



Elation groups of the Hermitian surface $H(3, q^2)$ over a finite field of characteristic 2

Robert L. Rostermundt

Abstract

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a finite generalized quadrangle (GQ) having order (s, t) . Let p be a point of \mathcal{S} . A *whorl* about p is a collineation of \mathcal{S} fixing all the lines through p . An *elation* about p is a whorl that does not fix any point not collinear with p , or is the identity. If \mathcal{S} has an elation group acting regularly on the set of points not collinear with p we say that \mathcal{S} is an elation generalized quadrangle (EGQ) with base point p . The following question has been posed: Can there be two non-isomorphic elation groups about the same point p ? In this presentation, we show that there are exactly two (up to isomorphism) elation groups of the Hermitian surface $H(3, q^2)$ over a finite field of characteristic 2.

Keywords: generalized quadrangles, elation groups, Hermitian surface

MSC 2000: 51E12

1 Introduction

The focus of this article is $H(3, q^2)$, the Hermitian surface in three-dimensional projective space over the field $\text{GF}(q^2)$, where $q = 2^e$. The first results were discovered by Tim Penttila, for $q = 2$ and $q = 4$, using the software package Magma [5]. In this paper we give a constructive proof for any $q = 2^e$. We introduce generalized quadrangles with some basic definitions.

Let \mathcal{P} and \mathcal{B} be two non-empty sets, called points and lines, with an incidence relation \mathcal{I} such there are two positive integers s and t satisfying

- (1) Each point is incident with $t+1$ lines; any two points are mutually incident with at most one line.

- (2) Each line is incident with $s + 1$ points; any two lines are mutually incident with at most one point.
- (3) Given a line L and a point x not incident with L there is a unique point y and a unique line M such that $x \mathcal{I} M \mathcal{I} y \mathcal{I} L$.

Such a collection $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is called a *generalized quadrangle of order (s, t)* written $\text{GQ}(s, t)$. The *dual* of a $\text{GQ}(s, t)$ is the $\text{GQ}(t, s)$ obtained by interchanging the roles of points and lines. Furthermore, any theorem or definition given for a GQ can be dualized by interchanging the words points and lines. It will therefore be assumed that whenever a definition or theorem is given, its dual has also been given.

Two points incident with a common line are said to be *collinear* and two lines incident with a common point are *concurrent*. If x and y are collinear we use the notation $x \sim y$. Similarly, if L and M are concurrent we denote this $L \sim M$.

If X is a set of points (respectively, lines) of \mathcal{S} , then X^\perp denotes the set of all points collinear (resp., lines concurrent) with everything in X . If $X = \{x\}$ is a singleton set, it is common to write X^\perp as x^\perp . The set $X^{\perp\perp}$, is the set of all points collinear (resp., lines concurrent) with all of X^\perp . By convention $x \in x^\perp$.

Let x, y be two noncollinear points of a $\text{GQ}(s, t)$. We say that $\{x, y\}$ is a *regular pair* provided $|\{x, y\}^{\perp\perp}| = t + 1$; that is, if $|\{x, y\}^{\perp\perp}|$ is as large as possible. If x is a point such that for every y , with $x \not\sim y$, we have $|\{x, y\}^{\perp\perp}| = t + 1$, then we say x is a *regular point*. A set $\{x, y, z\}$ of pairwise non-collinear points is called a *triad* of points. If $\{x, y, z\}$ is a triad of points, then all points in $\{x, y, z\}^\perp$ are called *centers*.

Recall that a GQ is *classical* if it is isomorphic to a GQ (or its dual) that can be embedded in a projective space.

2 Elation generalized quadrangles

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $\text{GQ}(s, t)$, $s \geq 1$, $t \geq 1$, and let $p \in \mathcal{P}$ be a point of \mathcal{S} . A *whorl* about p is a collineation of \mathcal{S} that leaves invariant each line incident with p . If there is a group of whorls acting transitively on the points not collinear with p we say that p is a *center of transitivity*. Let θ be a whorl about p . If $\theta = \text{id}$ or if θ fixes no point of $\mathcal{P} \setminus p^\perp$, then θ is an *elation* about p . If there is a group G of elations about p acting regularly on $\mathcal{P} \setminus p^\perp$, we say \mathcal{S} is an *elation generalized quadrangle* (EGQ) with *elation group* G and *base point* p . We will often denote this quadrangle as $(\mathcal{S}^{(p)}, G)$, or simply $\mathcal{S}^{(p)}$. A *skew-translation* GQ (STGQ), is an EGQ $(\mathcal{S}^{(p)}, G)$ where G contains a full group of symmetries about p . All known GQ with parameters (q^2, q) are in fact STGQ. Moreover, a result

obtained by X. Chen [2] and independently by D. Hachenberger [3] states that an STGQ must have both s and t powers of the same prime. If G is abelian we say $S^{(p)}$ is a *translation* generalized quadrangle, denoted TGQ.

