The nilpotence theorem for the algebraic K–theory of the sphere spectrum

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We prove that in the graded commutative ring $K_*(\mathbb{S})$, all positive degree elements are multiplicatively nilpotent. The analogous statements also hold for $\mathrm{TC}_*(\mathbb{S})^\wedge_p$ and $K_*(\mathbb{Z})$.

1 Introduction

Much of the most exciting work in algebraic K–theory over the past 15 years has been aimed at the verification of the Quillen–Lichtenbaum conjecture. The successful affirmation of this conjecture has led to the identification of the homotopy types of the K–theory of the integers $\mathbb{Z}$ and the K–theory of the sphere spectrum $\mathbb{S}$ at regular primes; see Dwyer and Mitchell [15], Rognes [29; 30] and Rognes and Weibel [31]. Since $H\mathbb{Z}$ and $\mathbb{S}$ are E_∞ ring spectra, $K(\mathbb{Z})$ and $K(\mathbb{S})$ are E_∞ ring spectra and the graded rings $K_*(\mathbb{S}) = \pi_* K(\mathbb{S})$ and $K_*(\mathbb{Z}) = \pi_* K(\mathbb{Z})$ are commutative. However, almost nothing is known about the multiplicative structure. The only work in this direction so far is the investigation of Bergsaker and Rognes [4] of the Dyer–Lashof operations on $\mathrm{TC}_*(\mathbb{S})$ at the prime 2. In this paper, we begin the study of the multiplicative structure on the homotopy groups of $K(\mathbb{S})$ by proving the analogue of Nishida's nilpotence theorem.

Theorem 1 *Positive degree elements of $K_*(\mathbb{S})$ are nilpotent.*

On the way to proving the preceding theorem, we show the corresponding nilpotence result for $K_*(\mathbb{Z})$. We deduce this by observing that $K_{2n(p-1)}(\mathbb{Z}) \otimes \mathbb{Z}_{(p)} = 0$ for odd primes p and $n > 0$; it can also be deduced from the multiplicative properties of the Quillen–Lichtenbaum spectral sequence.

Theorem 2 *Positive degree elements of $K_*(\mathbb{Z})$ are nilpotent.*

Much of the interest in $K(\mathbb{S})$ comes from its identification as $A(*)$, Waldhausen's algebraic K–theory of the one-point space. Work of Waldhausen and collaborators shows that $A(X)$ controls high-dimensional manifold theory (eg see Waldhausen,

Jahren and Rognes [37] and Weiss and Williams [40]) via the connection to the stable pseudoisotopy spectrum $\mathrm{Wh}(X)$. Rognes shows that the infinite loop space structure on $\mathrm{Wh}(*)$ that is relevant to the Hatcher–Waldhausen map $G/O \to \Omega\mathrm{Wh}(*)$, where G/O denotes the classifying spectrum for smooth normal invariants, is induced by the ring structure on $A(*)$; see Rognes [28]. Moreover, $A(X)$ is a module over $A(*)$; more generally, for any ring spectrum (or even any Waldhausen category that admits factorization; see Blumberg and Mandell [7; 8]), the algebraic K–theory spectrum is a module over $A(*)$.

Theorem 1 also has direct implications in the context of Kontsevich's noncommutative motives. The work of Blumberg, Gepner and Tabuada [5; 6] produces a candidate category of spectral motives $\mathcal{M}ot_{\mathrm{ex}}$, which is a symmetric monoidal category with objects the smooth and proper small stable idempotent-complete ∞–categories. The category of spectral motives is stable, which in particular implies that it has a tensor-triangulated homotopy category and is enriched over spectra; the mapping spectra are essentially bivariant algebraic K–theory. The endomorphism spectrum of the unit is precisely $K(\mathbb{S})$ (as an E_∞ ring spectrum).

The Devinatz–Hopkins–Smith nilpotence theorem and the Hopkins–Smith thick subcategory theorem teach us that to understand a triangulated category, we should look to its thick subcategories, which play the role of prime ideals in derived algebraic geometry; see Hopkins [21], Neeman [25] and Thomason [35]. More recently, Balmer [1; 2] proposes a systematic study of this in the setting of "tensor-triangulated geometry", defining the *triangulated spectrum* to be the space of prime proper thick triangulated tensor ideals (with the Zariski topology). Balmer observes that there is a canonical map from the triangulated spectrum to the spectrum of the graded ring of endomorphisms of the unit and that in many known examples, the spectrum of the endomorphism ring controls the triangulated spectrum of the tensor-triangulated category. Our main theorem is the first step in realizing this program for spectral motives.

