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Real line arrangements with the Hirzebruch property

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A line arrangement of $3n$ lines in $\mathbb{C}P^2$ satisfies the Hirzebruch property if each line intersect others in $n+1$ points. Hirzebruch asked in 1985 if all such arrangements are related to finite complex reflection groups. We give a positive answer to this question in the case when the line arrangement in $\mathbb{C}P^2$ is real, confirming that there exist exactly four such arrangements.

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1 Introduction and the main result

The goal of this article is to prove the following result:

Theorem 1.1 *There exist exactly four line arrangements in $\mathbb{R}P^2$ consisting of $3 \cdot n$ lines such that each line intersects others in $n+1$ points. These arrangements are reflection arrangements of the Coxeter groups corresponding to spherical triangles with angles $\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}\right)$, $\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}\right)$.*

Let us give a description of these four arrangements. The first arrangement is a union of three generic lines. The second arrangement is composed of three lines spanning the sides of a regular triangle in $\mathbb{R}^2$ together with three axes of symmetry of the triangle. The third arrangement is composed of four sides of a square in $\mathbb{R}^2$, four symmetry axes of the square, and the line at infinity. The fourth arrangement is composed of the sides of a regular pentagon in $\mathbb{R}^2$, five axes of symmetry and five diagonals of the pentagon.

Following Panov and Petrunin [8], we say that a line arrangement in $\mathbb{C}P^2$ satisfies the *Hirzebruch property* if it consists of $3n$ lines and each line intersects others in exactly $n+1$ points. Such arrangements were studied first by Hirzebruch and Höfer in the context of construction of complex ball quotients.[1] The ball quotients were obtained as desingularisations of ramified covers of $\mathbb{C}P^2$ with branching along line arrangements; the construction is described in Hirzebruch [4] and Barthel, Hirzebruch and Höfer [1].

[1] By this we mean complex projective surfaces that are quotients of the unit complex ball $B^2_{\mathbb{C}} = \{|z_1|^2 + |z_2|^2 < 1\}$ by a cocompact action of a discrete torsion free group.

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Contemplating the list of arrangements suitable for construction of ball quotients, Hirzebruch [5] asked the following question:

Question 1.2 Let $\mathcal{L}$ be a complex line arrangement in $\mathbb{C}P^2$ consisting of $3 \cdot n$ lines such that each line of $\mathcal{L}$ intersect others at exactly $n+1$ points. Is it true that $\mathcal{L}$ is a *complex reflection arrangement*?[2]

This question is still open, and Theorem 1.1 gives a positive answer to it in the case when the line arrangement in $\mathbb{C}P^2$ is real.

Apart from the context of ball quotients, arrangements with the Hirzebruch property appear in the setting of polyhedral Kähler manifolds; see Panov [7]. This was used in Panov and Petrunin [8] to prove that the complement to any complex line arrangement with the Hirzebruch property is aspherical.

One more context in which these arrangements appear is the theory of convex foliations on $\mathbb{C}P^2$, ie foliations whose leaves other than straight lines have no inflection points; see Section 5 and Marín and Pereira [6] for more details.

About the proof Theorem 1.1 is deduced from the existence of a special polyhedral metric with conical singularities on $\mathbb{R}P^2$ for which the lines of the arrangement are geodesics. The metric on $\mathbb{R}P^2$ is obtained by restricting the polyhedral Kähler metric on the complexification of $\mathbb{R}P^2$, constructed in [7] and whose properties are summarised in Section 2.2. To prove Theorem 1.1 we show that the arrangement cuts $\mathbb{R}P^2$ into a collection of isometric Euclidean triangles. Here we rely on a collection of elementary statements about spherical polygons, proven in Section 3.

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2 Polyhedral metrics

Recall the definition of polyhedral manifolds.

Definition 2.1 Let M be a piecewise linear manifold M with a complete metric g. We say that M is a *polyhedral manifold of curvature* κ if it admits a *compatible triangulation* for which each simplex equipped with g is isometric to a geodesic simplex

[2] A *complex reflection line arrangement* is a line arrangement in $\mathbb{C}P^2$ consisting of lines fixed by nontrivial elements of a finite complex reflection group acting on $\mathbb{C}P^2$.

in the space of constant curvature κ. Depending on the sign of κ the manifold M is called a polyhedral spherical, Euclidean or hyperbolic manifold. The complement to metric singularities of a polyhedral manifold is denoted by M°.

Any polyhedral metric is nonsingular in codimension 1. The set of metric singularities $M \setminus M^\circ$ is a union of some codimension-two faces of a compatible triangulation. Let Δ be one of codimension-two faces inside $M \setminus M^\circ$ and let x be an interior point of Δ. Then in a neighbourhood of x there is a totally geodesic surface orthogonal to Δ at x. The conical angle of such a surface at x is the same for the all interior points of Δ and is called the *conical angle at* Δ.