Let G be a group with order s^2t . Then let $F = \{A_0, A_1, \dots, A_t\}$ be a family of $t + 1$ subgroups of G , each with order s , and let $F^* = \{A_0^*, A_1^*, \dots, A_t^*\}$ be another family of $t + 1$ subgroups of G , each having order st where $A_i \leq A_i^*$ for each $0 \leq i \leq t$.

Using the group G we define a coset geometry, which we denote $S^{(\infty)}$, as follows. There are three types of points; (i) elements $g \in G$, (ii) cosets A_i^*g , (iii) a symbol (∞) . There are two types of lines; (i) cosets $A_i g$, (ii) symbols $[A_i]$. Incidence is as follows; the symbol (∞) is incident with the $t + 1$ lines of type (ii), the s cosets of A_i^* are the other s points on a line $[A_i]$, each point A_i^*g is incident with lines corresponding to the cosets $A_i h$ that are completely contained in the coset A_i^*g , the remaining points on a line $A_i h$ are the group elements contained in the coset $A_i h$.

Theorem 2.1. *Let G be a group of order s^2t and let $F = \{A_0, A_1, \dots, A_t\}$ be a family of $t + 1$ subgroups, each with order s , and let $F^* = \{A_0^*, A_1^*, \dots, A_t^*\}$ be another family of $t + 1$ subgroups, each having order st where $A_i \leq A_i^*$ for each $0 \leq i \leq t$. Then if we build the coset geometry $S^{(\infty)}$ as prescribed above, $S^{(\infty)}$ is a GQ, having order (s, t) , if and only if properties K1 and K2 hold, where*

$$K1 : A_j A_i \cap A_k = \{\text{id}\} \text{ for all distinct } i, j, k,$$

$$K2 : A_j^* \cap A_i = \{\text{id}\} \text{ for all } i \neq j.$$

In the previous theorem, we call F a 4-gonal family of G , and $\{G, F, F^*\}$ is called a Kantor family.

It is also well known that the set F^* is completely determined by the elements in F . Define $\Omega = \cup\{A_i : 0 \leq i \leq t\}$, then $A_i^* = A_i \cup \{A_i g : A_i g \cap \Omega = \emptyset\}$. The next theorem is also well known.

Theorem 2.2. *Let $S = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a GQ (s, t) . If G is an elation group about a point p , q a point in $\mathcal{P} \setminus p^\perp$, and $\{p, q\}^\perp = \{x_0, \dots, x_t\}$, for $0 \leq i \leq t$, let A_i be the stabilizer of the line through q and x_i , and A_i^* be the stabilizer of the point x_i . Then $F = \{A_i : 0 \leq i \leq t\}$ is a 4-gonal family of G and the coset geometry $S^{(\infty)}$ obtained from this Kantor family $\{G, F, F^*\}$ is a GQ isomorphic to S .*

Theorem 2.3 (S.E. Payne and K. Thas). *Let S be a GQ and let H be a group of whorls about the point x acting transitively on the set $X = P \setminus \{x\}^\perp$. The set of elations in H does not form a group if and only if (at least) one of the following conditions is satisfied:*

- (1) *There is a $j \geq 2$ for which $|\text{Fix}(\sigma)| = j$ for some $\sigma \in H$.*

- (2) *There is a proper thick sub-GQ of \mathcal{S} containing x (and all the lines through x) fixed pointwise by a non-identity element of H .*

3 Elation groups of $H(3, q^2)$

For this paper we will assume that $q = 2^e$, $\mathbb{F} = \text{GF}(q)$, and as usual the \mathbb{F} -trace function is defined as

$$\text{tr}(\alpha) = \sum_{i=0}^{e-1} \alpha^{2^i}.$$

We then choose $\delta \in \text{GF}(q)$ with $\text{tr}(\delta) = 1$, and let ζ be a root of the polynomial $x^2 + x + \delta$. Put $\mathbb{F}^2 = \{a + b\zeta : a, b \in \text{GF}(q)\}$; a quadratic extension of \mathbb{F} .