In a different direction, Morava has developed a conjectural program for studying a homotopy-theoretic analogue of Kontsevich's Grothendieck–Teichmüller group — see Kitchloo and Morava [22] and Morava [24] — in terms of homotopical descent for the category of spectral motives. These ideas revolve around understanding the structure of $\mathbb{S} \wedge^{\mathbf{L}}_{K(\mathbb{S})} \mathbb{S}$, which of course depends on the ring structure of $K(\mathbb{S})$. Morava notes that the calculation of this object is straightforward rationally and results in a concise description as a polynomial algebra on even degree generators: it is the polynomial algebra on the free Lie coalgebra $L\langle x_6, x_{10}, x_{14}, \dots \rangle$ on generators in degrees 6, 10, 14, etc. (It is a Hopf algebra with coalgebra the tensor coalgebra on $\pi_*\Sigma\mathrm{Wh}(*)_\mathbb{Q} \cong \pi_*\Sigma^6 ko_\mathbb{Q}$, where $\mathrm{Wh}(*)$ is the fiber of the map $K(S) \to S$.) Our results give the first progress in the direction of the torsion part of this theory.

2 Reduction of Theorems 1 and 2

Consider the arithmetic square

$$
\begin{array}{ccc}
K(\mathbb{S}) & \longrightarrow & \prod_p K(\mathbb{S})_p^\wedge \\
\downarrow & & \downarrow \\
K(\mathbb{S})_\mathbb{Q} & \longrightarrow & \left(\prod_p K(\mathbb{S})_p^\wedge\right)_\mathbb{Q}
\end{array}
$$

where $(-)_p^\wedge$ denotes p–completion (localization with respect to the mod p Moore spectrum) and $(-)_\mathbb{Q}$ denotes rationalization. To prove Theorem 1, it suffices to prove the analogous nilpotence results for $K(\mathbb{S})_\mathbb{Q}$ and $K(\mathbb{S})_p^\wedge$ for each prime p; this is easy to see for $K(\mathbb{S})$ because $\pi_* K(\mathbb{S})$ is finitely generated in each degree [14, 1.2], which implies that $\pi_*(K(\mathbb{S})_p^\wedge) \cong (\pi_* K(\mathbb{S})) \otimes \mathbb{Z}_p^\wedge$; see [12, 2.5]. (Similar observations apply to $K(\mathbb{Z})$ for Theorem 2; see [27].). The rational part is well understood: the natural map $K(\mathbb{S})_\mathbb{Q} \to K(\mathbb{Z})_\mathbb{Q}$ is an equivalence [36, 2.3.8], and classical results of Borel [11, 12.2] imply that the positive degree elements of $\pi_* K(\mathbb{Z})_\mathbb{Q}$ are concentrated in odd degrees and therefore square to zero. It remains to study the situation after p–completion.

Our strategy for studying the multiplicative structure on $K(\mathbb{S})_p^\wedge$ uses the cyclotomic trace map, which is a map of E_∞ ring spectra from $K(\mathbb{S})$ to the topological cyclic homology TC($\mathbb{S}$). The homotopy type of TC($\mathbb{S}$)$_p^\wedge$ (as a spectrum) is known by work of [9].

Theorem 2.1 [9, 5.16] *There is an equivalence of p–complete spectra*

$$\mathrm{TC}(\mathbb{S})_p^\wedge \simeq \mathbb{S}_p^\wedge \vee \mathrm{hofib}(\Sigma(\Sigma_+^\infty \mathbb{CP}^\infty) \to \mathbb{S})_p^\wedge \simeq \mathbb{S}_p^\wedge \vee (\mathbb{CP}_{-1}^\infty)_p^\wedge.$$

The Devinatz–Hopkins–Smith nilpotence theorems provide a criterion for determining when elements in the homotopy groups of a ring spectrum R are multiplicatively nilpotent. Specifically, an element $x \in \pi_* R$ is nilpotent if and only if the Hurewicz map takes it to a nilpotent element of $K(n)_* R$ for all $0 \le n \le \infty$ (and all primes p).

Although the previous theorem only identifies the homotopy type of the underlying spectrum and says nothing about the multiplication, it is enough to deduce a nilpotence result for $\mathrm{TC}(\mathbb{S})_p^{\wedge}$.

Proposition 2.2 *Let p be a prime, let $0 \leq n \leq \infty$, and let $\widetilde{\mathrm{TC}}(\mathbb{S}; p)$ be the homotopy fiber of the augmentation map $\mathrm{TC}(\mathbb{S})_p^{\wedge} \to \mathbb{S}_p^{\wedge}$ (obtained from the canonical map $\mathrm{TC}(\mathbb{S})_p^{\wedge} \to \mathrm{THH}(\mathbb{S})_p^{\wedge} \simeq \mathbb{S}_p^{\wedge}$). Then $K(n)_*(\widetilde{\mathrm{TC}}(\mathbb{S}; p))$ is concentrated in odd degrees.*

Proof As a consequence of Theorem 2.1, $\widetilde{\mathrm{TC}}(\mathbb{S}; p) \simeq \Sigma(\mathbb{C}P_{-1}^{\infty})_p^{\wedge}$. The spectrum $\mathbb{C}P_{-1}^{\infty}$ is the Thom spectrum of the virtual bundle $-\gamma$, for γ the tautological line bundle over $\mathbb{C}P^{\infty}$. The spectra $K(n)$ are all complex oriented; the proposition now follows from the Thom isomorphism. □

Since $\pi_*(\mathrm{TC}(\mathbb{S})_p^{\wedge})$ splits as $\pi_* \mathbb{S}_p^{\wedge} \oplus \pi_* \widetilde{\mathrm{TC}}(\mathbb{S}; p)$, with the first factor the image of the inclusion of the unit, we obtain the following as an immediate corollary of the previous proposition and the nilpotence theorem.

