even, then each 2-path is contained in a unique $(n+1)$-gon.*

(Indeed, if $x \sim y \sim z$ in Γ, then $d_\Sigma(x,y) = d_\Sigma(y,z) = n$ (since by Step 2 the unique point on y^σ that has distance $n-2$ to y is not in Z). Let $x = x_0\,I\,x_1^\sigma\,I\,x_2\,I\,\ldots\,I\,x_{n-1}^\sigma = z^\sigma$ be the unique path of length $n-1$ in Σ joining x to z^σ. Then

$$x_0\,I\,x_1^\sigma\,I\,x_2\,I\,\ldots\,I\,x_{n-1}^\sigma\,I\,y\,I\,x_0^\sigma\,I\,x_1\,I\,\ldots\,I\,x_{n-1}\,I\,y^\sigma\,I\,x_0$$

is a self-polar $(2n+2)$-circuit in Σ. Thus, by Step 1, $x_0 \sim x_1 \sim \ldots \sim x_{n-1} = z \sim y \sim x$ is the unique $(n+1)$-gon on the path $x \sim y \sim z$ in Γ.)

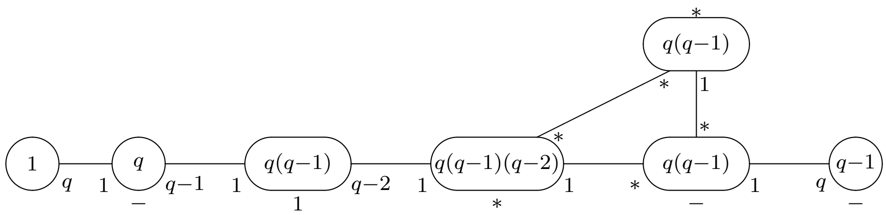
This completes the proof. □

If (X,L,I) is a $2m$-gon, then Γ is a single edge. If (X,L,I) is a $(2m+1)$-gon, then there are two possible polarities σ; for one choice of σ the graph Γ consists of a single vertex; for the other choice it is a $(2m+1)$-gon itself.

2 Pentagraphs

Now let us specialize to the finite case $n = 4$, i.e., let (X, L, I) be a generalized quadrangle of order q with a polarity σ. Then $2q$ is a square, cf. Payne [4]. Examples exist when q is an odd power of 2, cf. Tits [9]. We define the graph Γ as before. As we shall see, Γ is a *pentagraph*, that is, any 2-path in Γ is contained in a unique pentagon. (For this concept, and other examples, and some theory, see Perkel [5, 6, 7, 8] and Ivanov [3].)

Theorem 2.1. *Γ is a pentagraph of valency q on $q^3 + q$ vertices, and has distance distribution diagram*



Proof. Recall that a point or line is called *absolute* (for σ) if it is incident with its image (under σ). We shall use $\sim$ for adjacency in Γ, and $\perp$ for collinearity in (X, L).

Step 1. *Each line contains a unique absolute point, and, dually, each point is on a unique absolute line.*

(Indeed, if x is absolute, then x^σ is the only absolute line on x, and if x is not absolute then the unique line on x meeting x^σ is the only absolute line on x.)

Step 2. *The set A of absolute points under σ is an ovoid in (X, L). The graph Γ has $v = q(q^2 + 1)$ vertices.*

(Indeed, each $l \in L$ meets A in a unique point. It follows that $|A| = q^2 + 1$. But $|X| = (q + 1)(q^2 + 1)$.)

Step 3. *Γ is regular of valency q, and does not contain triangles. Adjacent vertices are non-collinear.*

(Indeed, the neighbours of x are the q nonabsolute points of x^σ.)

