Degenerations of quadratic differentials on $\mathbb{CP}^1$

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We describe the connected components of the complement of a natural "diagonal" of real codimension 1 in a stratum of quadratic differentials on $\mathbb{CP}^1$. We establish a natural bijection between the set of these connected components and the set of generic configurations that appear on such "flat spheres". We also prove that the stratum has only one topological end. Finally, we elaborate a necessary toolkit destined to evaluation of the Siegel–Veech constants.

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1 Introduction

The article deals with families of flat metric on surfaces of genus zero, where the flat metrics are assumed to have conical singularities, $\mathbb{Z}/2\mathbb{Z}$ linear holonomy and a fixed vertical direction. The moduli space of such metrics is isomorphic to the moduli space of meromorphic quadratic differential on $\mathbb{CP}^1$ with at most simple poles and is naturally stratified by the number of poles and by the orders of zeros of a quadratic differential.

Any stratum is non compact and a neighborhood of its boundary consists of flat surfaces that admit saddle connections of small length. The structure of the neighborhood of the boundary is also related to counting problems in a generic surface of the stratum (the "Siegel–Veech constants", see Eskin, Masur and Zorich [4] for the case of Abelian differentials).

When the length of a saddle connection tends to zero, some other saddle connections might also be forced to shrink. In the case of an Abelian differential this corresponds to homologous saddle connections. In the general case of quadratic differentials, the corresponding collections of saddle connections on a flat surface are said to be *ĥomologous*[1] (pronounced "hat-homologous"). Configurations associated to collections of ĥomologous saddle connections have been described for general strata by Masur

[1] The corresponding cycles are in fact homologous on the canonical double cover of S, usually denoted as $\widehat{S}$, see Section 1.2.

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and Zorich in [11] and more specifically for genus zero and in hyperelliptic connected components by the author in [1].

Usually, the study of the structure of the neighborhood of the boundary is restricted to a *thick part*, where all short saddle connections are pairwise $\hat{\text{h}}$omologous (see Masur and Smillie [9], and also [4; 11]). Following this idea, we will consider the complement of the codimension 1 subset Δ of flat surfaces that admit a pair of saddle connections that are both of minimal length, but which are not $\hat{\text{h}}$omologous.

For a flat surface in the complement of Δ, we can define the configuration of the maximal collection of $\hat{\text{h}}$omologous saddle connections that contains the smallest saddle connection of the surface. This defines a locally constant map outside Δ (see Section 5 for more details).

We will prove the following result.

Main Theorem *Let $\mathcal{Q}_1(k_1,\dots,k_r)$ be a stratum of quadratic differentials on $\mathbb{CP}^1$ with at most simple poles. There is a natural bijection between the configurations of $\hat{\text{h}}$omologous saddle connections existing in that stratum and the connected components of $\mathcal{Q}_1(k_1,\dots,k_r)\backslash\Delta$.*

We will call the connected components of $\mathcal{Q}_1(k_1,\dots,k_r)\backslash\Delta$ the *configuration domains* of the stratum. These configuration domains might be interesting to the extend that they are “almost” manifolds in the sense of the following corollary.

Corollary 1.1 *Let $\mathcal{D}$ be a configuration domain of a stratum of quadratic differentials on $\mathbb{CP}^1$. If $\mathcal{D}$ admits orbifoldic points, then the corresponding configuration is “symmetric” and the locus of such orbifoldic points are unions of copies (or coverings) of submanifolds of smaller strata.*

Restricting ourselves to the neighborhood of the boundary, we show the following proposition.

Proposition 1.2 *Let $\mathcal{D}$ be a configuration domain of a stratum of quadratic differentials on $\mathbb{CP}^1$. Then $\mathcal{D}$ has only one topological end.*

Corollary 1.3 *Any stratum of quadratic differentials on $\mathbb{CP}^1$ has only one topological end.*

Corollary 1.1 and Corollary 1.3 will be stated later as Corollary 5.4 and Corollary 5.5.

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1.1 Basic definitions

Here we first review standard facts about moduli spaces of quadratic differentials. We refer to Hubbard and Masur [5], Masur [8] and Veech [12] for proofs and details, and to Masur–Tabachnikov [10] or Zorich [15] for general surveys.