We say that a polyhedral Euclidean manifold M is *nonnegatively curved* if the conical angles at all its codimension-two faces are at most 2π.

2.1 Polyhedral surfaces

A polyhedral surface is a polyhedral manifold of dimension two. Such a surface S has a finite number of conical points $x_1, \dots, x_n$ and a complete metric g which has constant curvature κ on $S \setminus \{x_1, \dots, x_n\}$. We will only deal with the cases $\kappa = 1$ and $\kappa = 0$. In a neighbourhood of any conical point on S there are polar coordinates (r, θ) with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ in which the metric can be given by the formulas

$$g = dr^2 + \alpha^2 \sin(r)^2 d\theta^2 \quad \text{or} \quad g = dr^2 + \alpha^2 r^2 d\theta^2,$$

depending on whether $\kappa = 1$ or $\kappa = 0$. The conical angle at x is $2\pi\alpha$ in both cases.

Each oriented polyhedral surface has a unique complex structure for which the polyhedral metric is Kähler on the complement to conical points. We will mainly study positively curved polyhedral metrics on $\mathbb{C}P^1$, invariant under the complex conjugation on $\mathbb{C}P^1$. Such metrics can be constructed by the *doubling* of *spherical polygons*, which we will now describe.

Spherical polygons A *convex spherical polygon* is a closed convex subset of the sphere $\mathbb{S}^2_\kappa$ of curvature κ with boundary composed of a finite number of geodesic segments. The geodesic segments are called the *edges* of the polygon and the points where these edges meet are called the *vertices*. If P is a spherical (or Euclidean) polygon and A is its vertex, we will denote the angle of P at A either by $\angle_A(P)$ or just by $\angle A$ (when the latter notation is unambiguous). We will assume that no two adjacent edges of the polygon lie on one geodesic in $\mathbb{S}^2_\kappa$.

Doubling of polygons Let P be a convex spherical polygon and let P' be an isometric copy of it. The *doubling* of P is obtained by gluing P with P' along their boundaries by the natural isometry. The resulting polyhedral sphere has a natural involution.

Lemma 2.2 *There is a one-to-one correspondence between convex spherical polygons and polyhedral metrics of positive curvature on $\mathbb{C}P^1$ satisfying the following properties:*

- *The metric is invariant under the complex conjugation on $\mathbb{C}P^1$.*
- *All the conical points are real, ie belong to $\mathbb{R}P^1 \subset \mathbb{C}P^1$.*
- *All the conical angles are less than 2π.*

The proof is straightforward; one direction of the correspondence is given by the doubling construction. The other direction is given by taking the quotient of $\mathbb{C}P^1$ by the conjugation. Indeed, the conjugation is an isometry and so it leaves invariant a circle composed of geodesic segments.

2.2 Polyhedral Kähler manifolds

Here we recall some definitions and results from [7] concerning polyhedral Kähler manifolds.

Definition 2.3 Let M be an orientable nonnegatively curved Euclidean polyhedral manifold on dimension $2{\cdot}n$. We say that M is *polyhedral Kähler* if the holonomy of the metric on M° belongs to $\mathrm{U}(n) \subset \mathrm{SO}(2\cdot n)$.

Our proof of Theorem 1.1 relies heavily on the following theorem, proven in [7].

Theorem 2.4 *Let $\mathcal{L}$ be an arrangement of $3n$ lines $(n \geq 2)$ in $\mathbb{C}P^2$ with the Hirzebruch property. Then there exists a unique-up-to-scale polyhedral Kähler metric $g_{\mathcal{L}}^{\mathbb{C}}$ on $\mathbb{C}P^2$ which is singular along $\mathcal{L}$, nonsingular in the complement of $\mathcal{L}$ and has conical angle $2\pi \cdot \frac{n-1}{n}$ at each line of the arrangement.*

The existence part of this theorem is a partial case of Theorem 1.12 in [7]. The uniqueness of the metric up to scale follows from general results on unitary flat logarithmic connections.

The Euler field and the S^1–isometry It was proven in [7] that a polyhedral Kähler manifold complex dimension two has the structure of a smooth complex surface X such that $X \setminus X^\circ$ is a divisor in X. Since X is polyhedral, each point $x \in X$ has a conical ε–neighbourhood. It is obvious that on such a neighbourhood there is a real vector field e_r acting by radial dilatation. In [7, Section 3] it was explained that this field can be complexified to a holomorphic Euler field $e = e_r + i e_s$, and we sum up the properties of e in the following theorem. It will be convenient to set $\varepsilon = 2$, which can always be achieved by scaling the metric by a large factor.