Lemma 3.1. *The element $\zeta^q = \zeta + 1$ is also a root of the polynomial, and if $\alpha = a + b\zeta$, then $\text{tr}(b) = \text{tr}(\alpha + \alpha^q)$.*

Let $\mathcal{S} = H(3, q^2)$ be the Hermitian surface in projective 3-space. Its construction is well known. Consider the projective space $\text{PG}(V)$, where V is a 4-dimensional vector space over \mathbb{F}^2 . Without loss of generality we choose the Hermitian form $H : V \times V \rightarrow \mathbb{F}$ where

$$H(\bar{x}, \bar{y}) = x_1 y_4^q + x_2 y_3^q + x_3 y_2^q + x_4 y_1^q.$$

The set of all absolute points and totally isotropic lines of $\text{PG}(3, q^2)$ forms the Hermitian surface $H(3, q^2)$. This is a GQ(q^2, q).

Theorem 3.2 ([4]). *Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a GQ of order (s, t) , $s, t > 1$, with s and t powers of the same prime p . Suppose (∞) is a regular point that is a center of transitivity, and let W_∞ be the full group of whorls about the point (∞) . Let S_p be a Sylow p subgroup of W_∞ . Then we have*

1. $|S_p| = s^2 t$, or
2. $p = 2$, $|S_2| = 2s^2 t$, and \mathcal{S} contains a proper thick sub-GQ of order t isomorphic to the symplectic GQ, denoted $W(t)$; consequently, $s = t^2$.

Corollary 3.3. *Let $q = 2^e$ and $\mathcal{S} = H(3, q^2)$. Every point of \mathcal{S} is a regular point, and for each point p of \mathcal{S} , if S_2 is a Sylow 2 subgroup of the entire group of whorls about p , then $|S_2| = 2q^5$.*

Consider the point $p = (0, 0, 0, 1)$. If \mathcal{P} is the set of all points of $H(3, q^2)$, then $\mathcal{P} \setminus p^\perp$ is the set of q^5 points $(1, \alpha, \beta, \mu) \in \text{PG}(3, q^2)$ satisfying $\mu + \mu^q + \alpha\beta^q + \alpha^q\beta =$

0. The group of matrices

$$E_p = \left\{ \begin{pmatrix} 1 & \alpha & \beta & \mu \\ 0 & 1 & 0 & \beta^q \\ 0 & 0 & 1 & \alpha^q \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{GL}(4, q^2) : \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0 \right\}$$

is an elation group about p , as this group acts regularly on the set of points in $\mathcal{P} \setminus \{p^\perp\}$ (the E_p -orbit of $(1, 0, 0, 0)$ is the set of points $(1, \alpha, \beta, \mu)$ where $\mu + \bar{\mu} + \alpha\bar{\beta} + \bar{\alpha}\beta = 0$) and fixes every line through p . It is often more convenient to represent this group as the set of triples

$$E_p = \{[\alpha, \beta, \mu] : \alpha, \beta, \mu \in \mathbb{F}^2 \text{ and } \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0\}$$

with group operation

$$[\alpha, \beta, \mu] * [\alpha', \beta', \mu'] = [\alpha + \alpha', \beta + \beta', \mu + \mu' + \alpha\beta'^q + \beta\alpha'^q].$$

Next define the Hermitian preserving involution $\phi : \text{PG}(3, q^2) \rightarrow \text{PG}(3, q^2) : (x, y, z, w) \mapsto (x^q, y^q, z^q, w^q)$. Then ϕ induces a collineation that is a whorl about the point p . If we adjoin ϕ to E_p we form the group

$$W_2 = \{[\alpha, \beta, \mu] \circ \phi^i : \alpha, \beta, \mu \in \mathbb{F}^2 \text{ and } \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0\}$$

with group operation being composition of maps. That is, for $g = [\alpha, \beta, \mu] \circ \phi^i$ and $g' = [\alpha', \beta', \mu'] \circ \phi^j$ we have

$$g * g' = [\alpha + \alpha'^{q^i}, \beta + \beta'^{q^i}, \mu + \mu'^{q^i} + \alpha\beta'^{q^{i+1}} + \beta\alpha'^{q^{i+1}}] \circ \phi^{i+j}.$$

Then $|W_2| = 2q^5$ and Theorem 3.2 gaurantees W_2 is a Sylow₂ subgroup of the group of whorls about p .