Let S be a compact Riemann surface of genus g. A quadratic differential q on S is locally given by $q(z) = \phi(z)dz^2$, for (U,z) a local chart with ϕ a meromorphic function with at most simple poles. We define the poles and zeroes of q in a local chart to be the poles and zeroes of the corresponding meromorphic function ϕ. It is easy to check that they do not depend on the choice of the local chart. Slightly abusing notations, a marked point on the surface (resp. a pole) will be referred to as a zero of order 0 (resp. a zero of order -1). An Abelian differential on S is a holomorphic 1–form.

Outside its poles and zeros, q is locally the square of an Abelian differential. Integrating this 1–form gives a natural atlas such that the transition functions are of the kind $z \mapsto \pm z + c$. Thus S inherits a flat metric with singularities, where a zero of order $k \geq -1$ becomes a conical singularity of angle $(k+2)\pi$. The flat metric has trivial holonomy if and only if q is globally the square of any Abelian differential. If not, then the holonomy is $\mathbb{Z}/2\mathbb{Z}$ and (S,q) is sometimes called a *half-translation* surface since transition surfaces are either half-turns, or translations. In order to simplify the notation, we will usually denote by S a surface with a flat structure.

We can associate to a quadratic differential the set with multiplicities $\{k_1,\dots,k_r\}$ of orders of its poles and zeros. The Gauss–Bonnet formula asserts that $\sum_i k_i = 4g-4$. Conversely, if we fix a collection $\{k_1,\dots,k_r\}$ of integers, greater than or equal to -1 satisfying the previous equality, we denote by $\mathcal{Q}(k_1,\dots,k_r)$ the (possibly empty) moduli space of quadratic differential which are not globally squares of Abelian differential, and which have $\{k_1,\dots,k_r\}$ as orders of poles and zeros. It is well known that $\mathcal{Q}(k_1,\dots,k_r)$ is a complex analytic orbifold, which is usually called a *stratum* of the moduli space of quadratic differentials on a Riemann surface of genus g. We usually restrict ourselves to the subspace $\mathcal{Q}_1(k_1,\dots,k_r)$ of area one surfaces, where the area is given by the flat metric. In a similar way, we denote by $\mathcal{H}_1(n_1,\dots,n_s)$ the moduli space of Abelian differentials of area 1 having zeroes of degree $\{n_1,\dots,n_s\}$, where $n_i \geq 0$ and $\sum_{i=1}^{s} n_i = 2g-2$.

There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{Q}(k_1,\dots,k_r)$ that preserve its stratification: let $(U_i,\phi_i)_{i\in I}$ is a atlas of flat coordinates of S, with U_i open subset of S and $\phi_i(U_i)\subset\mathbb{R}^2$. An atlas of $A.S$ is given by $(U_i, A\circ\phi_i)_{i\in I}$. The action of the diagonal subgroup of $\mathrm{SL}_2(\mathbb{R})$ is called the Teichmüller geodesic flow. In order to specify notations, we denote by g_t and r_θ the following matrices of $\mathrm{SL}_2(\mathbb{R})$:

$$g_t=\begin{bmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}}\end{bmatrix} \qquad r_\theta=\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta)\end{bmatrix}.$$

A saddle connection is a geodesic segment (or geodesic loop) joining two singularities (or a singularity to itself) with no singularities in its interior. Even if q is not globally a square of an Abelian differential we can find a square root of it along the saddle connection. Integrating it along the saddle connection we get a complex number (defined up to multiplication by -1). Considered as a planar vector, this complex number represents the affine holonomy vector along the saddle connection. In particular, its euclidean length is the modulus of its holonomy vector. Note that a saddle connection persists under any small deformation of the surface.

Local coordinates for a stratum of Abelian differential are obtained by integrating the holomorphic 1–form along a basis of the relative homology $H_1(S,\{\mathrm{sing}\},\mathbb{Z})$, where $\{\mathrm{sing}\}$ denotes the set of conical singularities of S. Equivalently, this means that local coordinates are defined by the relative cohomology $H^1(S,\{\mathrm{sing}\},\mathbb{C})$.