Theorem 2.5 *Let $x \in X$ be a point, $B_x(2)$ be its conical neighbourhood of radius 2 and $S_x(2)$ be the boundary of this neighbourhood. There is a holomorphic Euler vector field $e = e_r + i e_s$ defined on $B_x(2)$ with the following properties:*

(1) *The field e_r is the real radial vector field acting by dilatations of the metric, it restricts to each ray of the cone as $r\frac{\partial}{\partial r}$.*

(2) *The field e_s is given by $e_s = J(e_r)$, where J is the operator of complex structure on TX. The field e_s acts by isometries on $B_x(2)$.*

(3) *Let x be a multiple point of an arrangement $\mathcal{L}$ from Theorem 2.4* of multiplicity[3] *$\mu(x) \geq 2$. Then e_s integrates to an isometric S^1–action on $B_x(2)$ which is free on $B_x(2) \setminus x$. The quotient $S_x(2)/S^1$ is a curvature-1 two-sphere with $\mu(x)$ conical singularities of angles $2\pi \cdot \frac{n-1}{n}$.*

Proof This theorem is a partial case of Theorem 1.7 in [7]. □

2.2.1 Polyhedral Kähler metric for real line arrangements From now on we will assume that $\{L_1, \dots, L_{3n}\} = \mathcal{L}$ is a real line arrangement in $\mathbb{R}P^2$ satisfying the Hirzebruch property and $\{L_1^{\mathbb{C}}, \dots, L_{3n}^{\mathbb{C}}\} = \mathcal{L}^{\mathbb{C}}$ is its complexification in $\mathbb{C}P^2$. Let σ be the involution on $\mathbb{C}P^2$ induced by the complex conjugation, and let $g_{\mathcal{L}}^{\mathbb{C}}$ be a polyhedral Kähler metric on $\mathbb{C}P^2$ given by Theorem 2.4, with conical singularities of angles $2\pi\frac{n-1}{n}$ at lines $L_i^{\mathbb{C}}$.

Corollary 2.6 (1) *The polyhedral Kähler metric $g_{\mathcal{L}}^{\mathbb{C}}$ is invariant under the complex conjugation σ on $\mathbb{C}P^2$.*

(2) *The metric $g_{\mathcal{L}}^{\mathbb{C}}$ restricts to a Euclidean polyhedral metric $g_{\mathcal{L}}^{\mathbb{R}}$ on $\mathbb{R}P^2$ and the lines L_i are geodesics on $\mathbb{R}P^2$ with respect to $g_{\mathcal{L}}^{\mathbb{R}}$.*

(3) *Let x be a real point $x \in \mathcal{L} \subset \mathcal{L}^{\mathbb{C}}$. Let $e = e_r + i e_s$ be the Euler field defined in a conical neighbourhood of x. Then $\sigma(e) = e_r - i e_s$.*

(4) *The involution σ descends to an isometry of the two-sphere $S_x(2)/S^1$, and $(S_x(2)/S^1)/\sigma$ is a convex spherical polygon of curvature 1.*

[3] The multiplicity of a point is the number of lines of the arrangement passing through the point.

Proof (1) The antiholomorphic involution sends the polyhedral Kähler metric $g_{\mathcal{L}}^{\mathbb{C}}$ to a polyhedral Kähler metric. Since such a metric is unique up to scale by Theorem 2.4, it is invariant under σ.

(2) For any polyhedral metric, the fixed set of any isometric involution is totally geodesic, so $\mathbb{R}P^2 \subset \mathbb{C}P^2$ is totally geodesic. Hence, the restriction of the metric to $\mathbb{R}P^2$ is a flat metric with conical singularities.

To see that the lines L_i are geodesic in $\mathbb{R}P^2$, note that each complex line $L_i^{\mathbb{C}}$ is totally geodesic in $\mathbb{C}P^2$, and L_i is the fixed locus of the isometric involution σ on $L_i^{\mathbb{C}}$.

(3) Let $e = e_r + i e_s$ be the holomorphic Euler field in a neighbourhood of x. Then $\sigma(e)$ is an antiholomorphic vector field. A the same time, since σ is an isometry preserving x, $\sigma(e_r) = e_r$. This proves the claim.

(4) Indeed, from (3) it follows that $\sigma(e_s) = -e_s$, hence σ sends S^1–orbits to S^1–orbits. □

Definition 2.7 For a real line arrangement $L_1, \dots, L_{3n}$ satisfying the Hirzebruch property let x be a multiple point. Denote by $\mathbb{D}(x)$ the convex spherical polygon $(S_x(2)/S^1)/\sigma$ from Corollary 2.6.