Theorem 3.4. *The set of all elations about p does not form a group.*

Proof. Since $\text{Fix}(\phi) \supset \{(1, a, b, c) : a, b, c \in \mathbb{F}\}$, Theorem 2.3 guarantees that the set of all elations about p does not form a group. \square

Theorem 3.5. *The only elements in W_2 that fix any points not collinear with $p = (0, 0, 0, 1)$ are the conjugates of ϕ .*

Proof. Suppose that x is a point opposite p that is fixed by ϕ . As $E_p \leq W_2$ the size of the orbit of x under W_2 is exactly q^5 . By the orbit-stabilizer theorem the size of the stabilizer of x in W_2 is 2; i.e., $|W_{2_x}| = 2$. Therefore, $W_{2_x} = \{\text{id}, \phi\}$. Now choose any point y opposite p . Because E_p acts regularly on points not collinear with p , there is a unique $g \in M_p$ such that $y^g = x$. So $(y)^{g\phi g^{-1}} = y$. Thus $g\phi g^{-1} \in W_{2_y}$ and by the orbit-stabilizer theorem we get $W_{2_y} = \{\text{id}, g\phi g^{-1}\}$. \square

Corollary 3.6. *A subgroup $E \leq W_2$, with $|E| = q^5$, is an elation group of $H(3, q^2)$ if and only if E contains no conjugates of ϕ .*

As usual, for $g, h \in W_2$, let $[g, h] = g^{-1}h^{-1}gh$; the commutator of g, h . Denote the commutator subgroup of W_2 as W'_2 . We observe that each conjugate of ϕ is in the coset of the commutator subgroup containing ϕ . That is,

$$g\phi g^{-1} = g\phi g^{-1}\phi^{-1}\phi = [g^{-1}, \phi] \cdot \phi.$$

Then since $W'_2 = \{[a, b, c] : a, b, c \in \mathbb{F}_q\}$ we see that $W'_2 \neq W'_2 \cdot \phi$.

We can choose the following $2q^2$ distinct coset representatives of the factor group W_2/W'_2 :

$$\{[\alpha, \beta, 0] \circ \phi^i : \alpha = 0 + ai, \beta = 0 + bi, i = 0, 1\}.$$

Put $\overline{W}_2 = W_2/W'_2$ and represent its elements as triples;

$$\overline{W}_2 = \{(a, b, i) : a, b \in \mathbb{F}_q, i = 0, 1\}.$$

Lemma 3.7 ([6]). *Let G be a finite p -group where $\Phi(G)$ is the Frattini subgroup. Then $\Phi(G) = G'G^p$ where G' is the commutator subgroup and G^p is the subgroup of G generated by all p^{th} powers. Moreover, $G/\Phi(G)$ is a vector space over $\text{GF}(p)$.*

Theorem 3.8. *The factor group \overline{W}_2 is a vector space over $\text{GF}(2)$.*

Proof. Every non-identity element in W_2 has order two or four. Moreover, if $|g| = 4$ it is easy to show that $g^2 \in S'_2$. It follows that $W_2^2 \leq W'_2$, and $W'_2 = \Phi(W_2)$. \square

We can treat \mathbb{F} as an e -dimensional vector space over $\text{GF}(2)$. Then for a fixed $\zeta \in \mathbb{F}^*$, the map

$$\text{tr}_\zeta(x) = \sum_{i=0}^{e-1} (\zeta x)^{2^i}$$

is a linear functional; i.e., $\text{tr}_\zeta : \mathbb{F} \mapsto \text{GF}(2)$. Letting ζ vary over \mathbb{F}^* gives us exactly $q - 1$ non-zero linear functionals on \mathbb{F} , and for each $\zeta \in \mathbb{F}^*$ there corresponds a unique hyperplane in \mathbb{F} giving $q - 1$ distinct hyperplanes. But every hyperplane of \mathbb{F} corresponds to a non-zero vector in \mathbb{F} . Hence $\{\text{tr}_\zeta : \zeta \in \mathbb{F}^*\}$ gives us all linear functionals from \mathbb{F} to $\text{GF}(2)$.