Local coordinates in a stratum of quadratic differentials are obtained in the following way: one can naturally associate to a quadratic differential $(S,q)\in\mathcal{Q}(k_1,\dots,k_r)$ a double cover $p\colon \widehat{S}\to S$ such that p^*q is the square of an Abelian differential ω. The surface $\widehat{S}$ admits a natural involution τ, that induces on the relative cohomology $H^1(\widehat{S},\{\mathrm{sing}\},\mathbb{C})$ an involution τ^*. It decomposes $H^1(\widehat{S},\{\mathrm{sing}\},\mathbb{C})$ into an invariant subspace $H^1_+(\widehat{S},\{\mathrm{sing}\},\mathbb{C})$ and an anti-invariant subspace $H^1_-(\widehat{S},\{\mathrm{sing}\},\mathbb{C})$. One can show that the anti-invariant subspace $H^1_-(\widehat{S},\{\mathrm{sing}\},\mathbb{C})$ gives local coordinates for the stratum $\mathcal{Q}(k_1,\dots,k_r)$.

1.2 Ĥomologous saddle connections

Let $S\in\mathcal{Q}(k_1,\dots,k_r)$ be a flat surface and denote by $p\colon \widehat{S}\to S$ its canonical double cover and τ its corresponding involution. Let Σ be the set of singularities of S and $\widehat{\Sigma}=p^{-1}(\Sigma)$.

To an oriented saddle connection γ on S, we can associate γ_1 and γ_2 its preimages by p. If the relative cycles $[\gamma_1]$ and $[\gamma_2]$ in $H_1(\widehat{S},\widehat{\Sigma},\mathbb{Z})$ satisfy $[\gamma_1]=-[\gamma_2]$, then we define $[\hat{\gamma}]=[\gamma_1]$. Otherwise, we define $[\hat{\gamma}]=[\gamma_1]-[\gamma_2]$. Note that in all cases, the cycle $[\hat{\gamma}]$ is anti-invariant with respect to the involution τ.

Definition 1.4 Two saddle connections γ and γ' are $\hat{\text{h}}$omologous if $[\hat{\gamma}] = \pm[\widehat{\gamma'}]$.

Example 1.5 Consider the flat surface $S \in \mathcal{Q}(-1,-1,-1,-1)$ given in Figure 1 (a "pillowcase"), it is easy to check from the definition that γ_1 and γ_2 are $\hat{\text{h}}$omologous since the corresponding cycles for the double cover $\widehat{S}$ are homologous.

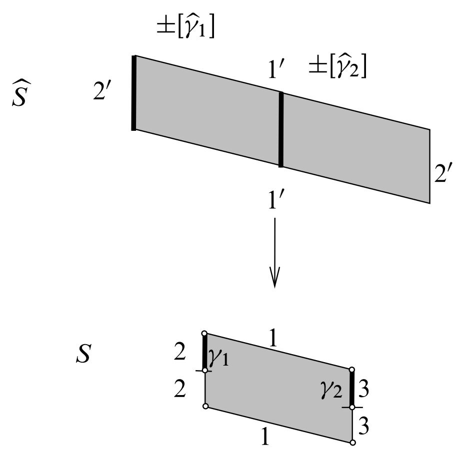

Figure 1: An unfolded flat surface S with two $\hat{\text{h}}$omologous saddle connections γ_1 and γ_2.

Example 1.6 Consider the flat surface given in Figure 2, the reader can check that the saddle connections γ_1, γ_2 and γ_3 are pairwise $\hat{\text{h}}$omologous.

The following theorem is due to Masur and Zorich [11]. It gives in particular a simple geometric criterion for deciding whether two saddle connections are $\hat{\text{h}}$omologous. We give in the appendix an alternative proof.

Theorem (H. Masur, A. Zorich) *Consider two distinct saddle connections γ, γ' on a half-translation surface. The following assertions are equivalent.*

- *The two saddle connections γ and γ' are $\hat{\text{h}}$omologous.*
- *The ratio of their length is constant under any small deformation of the surface inside the ambient stratum.*
- *They have no interior intersection and one of the connected component of $S \backslash \{\gamma \cup \gamma'\}$ has trivial linear holonomy.*

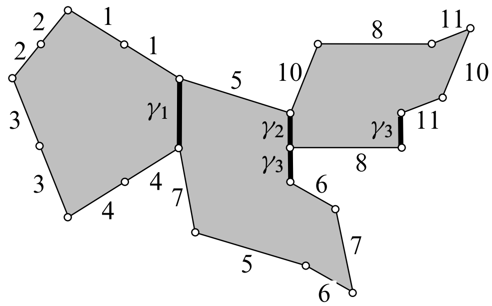

Figure 2: Unfolded flat surface with three $\hat{\text{h}}$omologous saddle connections γ_1, γ_2, and γ_3.