Theorem 2.3 *For any prime p, all the nonzero degree elements of $\pi_* \mathrm{TC}(\mathbb{S})_p^{\wedge}$ are nilpotent.*

In light of the previous result, Theorem 1 becomes an immediate consequence of the following lemma. We prove this lemma for odd p in later sections; for $p = 2$ it is a special case of [29, 3.16].

Lemma 1 *For $p = 2$, let $d = 8$, and for p odd, let $d = 2(p-1)$. The homotopy fiber of the cyclotomic trace map $trc_p\colon K(\mathbb{S})_p^{\wedge} \to \mathrm{TC}(\mathbb{S})_p^{\wedge}$ has trivial homotopy groups in degrees kd for $k > 0$.*

Proof of Theorem 1 from Lemma 1 Given $x \in \pi_k K(\mathbb{S})_p^{\wedge}$, $x^d \in \pi_{kd} K(\mathbb{S})_p^{\wedge}$. When $k > 0$, we then know that for some power n, $(x^d)^n$ maps to zero in $\pi_{kdn}(\mathrm{TC}(\mathbb{S})_p^{\wedge})$ under the trace map by Theorem 2.3. By Lemma 1, the kernel of the trace is zero in degree kdn, and so $x^{kdn} = 0$. □

As we used in the proof, Lemma 1 implies that the cyclotomic trace $K(\mathbb{S}) \to \mathrm{TC}(\mathbb{S})$ is injective in certain degrees. In fact, for odd regular primes, the cyclotomic trace is injective in all degrees. This follows from the work of Rognes on $\mathrm{Wh}(*)$ at odd regular primes, specifically [30, 3.6 and 3.8]. In the case of irregular primes, we expect that the trace fails to be injective; we hope to return to this question in a future paper.

On the way to proving Lemma 1, we also prove the following lemma. It is well known that $\pi_{4k} K(\mathbb{Z}) \otimes \mathbb{Z}_{(p)} = 0$ at regular primes, including $p = 2$ (see [39, 10.1], for example), and this combined with the following lemma now proves Theorem 2.

Lemma 2 *For p an odd prime, $\pi_{2(p-1)k} K(\mathbb{Z}) \otimes \mathbb{Z}_{(p)} = 0$ for $k > 0$.*

3   Reduction of Lemmas 1 and 2

The basic strategy for the proof of Lemmas 1 and 2 is to reduce the study of the homotopy fiber of the cyclotomic trace $K(\mathbb{S})^\wedge_p \to \mathrm{TC}(\mathbb{S})^\wedge_p$ to the study of the p–completion map $\mathbb{Z}[1/p] \to \mathbb{Q}^\wedge_p$ in étale cohomology. (This is now a fairly standard approach; for instance, see [30, Sections 2–3; 17; 19].) As indicated above, from here on we assume that p is odd (though all of what we say would also apply in the case $p = 2$ until (3.6)). First, we apply Dundas' theorem [13] about the cyclotomic trace: the square

$$
\begin{array}{ccc}
K(\mathbb{S})^\wedge_p & \longrightarrow & K(\mathbb{Z})^\wedge_p \\
{\scriptstyle trc_p}\downarrow & & \downarrow{\scriptstyle trc^\mathbb{Z}_p} \\
\mathrm{TC}(\mathbb{S})^\wedge_p & \longrightarrow & \mathrm{TC}(\mathbb{Z})^\wedge_p
\end{array}
$$

is homotopy cocartesian, where the horizontal maps arise from linearization. As a consequence, we have the following lemma:

Proposition 3.1   (Dundas [13])   *The induced map* $\mathrm{hofib}(trc_p) \to \mathrm{hofib}(trc^\mathbb{Z}_p)$ *is an equivalence.*