In the next lemma we summarise what we need to know about polyhedral Kähler metrics in order to prove Theorem 1.1.

Let $\mathcal{L} = \{L_1, \dots, L_{3n}\}$ be a real arrangement with the Hirzebruch property. Suppose x is a multiple point of $\mathcal{L}$ and assume that k lines pass through x, ie $\mu(x) = k$. After a possible reenumeration assume that the lines passing through x are $L_1, \dots, L_k$ and they go in a cyclic order at x on $\mathbb{R}P^2$. The spherical polygon $\mathbb{D}(x)$ associated to x by Definition 2.7 has k vertices $A_1, \dots, A_k$ corresponding to the lines $L_1, \dots, L_k$.

Lemma 2.8 *The angle of the spherical polygon $\mathbb{D}(x)$ at each vertex A_i is equal to $\pi\frac{n-1}{n}$. Both angles between geodesics L_i and L_{i+1} on $\mathbb{R}P^2$ at the point x with respect to the metric $g_{\mathcal{L}}^{\mathbb{R}}$ are equal to $\frac{1}{2}|A_iA_{i+1}|$ for all $i \in \{1, \dots, k\}$* (here $A_{k+1} = A_1$).

Proof Let $B_x(2)$ be a conical 2–neighbourhood of x in $\mathbb{C}P^2$ with respect to the metric $g_{\mathcal{L}}^{\mathbb{C}}$. Consider its intersection with $\mathbb{R}P^2$, and let S^1 be the boundary of this intersection. Each line L_i for $i \in \{1, \dots, k\}$ intersects S^1 in two points and we can denote them by B_i and B_{i+k}, so that points $B_1, \dots, B_{2k}$ go along S^1 in a cyclic order.

Denote by π the quotient map $S_x(2) \to \mathbb{D}(x)$. Note that the map $\pi\colon S^1 \to \partial(\mathbb{D}(x))$ is a locally isometric cover of degree two, and for any $i \in \{1,\dots,k-1\}$ the segment of S^1 included between B_i and B_{i+1} is sent isometrically to the edge A_iA_{i+1} of $\mathbb{D}(x)$. Note finally that the length of B_iB_{i+1} is twice the angle between L_i and L_{i+1} on $\mathbb{R}P^2$. □

3 Equiangular spherical polygons

From now on, by spherical polygons we mean polygons on the unit sphere $\mathbb{S}^2$. In view of Lemma 2.8 we will need to study equiangular spherical polygons.

Definition 3.1 A convex spherical polygon is called *equiangular* if the angles of the polygon at all vertices are equal. The polygon is called *equilateral* if all its edges are of the same length.

The goal of this section is to prove the following proposition and its refinement Lemma 3.8 on equiangular spherical polygons.

Proposition 3.2 *Let P^* be a convex equiangular spherical polygon with $n \ge 3$ vertices. The sum of lengths of any two consecutive edges of P is smaller than π if n is even and smaller than $2\pi - 2\arccos\big(\frac{1}{n-1}\big)$ if n is odd.*

To each convex spherical polygon $P \subset \mathbb{S}^2$ with vertices $A_1,\dots,A_n$, one can associate the *dual convex polygon* P^* with edges of lengths $\pi - \angle A_i$ and angles of values $|A_iA_{i+1}|$. To produce P^* one starts with the convex cone C_P in $\mathbb{R}^3$ over $P \subset \mathbb{S}^2$, takes its dual cone C_P^* and intersects it with $\mathbb{S}^2$, ie $P^* = C_P^* \cap \mathbb{S}^2$. Clearly, this duality defines a one-to-one correspondence between equiangular and equilateral polygons. So, Proposition 3.2 is equivalent to the following dual one, which we are going to prove.

Proposition 3.3 *Let P be a convex equilateral spherical polygon with $n \ge 3$ vertices. The sum of any two consecutive angles of P is larger than π if n is even and greater than $2\arccos\big(\frac{1}{n-1}\big)$ if n is odd.*

We will first reduce this statement to its Euclidean analogue by means of the following standard lemma:

Lemma 3.4 *For any convex spherical polygon P with vertices $A_1,\dots,A_n$, there is a convex Euclidean polygon P' with vertices $B_1,\dots,B_n$ such that, for all i, $|A_iA_{i+1}| = |B_iB_{i+1}|$ and $\angle A_i > \angle B_i$.*

Proof Cut P into $n-2$ convex triangles by diagonals A_1A_i. Replace each triangle by a flat one with sides of the same length and glue back to get a flat polygon. Since the angles of all $n-2$ triangles have decreased, the resulting Euclidean polygon satisfies the condition of the lemma. □

To prove Proposition 3.3 it remains to prove the following:

Proposition 3.5 *Let P be a convex equilateral Euclidean polygon with $n \ge 3$ vertices. The sum of any two consecutive angles of P is at least π if n is even and at least $2\arccos\left(\frac{1}{n-1}\right)$ if n is odd.*

This proposition in its turn will be deduced from the following two lemmas, the first of which is completely straightforward, and we omit its proof.