Furthermore, if γ *and* γ' *are* $\hat{\text{h}}$*omologous, then the ratio of their length belongs to* $\{1/2, 1, 2\}$ *and they are parallel.*

A saddle connection γ_1 will be called *simple* if they are no other saddle connections $\hat{\text{h}}$omologous to γ_1. Now we consider a set of $\hat{\text{h}}$omologous saddle connections $\gamma = \{\gamma_1, \dots, \gamma_s\}$ on a flat surface S. Slightly abusing notation, we will denote by $S\backslash\gamma$ the subset $S\backslash\big(\cup_{i=1}^{s}\gamma_i\big)$. This subset is a finite union of connected half-translation surfaces with boundary. We define a graph $\Gamma(S,\gamma)$ called the graph of connected components in the following way (see [11]): the vertices are the connected components of $S\backslash\gamma$, labelled as "$\circ$" if the corresponding surface is a cylinder, as "$+$" if it has trivial holonomy (but is not a cylinder), and as "$-$" if it has non-trivial holonomy. The edges are given by the saddle connections in γ. Each γ_i is on the boundary of one or two connected components of $S\backslash\gamma$. In the first case it becomes an edge joining the corresponding vertex to itself. In the second case, it becomes an edge joining the two corresponding vertices.

Each connected components of $S\backslash\gamma$ is a non-compact surface but can be naturally compactified (for example considering the distance induced by the flat metric on a connected component of $S\backslash\gamma$, and the corresponding completion). We denote this compactification by S_j. We warn the reader that S_j might differ from the closure of the component in the surface S: for example, if γ_i is on the boundary of just one connected component S_j of $S\backslash\gamma$, then the compactification of S_j contains two copies of γ_i in its boundary, while in the closure of S_j these two copies are identified. The boundary of each S_i is a union of saddle connections; it has one or several connected components. Each of them is homeomorphic to $\mathbb{S}^1$ and therefore the orientation of S defines a cyclic order in the set of boundary saddle connections. Each consecutive pair of saddle connections for that cyclic order defines a *boundary singularity* with an associated angle θ which is a integer multiple of π (because the boundary saddle connections are parallel). We call *order of the boundary singularity* the integer $\frac{\theta-\pi}{\pi}$. The surface with boundary S_i might have singularities in its interior. We call them *interior singularities*.

Definition 1.7 Let $\gamma = \{\gamma_1, \dots, \gamma_r\}$ be a maximal collection of $\hat{\text{h}}$omologous saddle connections on a flat surface. A *configuration* is the following combinatorial data.

- The graph $\Gamma(S, \gamma)$.
- For each vertex of this graph, a permutation of the edges adjacent to the vertex (encoding the cyclic order of the saddle connections on each connected component of the boundary of the S_i).
- For each pair of consecutive elements in that cyclic order, a nonnegative integer corresponding to the order of the boundary singularity defined by the two corresponding saddle connections.
- For each S_i, a collection of integers greater than or equal to -1 that are the orders of the interior singularities of S_i.

We refer to Masur and Zorich [11] for a more detailed definition of a configuration (see also the author's paper [1]).

1.3 Neighborhood of the boundary, thick-thin decomposition

For any compact subset K of a stratum, there exists a constant c_K such that the length of any saddle connection of any surface in K is greater than c_K. Therefore, we can define the δ–neighborhood of the boundary of the stratum to be the subset of area 1 surfaces that admit a saddle connection of length less than δ.

According to Masur and Smillie [9], one can split the δ–neigborhood of the boundary of a stratum into a *thin part* (of negligibly small measure) and a *thick part*. The thin part being for example the subset of surfaces with a pair of non$\hat{\text{h}}$omologous saddle connections of length respectively less than δ and $N\delta$, for some fixed $N \geq 1$ (the decomposition depends on the choice of N). We also refer to [4] for the case of Abelian differentials and to [11] for the case of quadratic differentials.