Next, for some $\zeta \in \mathbb{F}$ define the map $T_{\zeta, \sigma} : \overline{W}_2 \rightarrow \text{GF}(2)$ as $T_{\zeta, \sigma}(a, b, i) = \text{tr}_\zeta(a) + \text{tr}_\sigma(b) + i$. This is a linear functional from \overline{W}_2 onto $\text{GF}(2)$. The kernel is a hyperplane and it is easy to see that the vector $(0, 0, 1)$ is not in the kernel

of $T_{\zeta, \sigma}$. This gives us q^2 hyperplanes of S_2/S'_2 without the vector $(0, 0, 1)$ and these correspond to subgroups of W_2 with order q^5 that do not contain any conjugates of ϕ . That is, these hyperplanes correspond to elation groups about the point p . If we then define the map $T_{\zeta, \sigma}^*(a, b, i) = \text{tr}_{\zeta}(a) + \text{tr}_{\sigma}(b)$ we get the other q^2 hyperplanes of $\overline{W_2}$, each one containing the vector $(0, 0, 1)$. Each of these hyperplanes corresponds to a subgroup of W_2 of order q^5 that contains conjugates of ϕ .

We have shown that there are q^2 elation groups about the point $p = (0, 0, 0, 1)$. Furthermore, $\text{PGU}(4, q^2)$ is transitive on the points of $H(3, q^2)$ and this holds for all points on the Hermitian surface. We summarize in the following theorem.

Theorem 3.9. *Given a point p of $H(3, q^2)$, there are exactly q^2 elation groups of $H(3, q^2)$ about p .*

We represent these groups in W_2 as follows. For some $\zeta \in \mathbb{F}$ and $\alpha \in \mathbb{F}^2$, define the map $T_{\zeta}(\alpha) = \text{tr}_{\zeta}(\alpha + \alpha^q) \in \text{GF}(2)$. Then using Observation 3.1 we see that for each pair $\zeta, \sigma \in \mathbb{F}$ we get an elation group

$$E_{\zeta, \sigma} = \left\{ [\alpha, \beta, \mu] \circ \phi^{T_{\zeta}(\alpha) + T_{\sigma}(\beta)} : \alpha\beta^q + \beta\alpha^q + \mu + \mu^q = 0 \right\}.$$

We refer to $E_{0,0}$ as the *familiar group* and the other $q^2 - 1$ groups as *exotic*.

Theorem 3.10. *All of the $q^2 - 1$ exotic elation groups are pairwise isomorphic.*

Proof. We represent $E_{1,0}$ as a subgroup of $\text{PGU}(4, q^2)$, the entire stabilizer of $H(3, q^2)$. Recall that if $A, B \in \text{PGU}(4, q^2)$ and $\sigma, \sigma' \in \text{Aut}(\mathbb{F}^2)$, then in $\text{PGU}(4, q^2)$ we have the group product

$$(A \circ \sigma) * (B \circ \sigma') = (A \cdot B^{\sigma^{-1}}) \circ (\sigma \cdot \sigma'),$$

where $B^{\sigma^{-1}} = [b_{ij}^{\sigma^{-1}}]$.

It is well known that $|\text{PGU}(4, q^2)| = 2e \cdot q^6(q+1)^2(q^3+1)(q^4-1)$. We first look at that particular Sylow₂ subgroup of $\text{PGU}(4, q^2)$ which is the subgroup of all upper-triangular matrices with ones on the diagonal. We denote this group as M_2 . For a matrix M to be in $\text{PGU}(4, q^2)$ it must satisfy $MB(M^q)^T = B$, where B is the bilinear form associated with $H(3, q^2)$. Therefore,

$$M_2 = \left\{ \left(\begin{array}{cccc} 1 & a & b & c \\ 0 & 1 & d & b^q + a^q d \\ 0 & 0 & 1 & a^q \\ 0 & 0 & 0 & 1 \end{array} \right) : \begin{array}{l} ab^q + a^q b + c + c^q = 0 \\ d = d^q \end{array} \right\}.$$

This subgroup has order q^6 and is the stabilizer of the flag (L, p) , where L is the line given by $x_1 = x_2 = 0$ and $p = (0, 0, 0, 1)$. If we adjoin each field

automorphism whose order is a power of two we get a Sylow₂ subgroup of $\text{P}\Gamma\text{U}(4, q^2)$. The normalizer in $\text{P}\Gamma\text{U}(4, q^2)$ of M_2 , which we denote by N_{M_2} , is the group of all upper-triangular matrices.

$$N_{M_2} = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & w^{-q} & r & y^q w + r x^q \\ 0 & 0 & w & x^q w^{-q} \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{array}{l} xy^q + x^q y + z + z^q = 0 \\ rw^q = wr^q \\ w \neq 0 \end{array} \right\}.$$

If we adjoin all the automorphisms of \mathbb{F}^2 this gives us the normalizer, which we denote as $N_{M_2}^*$, of the Sylow₂ subgroup of $\text{P}\Gamma\text{U}(4, q^2)$. Moreover, $|N_{M_2}^*| = 2e \cdot q^6(q^2 - 1)$. We next show that the index of the normalizer of an exotic elation group in $N_{M_2}^*$ is $q^2 - 1$. That is, we show that the normalizer of an exotic elation group in $N_{M_2}^*$ has order $2e \cdot q^6$.