Lemma 3.6 *For any convex Euclidean polygon P with $n \ge 5$ vertices $A_1, \dots, A_n$, there is an arbitrary small deformation of P that preserves the lengths of edges and decreases the value $\angle A_1 + \angle A_2$.*

Lemma 3.7 *Let $ABCD$ be a convex Euclidean quadrilateral with sides of integer lengths such that $|AB| = 1$ and $|AB|+|BC|+|CD|+|DA| = n$. Then $\angle A + \angle B \ge \pi$ if n is even and $\angle A + \angle B \ge 2\arccos\left(\frac{1}{n-1}\right)$ if n is odd.*

Proof Consider first the case when n is even. If $|CD| = 1$, $ABCD$ is a parallelogram, so we can assume $|CD| > 1$. There exists a unique parallelogram $ABC'D$ with $C'D = 1$. Clearly, $\angle_A(ABC'D) = \angle_A(ABCD)$, and it is not hard to check that $\angle_B(ABC'D) < \angle_B(ABCD)$. Since $ABC'D$ is a parallelogram, we conclude $\angle_A(ABCD) + \angle_B(ABCD) > \pi$.

Suppose now that n is odd and assume $\angle A + \angle B < \pi$. Let E be the intersection of the lines $\overline{AD}$ and $\overline{BC}$. Clearly

$$|AC| + |CB| < |AD| + |DC| + |CB| = n-1 < |AE| + |EB|,$$

so there is a point F in the segment EC such that $|AF| + |FB| = n-1$. Clearly, $(\angle_A + \angle_B)(ABCD) > (\angle_A + \angle_B)(ABF)$. Note finally that, among all possible triangles of perimeter n with one side of length 1, the sum of two angles at this side attains its minimum for the isosceles triangle, and this minimum is $2\arccos\left(\frac{1}{n-1}\right)$. □

Proof of Proposition 3.5 Let Π_n be the space of all convex equilateral polygons in $\mathbb{R}^2$ with sides of length 1. It has a natural compactification $\overline{\Pi}_n$ consisting of all convex polygons with sides of integer length. The function $(\angle_{A_1} + \angle_{A_2})(P)$ defined on Π_n extends continuously to $\overline{\Pi}_n$, and from Lemma 3.6 it follows that it attains its minimum on the part of $\overline{\Pi}_n$ consisting of quadrilaterals and triangles. Now the statement follows from Lemma 3.7. □

The next lemma is a slight refinement of Proposition 3.2 for pentagons.

Lemma 3.8 *Any convex spherical equiangular pentagon satisfying*

$$|A_{i-1}A_i| + |A_iA_{i+1}| > \tfrac{2\pi}{3}$$

for $i = 1, \dots, 5$ *satisfies* $|A_{i-1}A_i| + |A_iA_{i+1}| < \pi$.

Dually, any convex spherical equilateral pentagon satisfying $\angle A_i + \angle A_{i+1} < \frac{4\pi}{3}$ *for* $i = 1, \dots, 5$ *satisfies* $\angle A_i + \angle A_{i+1} > \pi$.

Proof Let us prove the dual statement. We will assume $\angle A_1 + \angle A_2 \le \pi$, and deduce that $\angle A_5 + \angle A_1 + \angle A_2 + \angle A_3 > \frac{8\pi}{3}$, which contradicts the conditions of the lemma.

Let us decompose the pentagon into the union of the triangle $A_5A_4A_3$ and the quadrilateral $A_5A_1A_2A_3$. The condition $\angle A_1 + \angle A_2 \le \pi$ implies $|A_1A_2| > |A_3A_5|$. So $|A_4A_5| = |A_4A_3| > |A_3A_5|$ and in the triangle $A_5A_4A_3$ the sum of angles at vertices A_5 and A_3 exceeds $\frac{2\pi}{3}$. Adding to this value the sum of all angles of the quadrilateral $A_5A_1A_2A_3$, which exceeds 2π, we get the contradiction. □

The next lemma is straightforward; we omit the proof.

Lemma 3.9 *Let* k *and* n *be two integers with* $n, k \ge 2$. *Let* P_k *be a regular* (ie equilateral and equiangular) *spherical* k*–gon and* P_n *be a regular spherical* n*–gon. Suppose that the angles and the sides of* P_k *have the same size as that of* P_n. *Then* $n = k$.