Let $N \geq 1$, we consider $\mathcal{Q}^N(k_1, k_2, \dots, k_r)$ the subset of flat surfaces such that, if γ_1 is the shortest saddle connection and γ_1' is another saddle connection non$\hat{\text{h}}$omologous to γ_1, then $|\gamma_1'| > N|\gamma_1|$. Similarly, we define $\mathcal{Q}_1^N(k_1, k_2, \dots, k_r)$ to be the intersection of $\mathcal{Q}^N(k_1, k_2, \dots, k_r)$ with the subset of area 1 flat surfaces.

For any surface in $\mathcal{Q}^N(k_1,k_2,\dots,k_r)$, we can define a maximal collection $\mathcal{F}$ of $\hat{\text{h}}$omologous saddle connections that contains the smallest one. This is well defined because if there exists two smallest saddle connections, they are necessary $\hat{\text{h}}$omologous. We will show in Section 5 that the associated configuration defines a locally constant map from $\mathcal{Q}_1^N(k_1,k_2,\dots,k_r)$ to the space of configurations. This leads to the following definition.

Definition 1.8 A *configuration domain* of $\mathcal{Q}_1(k_1,\dots,k_r)$ is a connected component of $\mathcal{Q}_1^N(k_1,\dots,k_r)$.

Remark 1.9 The previous definition of a configuration domain is a little more general than the one stated in the introduction that corresponds to the case $N=1$.

Definition 1.10 An *end* of a locally compact topological space W is a function

$$\epsilon\colon \{K,\ K\subset W \text{ is compact}\}\to\{X,\ X\subset W\}$$

such that:

- $\epsilon(K)$ is a (unbounded) component of $W\backslash K$ for each K
- if $K\subset L$, then $\epsilon(L)\subset\epsilon(K)$.

Proposition *If W is σ–compact, then the number of ends of W is the maximal number of unbounded components of $W\backslash K$, for K compact, when the number is bounded.*

We refer to the book of Hughes and Ranicki [6] for more details on the ends of a space.

1.4 Example on the moduli space of flat torus

If T is a flat torus (ie a Riemann surface of genus one with an Abelian differential ω), then, up to rescaling ω, we can assume that the holonomy vector of the shortest geodesic is 1. Then, choosing a second smallest non horizontal geodesic with a good choice of its orientation, this defines a complex number $z=x+iy$, with $y>0$, $-1/2\le x\le 1/2$ and $|z|\ge 1$. The corresponding domain $\mathcal{D}$ in $\mathbb{C}$ is a fundamental domain of $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$.

It is well know that this defines a map from the moduli space of flat torus with trivial holonomy (ie $\mathcal{H}(\varnothing)$), to $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ which is a bundle, with $\mathbb{C}^*$ as fiber. Orbifoldic points of $\mathcal{H}(\varnothing)$ are over the complex number $z_1=i$ and $z_2=\frac{1+i\sqrt{3}}{2}$. They correspond to Abelian differential on torus obtained by identifying the opposite sides of a square, or a regular hexagon.

Now with this representation, $\mathcal{H}^N(\varnothing)$ is obtained by restricting ourselves to the subdomain $\mathcal{D}^N = \mathcal{D} \cap \{z, |z| > N\}$ (see Figure 3). This subdomain contains neither z_1 nor z_2, so $\mathcal{H}^N(\varnothing)$ is a manifold. In the extreme case $N = 1$, the codimension one subset Δ is an arc joining z_1 to z_2.

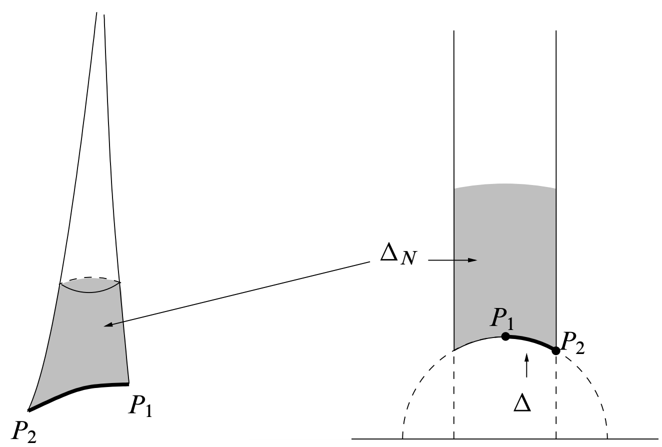

Figure 3: Configuration domain in $\mathcal{H}(\varnothing)$.