To understand $\mathrm{hofib}(trc^\mathbb{Z}_p)$, consider the commutative diagram

$$
\begin{array}{ccc}
K(\mathbb{Z})^\wedge_p & \xrightarrow{\ cmp\ } & K(\mathbb{Z}^\wedge_p)^\wedge_p \\
{\scriptstyle trc^\mathbb{Z}_p}\downarrow & & \downarrow{\scriptstyle trc^{\mathbb{Z}^\wedge_p}_p} \\
\mathrm{TC}(\mathbb{Z})^\wedge_p & \xrightarrow[\ cmp^{\mathrm{TC}}\ ]{} & \mathrm{TC}(\mathbb{Z}^\wedge_p)^\wedge_p
\end{array}
$$

where the horizontal maps cmp and cmp^{TC} are induced by the map of rings $\mathbb{Z} \to \mathbb{Z}^\wedge_p$. By work of Hesselholt and Madsen [20], the bottom map is a weak equivalence [20, Addendum 6.2] and the right-hand map induces a weak equivalence [20, Theorem D]

$$(3.2) \qquad\qquad K(\mathbb{Z}^\wedge_p)^\wedge_p \to \mathrm{TC}(\mathbb{Z}^\wedge_p)^\wedge_p[0, \infty)$$

(where $[0, \infty)$ denotes the connective cover). Thus, up to passing to a connective cover, we can identify the trace map $trc^\mathbb{Z}_p$ as the map $cmp\colon K(\mathbb{Z})^\wedge_p \to K(\mathbb{Z}^\wedge_p)^\wedge_p$. We then have the following relationship between $\mathrm{hofib}(trc_p) \simeq \mathrm{hofib}(trc^\mathbb{Z}_p)$ and $\mathrm{hofib}(cmp)$.

Proposition 3.3   *There is a cofiber sequence*

$$\mathrm{hofib}(cmp) \to \mathrm{hofib}(trc_p) \to \Sigma^{-2} H\mathbb{Z}^\wedge_p \to \Sigma \cdots$$

Proof   Using the equivalence of $\mathrm{hofib}(trc_p)$ and $\mathrm{hofib}(trc_p^{\mathbb{Z}})$ above, we get a diagram of cofiber sequences

$$
\begin{array}{ccccccc}
\mathrm{hofib}(cmp) & \longrightarrow & K(\mathbb{Z})_p^{\wedge} & \xrightarrow{\ cmp\ } & K(\mathbb{Z}_p^{\wedge})_p^{\wedge} & \longrightarrow & \Sigma\,\mathrm{hofib}(cmp) \\
\downarrow & & \| & & {\scriptstyle trc_p^{\mathbb{Z}_p^{\wedge}}}\big\downarrow & & \downarrow \\
\mathrm{hofib}(trc_p) & \longrightarrow & K(\mathbb{Z})_p^{\wedge} & \longrightarrow & \mathrm{TC}(\mathbb{Z}_p^{\wedge})_p^{\wedge} & \longrightarrow & \Sigma\,\mathrm{hofib}(trc_p)
\end{array}
$$

identifying the right-hand square as homotopy (co)cartesian. Since $\pi_{-1}\mathrm{TC}(\mathbb{Z})_p^{\wedge} = \mathbb{Z}_p^{\wedge}$ and $\pi_n\mathrm{TC}(\mathbb{Z})_p^{\wedge} = 0$ for $n < -1$, the homotopy cofiber of the map $trc_p^{\mathbb{Z}_p^{\wedge}}$ in the diagram is $\Sigma^{-1}H\mathbb{Z}_p^{\wedge}$. Desuspending, we see that the homotopy cofiber of

$$\mathrm{hofib}(cmp) \to \mathrm{hofib}(trc_p)$$

is $\Sigma^{-2}H\mathbb{Z}_p^{\wedge}$. $\square$

For Lemma 1 then, $\mathrm{hofib}(cmp)$ works just as well as $\mathrm{hofib}(trc_p)$. Quillen's localization sequence [26] gives cofiber sequences

$$
\begin{array}{ccccccc}
K(\mathbb{Z}/p) & \longrightarrow & K(\mathbb{Z}) & \longrightarrow & K(\mathbb{Z}[1/p]) & \longrightarrow & \Sigma \cdots \\
{\scriptstyle id}\big\downarrow & & \downarrow & & \downarrow & & \\
K(\mathbb{Z}/p) & \longrightarrow & K(\mathbb{Z}_p^{\wedge}) & \longrightarrow & K(\mathbb{Q}_p^{\wedge}) & \longrightarrow & \Sigma \cdots
\end{array}
$$

(3.4)

from which we can see that $\mathrm{hofib}(cmp)$ is equivalent to the homotopy fiber of the map

$$cmp'\colon K(\mathbb{Z}[1/p])_p^{\wedge} \to K(\mathbb{Q}_p^{\wedge})_p^{\wedge}.$$

Proposition 3.5   *There is a homotopy equivalence* $\mathrm{hofib}(cmp) \to \mathrm{hofib}(cmp')$.