4 Proof of Theorem 1.1

4.1 Properties of the polyhedral metric $g_{\mathcal{L}}^{\mathbb{R}}$ on $\mathbb{R}P^2$

Let us start the section by summarising the properties of the metric $g_{\mathcal{L}}^{\mathbb{R}}$ on $\mathbb{R}P^2$ induced from the polyhedral Kähler metric $g_{\mathcal{L}}^{\mathbb{C}}$ on $\mathbb{C}P^2$. First, we introduce some terminology. A real line arrangement $\mathcal{L}$ cuts $\mathbb{R}P^2$ into a collection of *polygons* whose edges are called the *edges* of the arrangement. Two multiple points of $\mathcal{L}$ are called *adjacent* if they are the endpoints points of one edge.

For each multiple point x of $\mathcal{L}$, by the *star* $S(x)$ of x we mean the union of all polygons adjacent to x. The intersection of a small neighbourhood of x with a star of x is a union of $2\mu(x)$ *sectors*.

Theorem 4.1 *Consider a real line arrangement $\mathcal{L}$ of $3n$ lines with the Hirzebruch property and let $g_{\mathcal{L}}^{\mathbb{R}}$ be the corresponding metric on $\mathbb{R}P^2$. Then the following properties hold:*

(1) *At any multiple point of $\mathcal{L}$, each sector has an acute angle unless the point is double, in which case all four sectors have angle $\frac{\pi}{2}$.*

(2) *There is a constant $a(n) < \frac{\pi}{3}$ such that the angles of sectors of all triple points of $\mathcal{L}$ are equal to $a(n)$.*

(3) *$\mathcal{L}$ is simplicial,*[4] *and no two vertices of multiplicity 2 are adjacent.*

(4) *Let x be a multiple point of $\mathcal{L}$. The sum of angles of any two adjacent sectors of x is less than $\frac{2\pi}{3}$ if $\mu(x) \geq 3$, and less than $\frac{\pi}{2}$ if $\mu = 4, 5$.*

(5) *The multiplicity of each multiple point of $\mathcal{L}$ is at most 5, and any point of multiplicity 5 has exactly 5 double points in the boundary of its star.*

(6) *For any multiple point of $\mathcal{L}$, the number of adjacent multiple points of multiplicity greater than 2 is at most five.*

Proof Let x be a multiple point of $\mathcal{L}$ and let $\mathbb{D}(x)$ be the associated spherical polygon. It is equiangular by Lemma 2.8.

(1) The length of any edge of a convex spherical polygon is at most π and it is equal to π only in the case when the polygon is a bigon. Hence, by Lemma 2.8, the angle of each sector is at most $\frac{\pi}{2}$ and it is equal to $\frac{\pi}{2}$ if and only if $\mathbb{D}(x)$ has exactly two vertices, ie x is a double point.

(2) If x is a triple point then $\mathbb{D}(x)$ is the unique regular spherical triangle with angles $\pi\frac{n-1}{n}$. The edges of such a triangle are shorter than $\frac{2\pi}{3}$, hence the statement holds by Lemma 2.8.

(3) Since by property (1) the angles of all polygons in which the arrangement cuts $\mathbb{R}P^2$ are not obtuse, the only polygons different from triangles that can be present in the decomposition are rectangles. Assume, by contradiction, that there is such a rectangle R in the decomposition. Applying again property (1), we see that all vertices of R are double points. If follows that all polygons sharing an edge with R are rectangles as well. Applying this reasoning repeatedly we come to a contradiction.

[4] That is, all the polygons of the decomposition are triangles.

(4) This is proven by applying Proposition 3.2 to the polygon $\mathbb{D}(x)$ if $\mu(x) \neq 5$ and applying Lemma 3.8 if $\mu(x) = 5$.

(5) Let x be a point of the arrangement of multiplicity d and let $S(x)$ be its star. This star is a union of triangles by property (3). Denote by $P_1, P_2, \dots, P_{2d}$ the vertices of these triangles lying on the boundary of $S(x)$, enumerated in a cyclic order. Note that unless the point P_i is a double point of the arrangement, by property (4) the angle of $S(x)$ at P_i is less than $\frac{2\pi}{3}$. We deduce from (3) that there are at least d points in the boundary of $S(x)$ with angle less than $\frac{2\pi}{3}$. Since the boundary of $S(x)$ is convex and the conical angle at x is less than 2π, applying the Gauss–Bonnet formula to the star $S(x)$ we conclude that $d \leq 5$.