1.5 Reader's guide

Now we sketch the proof of the Main Theorem.

(1) We first prove the theorem for the case of configuration domains defined by a simple saddle connection (we will refer to these configuration domains as *simple*). We will explain how we can shrink a simple saddle connection, when its length is small enough (therefore, describe the structure of the stratum in a neighborhood of an adjacent one). This is done in Section 4.
There is one easy case, when the shrinking process is done by local and canonical surgeries. The other case involves some non-local surgeries (hole transport) that depend on a choice of a path. We will have to describe the dependence of the choice of the path. More details on these surgeries appear in Section 3.

(2) The list of configurations was established by the author in [1]. The second step of the proof is to consider each configuration and to show that the subset of surface associated to this configuration is connected. This will be done in Section 5 and will use the "simple case".

2 Families of quadratic differentials defined by an involution

Consider a polygon whose sides come by pairs, and such that, for each pair, the corresponding sides are parallel and have the same length. Then identifying these pair of sides by appropriate isometries, this gives a flat surface. In this section we show that any flat surface can arise from such a polygon and give an explicit construction. We end by a technical lemma that will be one of the key arguments of Theorem 4.2.

The construction presented in this section is a natural generalization for the case of quadratic differentials of the well known *zippered rectangle construction*, due to Veech (see [12]). This idea was developed later by the author and Lanneau (see [2]).

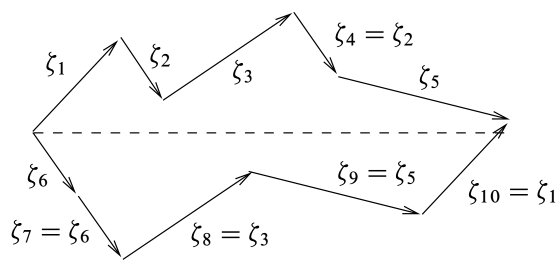

Figure 4: Flat surface unfolded into a polygon.

2.1 Constructions of a flat surface

Let σ be an involution of the set $\{1,\dots,l+m\}$, without fixed points.

We denote by $Q_{\sigma,l}$ the set of $\zeta=(\zeta_1,\dots,\zeta_{l+m})\in\mathbb{C}^{l+m}$ such that:

(1) $\forall i \quad \zeta_i=\zeta_{\sigma(i)}$

(2) $\forall i \quad Re(\zeta_i)>0$

(3) $\forall 1\le i\le l-1 \quad \mathrm{Im}(\sum_{k\le i}\zeta_k)>0$

(4) $\forall 1\le j\le m-1 \quad \mathrm{Im}(\sum_{1\le k\le j}\zeta_{l+k})<0$

(5) $\sum_{k\le l}\zeta_k=\sum_{1\le k\le m}\zeta_{l+k}$.

Now we will construct a map ZR from $Q_{\sigma,l}$ to the moduli space of quadratic differentials. Slightly abusing conventional terminology, we will call a surface in $ZR(Q_{\sigma,l})$ a *suspension* over (σ,l), and a vector in $Q_{\sigma,l}$ is then a *suspension data*.

Note that $Q_{\sigma,l}$ might be empty for some σ. Furthermore, since $Q_{\sigma,l}$ is convex, the connected component of the stratum is uniquely determined by (σ,l). This is discussed in detail in [2] .

Easy case Now we consider a broken line L_1 whose edge number i $(1 \leq i \leq l)$ is represented by the complex number ζ_i. Then we consider a second broken line L_2 which starts from the same point, and whose edge number j $(1 \leq j \leq m)$ is represented by ζ_{l+j}. The last condition implies that these two lines also end at the same point. If they have *no other intersection points*, then they form a polygon (see Figure 4). The sides of the polygon, enumerated by indices of the corresponding complex number, naturally come by pairs according to the involution σ. Gluing these pair of sides by isometries respecting the natural orientation of the polygon, this construction defines a flat surface which have trivial or non-trivial holonomy.

For this case, we will say that the suspension data defines a *suitable* polygon.