The advantage of this approach is that étale cohomology methods at the prime p can be applied in rings where p is a unit. Let R denote either $\mathbb{Z}[1/p]$ or $\mathbb{Q}_p^{\wedge}$; then R satisfies the "mild extra hypotheses" of Thomason [34, 0.1], which gives a spectral sequence

$$(3.6) \qquad E_2^{s,t} = H_{\text{ét}}^s\big(\mathrm{Spec}\,R; \mathbb{Z}/p^n\big(\tfrac{1}{2}t\big)\big) \implies \pi_{t-s}(K_{\text{ét}}(R); \mathbb{Z}/p^n)$$

from étale cohomology to the mod p^n homotopy groups of (Dwyer–Friedlander) étale K–theory. In the formula above

$$
\mathbb{Z}/p^n\big(\tfrac{1}{2}t\big) = \begin{cases} \mu_{p^n}^{\otimes(t/2)} & \text{if } t \text{ is even,} \\ 0 & \text{if } t \text{ is odd,} \end{cases}
$$

where μ_{p^n} denotes the $(p^n)^{\text{th}}$ roots of 1 (ie $\mu_{p^n}(A) = \{x \in A \mid x^{p^n} = 1\}$, a sheaf in the étale topology). In this case the affirmed Quillen–Lichtenbaum conjecture [39, VI.8.2] identifies

$$\pi_*(K(R); \mathbb{Z}/p^n) = \pi_*(K_{\text{ét}}(R); \mathbb{Z}/p^n)$$

for $* \geq 2$. Also, because we have assumed that p is odd, $H_{\text{ét}}^*(R; \mathbb{Z}/p^n(k)) = 0$ for $* > 2$ [32, Section III.1.3], and the spectral sequence collapses to give an isomorphism and a short exact sequence

$$\pi_{2k-1}(K(R); \mathbb{Z}/p^n) \cong H_{\text{ét}}^1(\operatorname{Spec} R; \mathbb{Z}/p^n(k)),$$

$$(3.7) \quad 0 \to H_{\text{ét}}^2(\operatorname{Spec} R; \mathbb{Z}/p^n(k+1)) \to \pi_{2k}(K(R); \mathbb{Z}/p^n)$$
$$\to H_{\text{ét}}^0(\operatorname{Spec} R; \mathbb{Z}/p^n(k)) \to 0$$

for $k > 1$. In fact, the calculation of the $H_{\text{ét}}^0$ term is well known:

Proposition 3.8 Let $R = \mathbb{Z}[1/p]$ or $\mathbb{Q}_p^\wedge$. Then $H_{\text{ét}}^0(\operatorname{Spec} R; \mathbb{Z}/p^n(k)) = 0$ unless $(p-1) \mid k$. If $k = m(p-1)$, then $H_{\text{ét}}^0(\operatorname{Spec} R; \mathbb{Z}/p^n(k)) \cong \mu_{p^i}^{\otimes k}(\bar{Q})$, where $p^i = \gcd(|m|p, p^n)$ (and $i = n$ if $m = 0$) and $\bar{Q}$ is the algebraic closure of the field of fractions of R.

Proof The inclusion of the generic point $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}[1/p]$ induces an isomorphism

$$H_{\text{ét}}^0(\operatorname{Spec} \mathbb{Z}[1/p], \mathbb{Z}/p^n(k)) \to H_{\text{ét}}^0(\operatorname{Spec} \mathbb{Q}, \mathbb{Z}/p^n(k));$$

see [32, Proposition 1]. This reduces to the case $Q = \mathbb{Q}$ or $\mathbb{Q}_p^\wedge$ and the étale cohomology $H_{\text{ét}}^0(\operatorname{Spec} Q; \mathbb{Z}/p^n(k))$ becomes the Galois cohomology $H_{\text{Gal}}^0(Q; \mu_{p^n}^{\otimes k}(\bar{Q}))$. (We will now fix $\bar{Q}$ and write μ_{p^n} for $\mu_{p^n}(\bar{Q})$.) Letting $G = \operatorname{Gal}(Q(\mu_{p^n})/Q)$, the action of $\operatorname{Gal}(\bar{Q}/Q)$ on $\mu_{p^n}^{\otimes k}$ factors through G, and we can identify $H_{\text{Gal}}^0(Q; \mu_{p^n}^{\otimes k})$ as the G–fixed point subgroup of $\mu_{p^n}^{\otimes k}$. We have a canonical isomorphism $G = (\mathbb{Z}/p^n)^\times$ given by letting $r \in (\mathbb{Z}/p^n)^\times$ act on $\alpha \in \mu_{p^n}$ by $\alpha \mapsto \alpha^r$; then r acts on $\mu_{p^n}^{\otimes k}$ by the r^k power map (ie multiplication by r^k when we write the group operation additively). Choosing r to be a generator of $(\mathbb{Z}/p^n)^\times$, the G–fixed point subgroup of $\mu_{p^n}^{\otimes k}$ is the subset where r acts by the identity, or equivalently, the subset $\alpha \in \mu_{p^n}^{\otimes k}$ such that $\alpha^{r^k-1} = 1$. If $p-1$ does not divide k, then $r^k - 1$ is not congruent to 0 mod p, and the only fixed point is the identity. On the other hand, $r^{m(p-1)} - 1$ is divisible by p^i (and for $i < n$ not p^{i+1}) where $p^i = \gcd(|m|p, p^n)$ (for $m \neq 0$ or $i = n$ if $m = 0$), and the G–fixed point subgroup is exactly the subgroup $\mu_{p^i}^{\otimes k}$. $\square$