First note that all of the field automorphisms normalize the elation group $E_{1,0}$. So we consider the normalizer of $E_{1,0}$ in N_{M_2} . Choose the element

$$g = M \circ \phi^{T_1(\alpha)} = \begin{pmatrix} 1 & \alpha & \beta & \mu \\ 0 & 1 & 0 & \bar{\beta} \\ 0 & 0 & 1 & \bar{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix} \circ \phi^{T_1(\alpha)}$$

in $E_{1,0}$ and choose an arbitrary matrix $A \in N_{M_2}$, so that $A \cdot g \cdot A^{-1} = B \circ \phi^{T_1(\alpha)}$, where $B = AM(A^{-1})^{q^{T_1(\alpha)}}$. First, when $T_1(\alpha) = 1$ we have $B_{21} = w^{q-q^2} = w^{q-1} = 1$ if and only if $w \in \mathbb{F}$. Then, with $w \in \mathbb{F}$ we have

$$B_{12} = w(x^{q^{T_1(\alpha)}} + x + \alpha).$$

It follows that

$$\begin{aligned} T_1[w(x^{q^{T_1(\alpha)}} + x + \alpha)] &= T_1[w(x^{q^{T_1(\alpha)}} + x) + w\alpha] \\ &= T_1[w(x^{q^{T_1(\alpha)}} + x)] + T_1(w\alpha) = T_1(w\alpha) \end{aligned}$$

for all $\alpha \in \mathbb{F}^2$, and that $w = 1$. Then, given $w = 1$, we must have $B_{23} = r + r^{q^{T_1(\alpha)}} = 0$, and $r \in \mathbb{F}$. Given these conditions on w and r we have $AgA^{-1} =$

$C \circ \phi^{T_1(\alpha)}$ where

$$C = \begin{pmatrix} 1 & x^{q^{T_1(\alpha)}} + x + \alpha & r(x^{q^{T_1(\alpha)}} + x + \alpha) + y + y^{q^{T_1(\alpha)}} + \beta & * \\ 0 & 1 & 0 & r(\bar{x}^{q^{T_1(\alpha)}} + \bar{x} + \bar{\alpha}) + \bar{y} + \bar{y}^{q^{T_1(\alpha)}} + \bar{\beta} \\ 0 & 0 & 1 & \bar{x}^{q^{T_1(\alpha)}} + \bar{x} + \bar{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$* = \mu + z + z^{q^{T_1(\alpha)}} + \bar{\beta}x + \beta\bar{x}^{q^{T_1(\alpha)}} + \bar{\alpha}y + \bar{\alpha}^{q^{T_1(\alpha)}}y + \alpha\bar{y}^{q^{T_1(\alpha)}} + x\bar{y}^{q^{T_1(\alpha)}} + \bar{x}^{q^{T_1(\alpha)}}y^{q^{T_1(\alpha)}} + x^{q^{T_1(\alpha)}}\bar{y}^{q^{T_1(\alpha)}}.$$

So $AgA^{-1} \in E_{1,0}$ for all choices x, y, z satisfying $x\bar{y} + y\bar{x} + z + \bar{z} = 0$, and the normalizer of $E_{1,0}$ in $N_{M_2}^*$ has order $2e \cdot q^6$. Hence, there are $q^2 - 1$ conjugates of $E_{1,0}$ in $N_{M_2}^*$. It follows that all $q^2 - 1$ exotic elation groups are conjugate and hence isomorphic. \square

From now put $E = E_{1,0}$ and $T = T_1$. Further, we will denote an element $[\alpha, \beta, \mu] \circ \phi^{T(\alpha)}$ as simply $[\alpha, \beta, \mu]$. Let $\bar{x} = x^q$, and if we define the map $\hat{\alpha} : x \mapsto x^{\hat{\alpha}} = x^{q^{T(\alpha)}}$ we have the following binary operation in E .

$$[\alpha, \beta, \mu] * [\alpha', \beta', \mu'] = [\alpha + \alpha'^{\hat{\alpha}}, \beta + \beta'^{\hat{\alpha}}, \mu + \mu'^{\hat{\alpha}} + \alpha(\bar{\beta}')^{\hat{\alpha}} + \beta(\bar{\alpha}')^{\hat{\alpha}}].$$

Moreover, straightforward computations give us

$$[\alpha, \beta, \mu]^{-1} = [\alpha^{\hat{\alpha}}, \beta^{\hat{\alpha}}, \mu^{\hat{\alpha}} + (\bar{\alpha}\beta)^{\hat{\alpha}} + (\bar{\beta}\alpha)^{\hat{\alpha}}].$$

The following properties of the function $\hat{\alpha}$ will help with computations.