First return map on a horizontal segment Let S be a flat surface and X be a horizontal segment with a choice of a positive vertical direction (or equivalently, a choice of left and right ends). We consider the first return map $T_1\colon X \to X$ for geodesics starting from X in the positive direction (with speed one). Any such geodesic which is infinite will intersect X again. Therefore, the map T_1 is well defined outside a finite number of points that correspond to vertical geodesics that stop at a singularity before intersecting the interval X again. This set $X \backslash \{\text{sing}\}$ is a finite union $X_1, \dots, X_l$ of open intervals and the restriction of T_1 on each X_i is of the kind $x \mapsto \pm x + c_i$. For each i, the first return time for the vertical geodesics starting from X_i (in the positive direction) is constant. Similarly, we define T_2 to be the first return map for geodesics in the negative direction and denote by $X_{l+1}, \dots, X_{l+m}$ the corresponding intervals. Remark that for $i \leq l$ (resp. $i > l$), $T_1(X_i) = X_j$ (resp. $T_2(X_i) = X_j$) for some $1 \leq j \leq l+m$. Therefore, (T_1, T_2) induce a permutation σ_X of $\{1, l+m\}$, and it is easy to check that σ_X is an involution without fixed points. When S is a translation surface, $T_2 = T_1^{-1}$ and T_1 is called an *interval exchange transformation*.

Note that the pair (T_1, T_2) can also be seen as a particular case of a *linear involution*, which was introduced by Danthony and Nogueira [3] in order to encode the first return map of a measured foliation on a transverse segment. See also [2].

If $S \in ZR(Q_{\sigma,l})$, constructed as previously, we choose X to be the horizontal line whose left end is the starting point of the broken lines, and of length $\mathrm{Re}(\sum_{k \leq l} \zeta_k)$. Then it is easy to check that $\sigma_X = \sigma$.

Veech zippered rectangle construction The broken lines L_1 and L_2 might intersect at other points (see Figure 5).

However, we can still define a flat surface by using an analogous construction as the well known zippered rectangles construction due to Veech. We give a description of this construction and refer to Veech [12] and Yoccoz [14] for the case of Abelian differentials. This construction is very similar to the usual one, although its precise description is quite technical. Still, for completeness, we give an equivalent but rather implicit formulation.

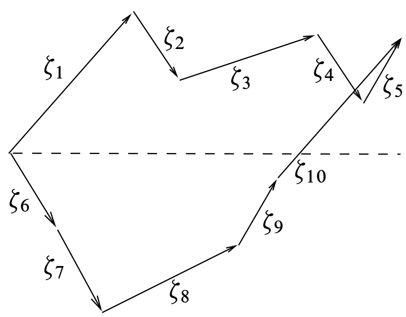

Figure 5: Suspension data that does not give a "suitable" polygon.

We first consider the previous case when L_1 and L_2 define an suitable polygon. For each pair of interval $X_i, X_{\sigma(i)}$ on X, the return time $h_i = h_{\sigma(i)}$ for the corresponding geodesics starting from $x \in X_i$ and returning in $y \in X_{\sigma(i)}$ is constant. This value depends only on (σ, l) and on the imaginary part of ζ. For each pair $\alpha = \{i, \sigma(i)\}$ there is a natural embedding of the open rectangle $R_\alpha = (0, \mathrm{Re}(\zeta_i)) \times (0, h_i)$ into the flat surface S (see Figure 6). For each R_α, we glue a horizontal side to X_i and the other to $X_{\sigma(i)}$. The surface S is then obtained after suitable identifications of the vertical sides of the the rectangles $\{R_\alpha\}_\alpha$. These vertical identifications only depend on (σ, l) and on the imaginary part of ζ.

For the general case, we construct the rectangles $\{R_\alpha\}_\alpha$ by using the same formulas. Identifications for the horizontal sides are straightforward. Identifications for the vertical sides do not depends on the horizontal parameters, and will be the same as for a suspension data ζ' that have the same imaginary part as ζ, but which correspond to a suitable polygon. This will be well defined after the following lemma.

Lemma 2.1 *Let ζ be a collection of complex numbers in $Q_{\sigma,l}$ then there exists $\zeta' \in Q_{\sigma,l}$ with the same imaginary part as ζ, that defines a suitable polygon.*

Proof We can assume that $\sum_{k=1}^{l} \mathrm{Im}(\zeta_k) > 0$ (the negative case is analogous and there is nothing to prove when the sum is zero). If we find a suspension data ζ' with the same imaginary part as ζ, and such that $\mathrm{Re}(\zeta'_{l+m}) < \mathrm{Re}(\zeta'_l) + \varepsilon$, for ε small enough. Then such suspension data defines a suitable polygon.