Defining $H_{\text{ét}}^*(-; \mathbb{Z}_p^\wedge(k))$ as the inverse limit of $H_{\text{ét}}^*(-; \mathbb{Z}/p^n(k))$, we see from the preceding proposition that for $R = \mathbb{Z}[1/p]$ or $\mathbb{Q}_p^\wedge$, we have $H_{\text{ét}}^0(R; \mathbb{Z}_p^\wedge(k)) = 0$ for

$k \neq 0$. Since in these cases the homotopy groups of $K(R)$ are finitely generated $\mathbb{Z}_p^\wedge$–modules in each degree (see [14, Section 4; 20, Theorem D; 10, 0.7]), $\pi_* K(R)_p^\wedge \cong \lim \pi_*(K(R); \mathbb{Z}/p^n)$. Combining these observations and the left exactness of $\lim$, we then get isomorphisms

$$
\begin{aligned}
\pi_{2k-1}(K(R)_p^\wedge) &\cong H^1_{\text{ét}}(\operatorname{Spec} R; \mathbb{Z}_p^\wedge(k)), \\
\pi_{2k}(K(R)_p^\wedge) &\cong H^2_{\text{ét}}(\operatorname{Spec} R; \mathbb{Z}_p^\wedge(k+1)),
\end{aligned}
$$

(3.9)

for $k > 1$. Combining these isomorphisms with the fact that

$$
\pi_{2m(p-1)} \operatorname{hofib}(trc_p) \cong \pi_{2m(p-1)} \operatorname{hofib}(cmp) \cong \pi_{2m(p-1)} \operatorname{hofib}(cmp')
$$

and $\pi_{2m(p-1)} \operatorname{hofib}(cmp')$ fits in an exact sequence

$$
\pi_{2m(p-1)+1} K(\mathbb{Z}[1/p])_p^\wedge \to \pi_{2m(p-1)+1} K(\mathbb{Q}_p^\wedge)_p^\wedge
$$
$$
\to \pi_{2m(p-1)} \operatorname{hofib}(cmp') \longrightarrow \pi_{2m(p-1)} K(\mathbb{Z}[1/p])_p^\wedge,
$$

Lemma 1 is now an immediate consequence of the following pair of lemmas, proved in the next section.

Lemma 3 *Let* p *be an odd prime. The map of rings* $\mathbb{Z}[1/p] \to \mathbb{Q}_p^\wedge$ *induces a surjection*

$$
H^1_{\text{ét}}(\operatorname{Spec} \mathbb{Z}[1/p]; \mathbb{Z}_p^\wedge(m(p-1)+1)) \to H^1_{\text{ét}}(\operatorname{Spec} \mathbb{Q}_p^\wedge; \mathbb{Z}_p^\wedge(m(p-1)+1))
$$

for all $m > 0$.

Lemma 4 *Let* p *be an odd prime.* $H^2_{\text{ét}}(\operatorname{Spec} \mathbb{Z}[1/p]; \mathbb{Z}_p^\wedge(m(p-1)+1)) = 0$ *for all* $m > 0$.

We can also deduce Lemma 2: Quillen's computation of the K–theory of finite fields implies in particular that $K(\mathbb{Z}/p)_p^\wedge \simeq H\mathbb{Z}_p^\wedge$. It then follows from Quillen's localization sequence (3.4) that the map $K(\mathbb{Z})_p^\wedge \to K(\mathbb{Z}[1/p])_p^\wedge$ induces an isomorphism in homotopy groups above degree 1. Lemma 2 now follows from the isomorphisms (3.9) and Lemma 4.

4 Proof of Lemmas 3 and 4

In this section, we prove Lemmas 3 and 4. Lemma 3 is about the p–completion map in étale cohomology and the basic tool for studying this is the Tate–Poitou duality

long exact sequence [33]. (Again, for examples applied to K–theory, see [30, 3.1; 17, Section 4; 19].)