Step 4. *Γ does not have quadrangles, and any two vertices at distance 2 determine a unique pentagon. Two vertices have distance 2 if and only if they are collinear and the line joining them is non-absolute.*

(Indeed, if $x \sim y \sim z$, then x and z are joined by the line y^σ. In particular, y is the only common neighbour of x and z. Let $z \perp p \in x^\sigma$. Then $p \notin A$ because the unique absolute point on x^σ is collinear to x. Also the line $l = zp$ is not absolute because z^σ passes through y and $p \neq y$. It follows that $x \sim y \sim z \sim l^\sigma \sim p \sim x$ is the unique pentagon on x and z.)

Let us describe the distribution of vertices in Γ around a vertex x. Let m be the absolute line on x, and let $x' = m^\sigma = x^\sigma \cap A$ be its absolute point. The vertex set of Γ is partitioned into the following seven parts: $X_0 = \{x\}$, $X_1 = x^\sigma \setminus A$, $X_2 = x^\perp \setminus (A \cup m)$, $X_5 = m \setminus (A \cup \{x\})$, $X_{4a} = \{x'\}^\perp \setminus (A \cup m \cup x^\sigma)$, $X_{4b} = \{y \in X \setminus A \mid y \sim z \in X_1 \text{ and } yz \text{ is absolute}\}$, and X_3, consisting of the remaining points. Our aim is to show that X_i consists of the vertices at distance i from x in Γ, where X_{4a} and X_{4b} are distinguished by the fact that points in X_{4a} have neighbours in X_5. (Note however that for $q = 2$ we have $X_3 = \emptyset$, and the graph Γ is the disjoint union of two pentagons. If p is in the relation $4a$ to x, then x is in relation $4b$ to p, i.e., relations $4a$ and $4b$ are paired, while the remaining relations are self-paired.)

Step 5. *We have* $|X_0| = 1$, $|X_1| = q$, $|X_2| = q(q-1)$, $|X_3| = q(q-1)(q-2)$, $|X_{4a}| = |X_{4b}| = q(q-1)$, $|X_5| = q-1$.

(Indeed, the claims are clear for X_i with $i \leq 2$. The only vertices that do not have distance 2 to some vertex of X_1, are the vertices that either are collinear to the point $x' = x^\sigma \cap A$ (i.e., are in $X_{4a} \cup X_5$), or are joined to a vertex on x^σ by an absolute line (i.e., are in X_{4b}). The absolute line m on x contains q vertices, $q-1$ other than x, and none of them is collinear to a point in $X_0 \cup X_1 \cup X_2$, so these vertices have distance at least 5 to x. The vertices adjacent to some vertex in X_5 are the $q(q-1)$ vertices of X_{4a}. The vertices of X_3 are collinear to a unique vertex of x^σ, so this determines $|X_3|$.)

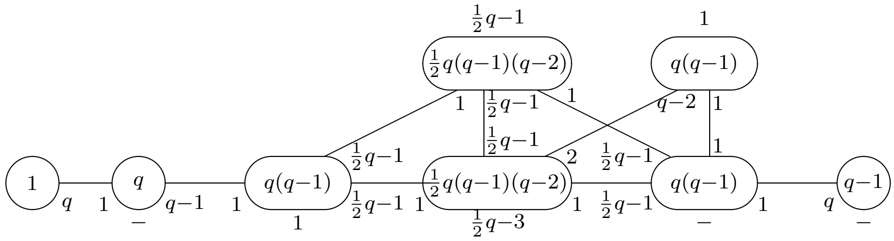
Step 6. *Each vertex in* $X_3 \cup X_{4b}$ *has a unique neighbour in* X_{4a}.

(Indeed, let $p \in X_3 \cup X_{4a}$. Then p^σ does not pass through x' (since $p \notin m$, i.e., $p \notin X_0 \cup X_5$), so x' is collinear with a unique point $z \in p^\sigma$. The line $x'z$ is not absolute (since $z \notin m$ because $p \notin X_{4a} \cup X_1$) and the point z is not absolute (since the line $x'z$ contains only one absolute point), so z is the unique neighbour of p in $X_1 \cup X_{4a}$. Clearly $z \in X_1$ iff $p \in X_2$.)