(6) The proof of this statement repeats the proof of statement (5). □

4.2 Proof of Theorem 1.1

To prove Theorem 1.1 we will show that all the triangles in the decomposition of $\mathbb{R}P^2$ by $\mathcal{L}$ are isometric with respect to the metric $g_{\mathcal{L}}^{\mathbb{R}}$. We will start with the following lemma:

Lemma 4.2 *Let x and y be two adjacent multiple points in a real arrangement satisfying the Hirzebruch property. Suppose $\mu(x), \mu(y) \geq 3$. Then $\mu(x) = 3$ or $\mu(y) = 3$.*

Proof Consider triangles Δ_1 and Δ_2 of the decomposition that contain the edge xy and let Q_1 and Q_2 be their vertices opposite to xy. Since the angles at points Q_1 and Q_2 can not be obtuse by Theorem 4.1(1), in quadrilateral xQ_1yQ_2 we have $\angle x + \angle y \geq \pi$. Hence, either $\angle x \geq \frac{\pi}{2}$ or $\angle y \geq \frac{\pi}{2}$, and the corresponding point is of multiplicity 3 by Theorem 4.1(4)–(5). □

The next two corollaries give a complete description of stars of vertices having multiplicities 4 and 5.

Corollary 4.3 *Let x be a point of multiplicity 5 of a real arrangement with the Hirzebruch property. Let $P_1, \dots, P_{10}$ be the multiple points of the arrangement at the boundary of $S(x)$ and assume that $\mu(P_1) = 2$. Then for $i = 1, \dots, 5$ we have $\mu(P_{2i-1}) = 2$ and $\mu(P_{2i}) = 3$.*

Proof By Theorem 4.1(5), five of the points $P_1, \dots, P_{10}$ have multiplicity 2. Hence, it follows from Theorem 4.1(3) that the points P_{2i-1} have multiplicity 2. The remaining five points have multiplicity 3 by Lemma 4.2. □

Corollary 4.4 *Suppose x is a point of multiplicity 4 of a real arrangement with the Hirzebruch property, and let $P_1,\dots,P_8$ be the vertices of its star. Then at least one of the points P_i, say P_1, has multiplicity 2. In such a case, for $i=1,\dots,4$ we have $\mu(P_{2i-1})=2$ and $\mu(P_{2i})=3$.*

Proof By Theorem 4.1(6), x has at least one adjacent point of multiplicity 2. Let us denote it by P_1. By Lemma 4.2, points $P_1,\dots,P_8$ cannot have multiplicity 4 or 5. So it is enough to show that there cannot be five points of multiplicity 3 in the star of x. Since points of multiplicity 2 cannot be adjacent, this will follow if we show that no two consecutive points P_i are simultaneously of multiplicity 3.

Suppose by contradiction that P_2 and P_3 have multiplicity 3 and let us deduce that P_6 and P_7 have multiplicity 3.

Consider two triangles xP_2P_3 and xP_6P_7. By Lemma 2.8, the angles at x of these two triangles are the same. Hence, we should have

$$(\angle_{P_2}+\angle_{P_3})(xP_2P_3)=(\angle_{P_6}+\angle_{P_7})(xP_6P_7).$$

So, using Theorem 4.1(1)–(2), we see that both points P_6 and P_7 should be of multiplicity 3. To get a contradiction notice that P_8 is of multiplicity 3 and either P_4 or P_5 has multiplicity 3. So we get at least 6 points of multiplicity 3 among P_i. □

An immediate consequence of Corollaries 4.3 and 4.4 is the following statement:

Corollary 4.5 *Let $\mathcal{L}$ be a real line arrangement with the Hirzebruch property and let x be its multiple point. All sectors at x have the same angle at x with respect to the metric $g^{\mathbb{R}}_{\mathcal{L}}$.*

Proof If x is a double or triple point then this statement holds by Theorem 4.1.

Suppose x is a point of multiplicity 4. Using the notation of Corollary 4.4, we see that for any $i=1,\dots,7$ triangles xP_iP_{i+1} and $xP_{i+1}P_{i+2}$ (with $P_9=P_1$) are isometric by an isometry that sends P_i to P_{i+2} and fixes P_{i+1} and x. Hence all 8 sectors at x have the same angle.

The case $\mu(x)=5$ follows from Corollary 4.3 in the same way. □

Corollary 4.6 *Suppose that x, y and z are adjacent points of a real arrangement with the Hirzebruch property. Then the multiplicities of these points belong to the following list (up to a permutation): $(2,3,3)$, $(2,3,4)$, $(2,3,5)$.*

Proof By Lemma 4.2, at most one of the points x, y or z can have multiplicity 4 or 5. Assume that this point is z. Then applying to the star of z either Corollary 4.3 or Corollary 4.4, we see that multiplicities of x and y are $(2,3)$ up to a permutation.