In our context, the Tate–Poitou sequence takes the following form. Let M be a finite abelian p–group with an action of the Galois group G of the maximal extension of $\mathbb{Q}$ unramified except at p (eg $M = \mathbb{Z}/p^n(k)$) and let $(-)^*$ denote the Pontryagin dual, $A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$; then $M^*(1)$ is the G–module $\mathrm{Hom}(M, \mu_\infty)$, where μ_∞ denotes the G–module of all roots of 1 in the algebraic closure of $\mathbb{Q}$. The low-dimensional part of Tate–Poitou duality in the case at hand is then summarized by the following long exact sequence [33, 3.1]:

$$0 \longrightarrow H^0_{\text{ét}}(\mathbb{Z}[1/p]; M) \longrightarrow H^0_{\text{ét}}(\mathbb{Q}_p^\wedge; M) \longrightarrow \left(H^2_{\text{ét}}(\mathbb{Z}[1/p], M^*(1))\right)^*$$

(4.1)
$$\longrightarrow H^1_{\text{ét}}(\mathbb{Z}[1/p]; M) \longrightarrow H^1_{\text{ét}}(\mathbb{Q}_p^\wedge; M) \longrightarrow \left(H^1_{\text{ét}}(\mathbb{Z}[1/p], M^*(1))\right)^*$$

$$\longrightarrow H^2_{\text{ét}}(\mathbb{Z}[1/p]; M) \longrightarrow H^2_{\text{ét}}(\mathbb{Q}_p^\wedge; M) \longrightarrow \left(H^0_{\text{ét}}(\mathbb{Z}[1/p], M^*(1))\right)^* \longrightarrow 0$$

When $M = \mathbb{Z}/p^n(k)$, the first map in the sequence above,

$$H^0_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^n(k)) \longrightarrow H^0_{\text{ét}}(\mathbb{Q}_p^\wedge; \mathbb{Z}/p^n(k)),$$

is an isomorphism by Proposition 3.8. Likewise, when $M = \mathbb{Z}/p^n(k)$ for $k > 1$, we see from (3.7) that $H^i_{\text{ét}}(R; \mathbb{Z}/p^n(k))$ is finite for $R = \mathbb{Z}[1/p]$ or $\mathbb{Q}_p^\wedge$, and it follows that the above is an exact sequence of finite groups. Taking the inverse limit over n is then exact and we get the following Tate–Poitou sequence:

$$0 \to \left(H^2_{\text{ét}}(\mathbb{Z}[1/p], \mathbb{Z}/p^\infty(1-k))\right)^*$$

(4.2)
$$\hookrightarrow H^1_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}_p^\wedge(k)) \to H^1_{\text{ét}}(\mathbb{Q}_p^\wedge; \mathbb{Z}_p^\wedge(k)) \to \left(H^1_{\text{ét}}(\mathbb{Z}[1/p], \mathbb{Z}/p^\infty(1-k))\right)^*$$

$$\hookrightarrow H^2_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}_p^\wedge(k)) \to H^2_{\text{ét}}(\mathbb{Q}_p^\wedge; \mathbb{Z}_p^\wedge(k)) \to \left(H^0_{\text{ét}}(\mathbb{Z}[1/p], \mathbb{Z}/p^\infty(1-k))\right)^* \to 0$$

For Lemmas 3 and 4, we apply (4.2) with $k = m(p-1)+1$, combined with the main theorem of Bayer and Neukirch [3], which relates the values of the Iwasawa p–adic ζ–function with the size of étale cohomology groups. In the following theorem, $|\cdot|_p$ denotes the p–adic valuation on $\mathbb{Q}_p^\wedge$, normalized so that $|p^n u|_p = p^{-n}$, where u is a unit in $\mathbb{Z}_p^\wedge$.

Theorem 4.3 (Bayer and Neukirch [3, 6.1]) *Let $\zeta_I(\omega^0, s)$ denote the Iwasawa zeta function of [3, 5.1] associated to the trivial character ω^0 and the field $\mathbb{Q}$. Let $k =$*

$m(p-1)+1$ for $m \neq 0$. If $\zeta_I(\omega^0, k) \neq 0$ then the groups $H^*_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k))$ are all finite (zero for $* \geq 2$) and

$$|\zeta_I(\omega^0, k)|_p = \frac{\#\big(H^0_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k))\big)}{\#\big(H^1_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k))\big)}.$$

The following computation of $|\zeta_I(\omega^0, m(p-1)+1)|_p$ is well known.

Proposition 4.4 *For $m \neq 0$ and $k = m(p-1)+1$,*

$$|\zeta_I(\omega^0, k)|_p = \left|\frac{1}{mp}\right|_p.$$

Proof The Iwasawa zeta function used by Bayer and Neukirch [3, 5.1] depends on a choice of $q \in \mathbb{Z}^\wedge_p$ with $q \equiv 1 \bmod p$. For the trivial character, the formula is then