Lemma 3.11. *Suppose $a, b \in \mathbb{F}^2$ and $c \in \mathbb{F}$. Then for any $x \in \mathbb{F}^2$ we have*

- (i) $x^{\hat{0}} = x$.
- (ii) $x^{\hat{a}} = x^{\hat{\hat{a}}}$.
- (iii) $x^{\hat{c}} = x$.
- (iv) $x^{\widehat{(a^b)}} = x^{\hat{a}}$.
- (v) $x^{\widehat{a+b}} = (x^{\hat{a}})^{\hat{b}} = (x^{\hat{b}})^{\hat{a}}$.

Theorem 3.12. *Each of the exotic elation groups of $H(3, q^2)$ is not isomorphic to the familiar example.*

Proof. In the familiar example the center of the group is equal to the commutator subgroup and $E_{0,0}$ has nilpotency class 2. However, each of the exotic elation groups has nilpotency class 3. To see this choose $g = [\alpha, \beta, \mu]$ and $h = [\alpha', \beta', \mu']$ in E and compute $[g, h]$. Using the facts in Observation 3.11 we arrive at $[g, h] = [a, b, c]$, where

$$\begin{aligned} a &= \left(\alpha + \alpha^{\widehat{\alpha}'}\right)^{\widehat{\alpha}} + \left(\alpha' + \alpha'^{\widehat{\alpha}'}\right)^{\widehat{\alpha}'}, \\ b &= \left(\beta + \beta^{\widehat{\alpha}'}\right)^{\widehat{\alpha}} + \left(\beta' + \beta'^{\widehat{\alpha}'}\right)^{\widehat{\alpha}'}. \end{aligned}$$

Now consider the following cases:

- (i) $\widehat{\alpha} = \widehat{\alpha}' = 1$: Then $[g, h] = [0, 0, c]$ with $c \in \mathbb{F}$.
- (ii) $\widehat{\alpha} = \widehat{\alpha}' = q$: Then $[g, h] = [\alpha + \bar{\alpha} + \alpha' + \bar{\alpha}', \beta + \bar{\beta} + \beta' + \bar{\beta}', c]$ with $c \in \mathbb{F}$.
- (iii) $\widehat{\alpha} = 1$ and $\widehat{\alpha}' = q$: Then $[g, h] = [\alpha + \bar{\alpha}, \beta + \bar{\beta}, c]$ with $c \in \mathbb{F}$.
- (iv) $\widehat{\alpha} = q$ and $\widehat{\alpha}' = 1$: Then $[g, h] = [\alpha' + \bar{\alpha}', \beta' + \bar{\beta}', c]$ with $c \in \mathbb{F}$.

Clearly, in case (ii) we have $\widehat{\alpha} = 1$, where $a = (\alpha + \alpha') + \overline{(\alpha + \alpha')}$. So the possible values taken by the first coordinate in this case are a subset of the possible values of the first coordinate in cases (iii) and (iv). Then since the relative trace function $\text{tr}_q : x \mapsto x + x^q$ is a homomorphism from $\mathbb{F}^2 \rightarrow \mathbb{F}$, it follows that there are $q/2$ possible choices for the first coordinate of any commutator. It is then easy to see that for $[g, h] = [a, b, c]$ the coordinate b can take on all possible elements of \mathbb{F} . Next choose, $\alpha, \alpha', \beta' \in \mathbb{F}$, and we get $[g, h] = [0, 0, \alpha'(\beta + \bar{\beta})]$, and we have all of \mathbb{F} as possible entries in the third coordinate c . Then since the product of two commutators is again a commutator we get $E' = \{[\alpha + \bar{\alpha}, b, c] : b, c \in \mathbb{F}; \alpha \in \mathbb{F}^2 \text{ and } \widehat{\alpha} = 1\}$ and $|E'| = q^3/2$.