All three points of the triangle xyz cannot be of multiplicity 3 since in this case $\angle x = \angle y = \angle z < \frac{\pi}{3}$ by Theorem 4.1(2), which contradicts Gauss–Bonnet. □

Corollary 4.7 *Let $\mathcal{L}$ be a real line arrangement with the Hirzebruch property.*

(1) *The lines of $\mathcal{L}$ cut $\mathbb{R}P^2$ into isometric triangles with respect to the metric $g_{\mathcal{L}}^{\mathbb{R}}$.*

(2) *There is some $d \in \{3,4,5\}$ such that the multiplicities of vertices of each triangle are $(2,3,d)$ up to a permutation.*

Proof (1) Let xyz and xyt be two triangles of the decomposition that share the side xy. Then, by Corollary 4.5, these triangles have the same angles at x and y. Hence, they are isometric. Hence, all triangles of the decomposition are isometric.

(2) By Corollary 4.6, for any two triangles of the decomposition, their vertices can be denoted by x, y, z and x', y', z' in such a way that

$$\mu(x) = \mu(x') = 2, \quad \mu(y) = \mu(y') = 3, \quad \mu(z) = d, \quad \mu(z') = d', \quad d, d' \geq 3.$$

In this case, by (1) there is an isometry between the triangles that sends x to x', y to y' and z to z'. By Corollary 4.5, the spherical polygons $\mathbb{D}(x)$ and $\mathbb{D}(x')$ are regular. Moreover, since $\angle_z = \angle_{z'}$, the polygons have sides of the same length and additionally they have angles of size $\pi\frac{n-1}{n}$ by Lemma 2.8. Hence, $d = d'$ by Lemma 3.9. □

Proof of Theorem 1.1 According to Corollary 4.7 we have three cases, $d = 3, 4, 5$. Replace each triangle in $\mathbb{R}P^2$ by a spherical triangle (of curvature 1) with angles $\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{d}\right)$. As a result, we obtain an $\mathbb{R}P^2$ with curvature-1 metric and a Coxeter arrangement in it. □

5 Discussion

Hirzebruch [5] gives the list of complex reflection arrangements of $3n$ lines such that each line intersects others in $n+1$ points. This list consists of two infinite series and five exceptional examples. The infinite series are called A_m^0 or Ceva arrangements $(m \geq 3)$ and A_m^3 $(m \geq 2)$ (or extended Ceva arrangements) and correspond to reflection groups $G(m,m,3)$ and $G(m,p,3)$ $(p<m)$ from the Shephard–Todd classification. The arrangements A_m^0 and A_m^3 are given in homogeneous coordinates by equations

$$(z_0^m - z_1^m)(z_1^m - z_2^m)(z_2^m - z_0^m) = 0,$$
$$z_0 z_1 z_2 (z_0^m - z_1^m)(z_1^m - z_2^m)(z_2^m - z_0^m) = 0,$$

respectively. The five exceptional examples are associated to reflection groups G_{23}, G_{24}, G_{25}, G_{26} and G_{27}. The corresponding arrangements are called the icosahedron configuration (15 lines), the configuration G_{168} or Klein configuration (21 lines), the Hesse configuration (12 lines), the configuration G_{216} or extended Hesse configuration (21 lines), and the configuration G_{360} or Valentiner configuration (45 lines); see [5].

I believe that in view of Theorem 1.1 one can restate Hirzebruch's question as a conjecture:

Conjecture 5.1 *All arrangements satisfying the Hirzebruch property are complex reflection arrangements.*

Convex foliations Line arrangements with the Hirzebruch property have an interesting relation to *reduced convex foliations* in $\mathbb{C}P^2$. A foliation in $\mathbb{C}P^2$ is called *convex* if its leaves other than straight lines have no inflection points. A foliation is called *reduced* if its inflection divisor is reduced [6]. It turns out that any arrangement which can be realised as the union of all lines tangent to a reduced convex foliation satisfies the Hirzebruch property. Moreover, all arrangements from Hirzebruch's list apart from G_{169} and G_{360} are indeed realised as line arrangements of reduced convex foliations (see [6] for more details).

It was explained in [9] that any real line arrangement realisable as the line arrangement of a convex foliation is simplicial, which can be seen as a partial case of Theorem 4.1(3). Note that at the present only a conjectural classification of simplicial arrangements in $\mathbb{R}P^2$ is known; see [2; 3].

Real polyhedral Kähler metrics Theorem 1.1 can be seen as a first step toward a solution of the following classification problem:

Definition 5.2 A polyhedral Kähler metric on $\mathbb{C}P^2$ is called real if it is invariant under the conjugation of $\mathbb{C}P^2$. We call this metric *maximally real* if the divisor of singularities of the metric is smooth in the complement of $\mathbb{R}P^2$.

Problem 5.3 Classify all positively curved maximally real polyhedral Kähler metrics on $\mathbb{C}P^2$.

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