$$\zeta_I(\omega^0, s) = \frac{p^{\mu_0} g_0(q^{1-s} - 1)}{1 - q^{1-s}},$$

where, for $\mathbb{Q}$ (and any abelian extension thereof), $\mu_0 = 0$ as a case of the Iwasawa "$\mu = 0$" conjecture proved by Ferrero and Washington [16] (see [3, 5.3]) and $g_0(x)$ is the characteristic polynomial of the action of $T \in \mathbb{Z}^\wedge_p[\![T]\!] \cong \Lambda$ on a Λ–module denoted as $e_0 \mathcal{M}$ in [3]. (Here Λ is the Iwasawa algebra [38, 7.1] for $\mathbb{Z}^\wedge_p \cong \Gamma < \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ with topological generator $\gamma \leftrightarrow 1 + T$ acting by $\alpha \mapsto \alpha^q$ for $x \in \mu_{p^\infty}$.) Since, for $k = m(p-1)+1$ with $m \neq 0$,

$$|1 - q^{1-k}|_p = |1 - q^{-m(p-1)}|_p = |1 - q^{|m|(p-1)}|_p = \left|\frac{1}{m(p-1)p}\right|_p = \left|\frac{1}{mp}\right|_p,$$

it suffices to show that $g_0(x) = 1$. This is a special case of the main conjecture of Iwasawa theory [23, Section 6, Conjecture] for the trivial character. Though the exposition preceding [23, Section 9, Theorem] makes the statement appear ambiguous in the case of the trivial character, this case was known at least as far back as [18], as we now discuss for the benefit of those (like the authors) not expert in this theory.

Washington [38, 15.37] denotes $e_0 \mathcal{M}$ as $\epsilon_0 \mathcal{X}$ and $\epsilon_0 \mathcal{X}_\infty$ and shows that

$$g_0(q(1+T)^{-1} - 1) = f(T)u(T)$$

in $\mathbb{Z}^\wedge_p[\![T]\!]$ for $u(T)$ a unit power series and $f(x)$ the characteristic polynomial of $\epsilon_1 X$, where X is the inverse limit of X_n and $X_n \cong A_n$ is the p–Sylow subgroup of the class group of $\mathbb{Q}(\mu_{p^n})$. Greenberg [18] denotes X as X_K, $\epsilon_1 X$ as $X_K^{[1]}$, and defines $V^{[1]} = \epsilon_1 X \otimes_{\mathbb{Z}^\wedge_p} \Omega_p$ where $\Omega_p = \overline{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$. The characteristic polynomial of $\epsilon_1 X$ and $V^{[1]}$ are therefore equal, and Greenberg [18, Corollary 1]

shows that $V^{[1]} = 0$. Thus, $f(x) = 1$ and we conclude that $g_0(x) = 1$. (In fact, $\epsilon_1 X = 0$ and $\epsilon_1 X_n = 0$ for all n as can be seen from [38, 6.16, 13.22] and Nakayama's lemma.) □

We can also compute $H^0_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k))$ using Proposition 3.8, and for $k = m(p-1) + 1$ we get

$$H^0_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k)) = H^0_{\text{ét}}\big(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(-m(p-1))\big)$$
$$= \mu_{p^i}^{\otimes(1-k)}(\overline{\mathbb{Q}}) \cong \mathbb{Z}_p^\wedge/(mp),$$

where $mp = p^i r$ for r relatively prime to p, or more concisely, $p^i = |1/(mp)|_p$. The following proposition is now immediate.

Proposition 4.5 $H^1_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k)) = 0$ for $m \neq 0$ and $k = m(p-1) + 1$.

Combining the previous proposition with the Tate–Poitou sequence (4.2), the proof of Lemma 3 is now clear. For Lemma 4, we need the following K–theory computation of Bökstedt and Madsen [10] and Hesselholt and Madsen [20].

Theorem 4.6   (Hesselholt and Madsen [20, Theorem D], Bökstedt and Madsen [10, 0.7])   *For $m > 0$,*

$$\pi_{2m(p-1)}(K(\mathbb{Q}_p^\wedge)_p^\wedge) \cong \mathbb{Z}_p^\wedge/(mp).$$

Proof of Lemma 4   Let $k = m(p-1) + 1$. By the previous theorem and (3.9), we have

$$\#\big(H^2_{\text{ét}}(\mathbb{Q}_p^\wedge; \mathbb{Z}_p^\wedge(k))\big) = \left|\frac{1}{mp}\right|_p$$

and by Proposition 3.8, we have

$$\#\big((H^0_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k)))^*\big) = \#\big(H^0_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k))\big) = \left|\frac{1}{mp}\right|_p.$$

Because the map

$$H^2_{\text{ét}}(\mathbb{Q}_p^\wedge; \mathbb{Z}_p^\wedge(k)) \to \big(H^0_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k))\big)^*$$

in the Tate–Poitou sequence (4.2) is surjective and the groups are the same finite cardinality, it must therefore also be injective. The map

$$\big(H^1_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}/p^\infty(1-k))\big)^* \to H^2_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}_p^\wedge(k))$$

is therefore surjective, and Proposition 4.5 then shows that $H^2_{\text{ét}}(\mathbb{Z}[1/p]; \mathbb{Z}_p^\wedge(k))$ is zero. □

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