Finally, using case (iv) from above, we see that if $h \in E'$, then $[g, h]$ is in $Z(E)$ and E has nilpotency class 3. \square

We summarize the above with the following main result.

Theorem 3.13. *Let $S = H(3, q^2)$ and p be any point of S . Then, up to isomorphism, there are exactly two elation groups about p .*

We also observe that $Z(E) = \{[0, 0, c] : c \in \text{GF}(q)\} = Z(E_{0,0})$.

4 Looking forward

Consider the important theorem due to R.C. Bose.

Theorem 4.1 (Bose). *Let $S = (P, B, I)$ be a GQ of order (q^2, q) . Then every set of three pairwise non-concurrent lines has exactly $q + 1$ transversals.*

We can now give the following definition.

Definition 4.2. A GQ S with parameters (q^2, q) has Property (G) at the point p provided the following holds. Let L_1 and M_1 be distinct lines incident with the point p . Let M_2, M_3, M_4 be distinct lines and L_2, L_3, L_4 be distinct lines for which $L_i \sim M_j$ whenever $i + j \leq 7$. Then $L_4 \sim M_4$.

One of the most powerful recent results characterizing flock-GQ by Property (G) was given by J. A. Thas in [7], with one missing case in characteristic two, which was completed by M. Brown in [1].

Theorem 4.3 (J. A. Thas and M. Brown). *Let $S = (\mathcal{P}, \mathcal{B}, I)$ be a GQ (q, q^2) , $q > 1$, and assume that S satisfies Property (G) at some line l . Then S is the dual of a flock-GQ.*

Theorem 4.4 (S.E. Payne and K. Thas). *Let $S(\mathcal{F})$ be a non-classical flock generalized quadrangle of order (q^2, q) , $q > 1$, q even. Then the set of all elations about (∞) does form a group.*

The previous theorems should be helpful in characterizing any EGQ admitting an exotic elation group. This is the most obvious question. Another problem worth considering is the following.

Question 4.5. *Are there any non-classical EGQ admitting one of the exotic elation groups?*

Another problem worth considering is the following.

Question 4.6. *If G is an elation group, is there a bound on the nilpotency class of G ?*

We should remark that a very different and independent approach to the problem of whether an EGQ admits a non-classical elation group with base point p appears in a paper by K. Thas [8]. The author offers the following conjecture.

Conjecture 4.7. *Let $S = (S^{(p)}, G)$ be an EGQ with Kantor-family $\{G, F, F^*\}$ and suppose that, for some $A^* \in F^*$, A^* is not normal in G . Then S (which has order (s, t)) has non-isomorphic (full) elation groups, and S has a sub-GQ of order $(s/t, t)$ fixed pointwise by some non-trivial collineation (possibly under some mild extra assumption).*

5 Acknowledgments

The author would like to thank Tim Penttila for suggesting this problem, and also Stan Payne for his supporting role as the author's Ph.D. advisor, as this work comes from the author's Ph.D. dissertation. We also thank Warren Bateman and his support for the Lynn Bateman Fellowship, given in the mathematics department at University of Colorado at Denver, which allowed the author to focus on research while under its support as a Ph.D. student.

References

- [1] **M. Brown**, Projective ovoids and generalized quadrangles, *Adv. Geom.* **7** (2007), 65–81.
- [2] **X. Chen**, On the groups that generate skew-translation generalized quadrangles, unpublished manuscript.
- [3] **D. Hachenberger**, Groups admitting a Kantor family and a factorized normal subgroup, *Des. Codes Cryptogr.* **8** (1996), 135–143.
- [4] **S. E. Payne** and **K. Thas**, Notes on elation generalized quadrangles, *European J. Combin.* **24** (2003), 969–981.
- [5] **T. Penttila**, private communication (2004).
- [6] **J. Rotman**, *An Introduction to the Theory of Groups*, Fourth edition, Springer-Verlag New York, Inc., 1995.
- [7] **J. A. Thas**, Generalized quadrangles of order (s, s^2) , III, *J. Combin. Theory Ser. A* **87** (1999), 247–272.
- [8] **K. Thas**, Some basic questions and conjectures on elation generalized quadrangles, and their solutions, *Bull. Belg. Math. Soc. Simon Stevin* **12** (2006), 909–918.

Robert L. Rostermundt

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO AT DENVER, AND HEALTH SCIENCES CENTER, CAMPUS BOX 170, P.O. BOX 173364, DENVER, CO 80217-3364

e-mail: riverrasta@yahoo.com