This proves everything claimed in the diagram. □

Now let us look at the special case where $q = 2^{2e} + 1$ and (X, L) is the $Sp(4, q)$ generalized quadrangle. The centralizer in $Sp(4, q)$ of the polarity σ is the Suzuki group $Sz(q)$ of order $(q^2 + 1)q^2(q-1)$. This group is 2-transitive on the set A of absolute points, and 2-arc transitive on the graph Γ (cf. Tits [9, Th. 6.1]). In this case we can be more precise about the stars in the diagram above.

Theorem 2.2. *The graph Γ is a 2-arc transitive pentagraph with distance distribution diagram*



Proof. If p and x are two non-collinear points, then $\{p,x\}^\perp$ is a hyperbolic line that meets A in either 0 or 2 points (since A is an ovoid, and all tangents to A are totally isotropic lines). We shall talk about *exterior* and *secant* (hyperbolic) lines, respectively.

Step 1. *Each vertex in $X_{4a} \cup X_{4b}$ has a unique neighbour in X_{4b}.*

(Indeed, if $p \in X_{4a}$ or $p \in X_{4b}$, then $\{p,x\}^\perp \cap A$ contains the point x' (or p', respectively), so this hyperbolic line is a secant, and there are precisely $q-2$ points in Γ at distance 2 from both p and x.)

If $p \in X_3$, then $\{p,x\}^\perp \cap A$ contains either 0 or 2 points, so that p has either 0 or 2 neighbours in X_{4b}. Let us call the set of vertices of the former (latter) kind X_{3a} (X_{3b}, respectively).

Step 2. *X_{3a} is the set of vertices p such that the line xp is exterior. We have $|X_{3a}| = |X_{3b}| = \frac{1}{2}q(q-1)(q-2)$.*

(Indeed, the lines joining x to a point of $X_2 \cup X_5$ are the tangents (totally isotropic lines) on x, the lines joining x to a point of $X_1 \cup X_{3b} \cup X_4$ are the exterior lines on x, and the lines joining x to a point of X_{3a} are the secants on x. But A has $\frac{1}{2}q^2(q^2+1)$ secants, and the same number of exterior lines. (In fact, l is secant iff $l^\perp$ is exterior.))

The planes meet the set A either in one point: *tangent* planes, or in an oval (having $q+1$ points): *secant* planes.

Step 3. *If* $p \in X_2 \cup X_3 \cup X_{4a}$ *then* p *has* $\frac{1}{2}q - 1$ *neighbours in* X_{3a}.

(Indeed, if $p \in X_2 \cup X_3 \cup X_{4a}$, then $x \notin p^\sigma$, and the plane $\langle x, p^\sigma \rangle$ is a secant plane. In this plane, the point x is on one tangent, and on $\frac{1}{2}q$ secants. One of these secants contains p'; the remaining $\frac{1}{2}q - 1$ contain each one neighbour of p.)

This determines the entire diagram. □

Remark 2.3. The graph Γ, and the fact that it is 2-arc transitive for $Sz(q)$, was found independently by Fang Xin Gui, a student of Cheryl Praeger.

Remark 2.4. $\mathrm{Aut}\,\Gamma$ is not primitive: the spread $\{a^\sigma \mid a \in A\}$ is a system of blocks of imprimitivity. However, $\mathrm{Aut}\,\Gamma$ acts 2-transitively on the set of blocks, so that we do not find a nontrivial graph structure on the quotient.

Remark 2.5. Of course we also get finite heptagraphs (of valency $q = 3^{2e} + 1$) starting from a generalized hexagon (of type $G_2(q)$) with a polarity.

3 Addendum

The above was written in April 1992. In the meantime, Xin Gui Fang & C. E. Praeger [1, 2] appeared where the above graphs are found in the classification of certain 2-arc transitive graphs (and they refer to this work). As far as we know, the relation to generalized polygons with polarity still does not appear in the literature.

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