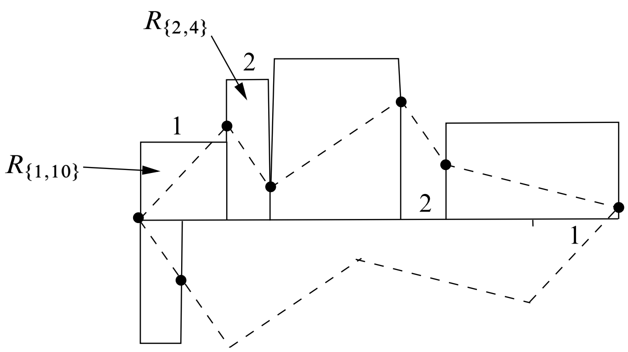

Figure 6: Zippered rectangle construction, for the case the flat surface of Figure 4.

It is clear that $\sigma(l+m)\neq l$ otherwise there would be no possible suspension data. If $\sigma(l+m)<l$, then we can shorten the real part of ζ_{l+m} and of $\zeta_{\sigma(l+m)}$, keeping conditions (1)—(5) satisfied, and get a suspension data ζ' with the same imaginary part as ζ, and such that $\mathrm{Re}(\zeta'_{l+m})$ is less than $\mathrm{Re}(\zeta'_l)$. This last condition implies that ζ' defines a suitable polygon.

Similarly, if $\sigma(l)>l$, then one can freely increase the real part of ζ_l and $\zeta_{\sigma(l)}$, keeping conditions (1)—(5) satisfied and get a suspension data ζ' with the same imaginary part as ζ, and such that ζ' defines a suitable polygon.

Now we assume that $\sigma(l+m)>l$. If there exists $i,\sigma(i)>l$, such that $\{i,\sigma(i)\}\neq\{l+m,\sigma(l+m)\}$, then we define ζ' by decreasing arbitrarily the real part of the corresponding $\zeta_{l+m},\zeta_{\sigma(l+m)}$, and increasing the real parts of $\zeta_i,\zeta_{\sigma(i)}$ such that the sum $\sum_{l<k\leq l+m}\zeta_k$ is constant. More precisely:

$$\begin{aligned}
\mathrm{Re}(\zeta'_{l+m}) &= \mathrm{Re}(\zeta'_{\sigma(l+m)}) = x\\
\mathrm{Re}(\zeta'_i) &= \mathrm{Re}(\zeta'_{\sigma(i)}) = \mathrm{Re}(\zeta_i)+\mathrm{Re}(\zeta_{l+m})-x\\
\mathrm{Re}(\zeta'_k) &= \mathrm{Re}(\zeta_k) \text{ for all } k\notin\{i,\sigma(i),l+m,\sigma(l+m)\}\\
\mathrm{Im}(\zeta'_k) &= \mathrm{Im}(\zeta_k) \text{ for all } k.
\end{aligned}$$

Then ζ' satisfy condition (1)—(5) and defines a suitable polygon for instance for $x<\mathrm{Re}(\zeta_l)$.

The last remaining case corresponds to when $\{l+m,\sigma(l+m)\}$ is the only pair $\{k,\sigma(k)\}$ such that $k,\sigma(k)>l$, and when $\sigma(l)<l$. There exists $i_0,\sigma(i_0)<l$, such that $\{i_0,\sigma(i_0)\}\neq\{l,\sigma(l)\}$ otherwise condition (5) implies that $\zeta_l=\zeta_{l+m}$, and ζ is

not a suspension data. Now for each pair $\{i, \sigma(i)\}$, with $i, \sigma(i) < l$ and different from $\{l, \sigma(l)\}$ we can shorten arbitrarily the real part of the corresponding $\zeta_i, \zeta_{\sigma(i)}$, and increase the real parts of $\zeta_l, \zeta_{\sigma(l)}$ such that the sum $\sum_{k\leq l} \zeta_k$ is constant, in a similar way as previously. If we do this operation for each pair $i, \sigma(i) < l$, then we get a new suspension data ζ' such that $\mathrm{Re}(\zeta'_{l+m}) < \mathrm{Re}(\zeta'_l) + \varepsilon$, for ε arbitrarily small. This gives a suitable polygon. □

2.2 The converse: construction of suspension data from a flat surface

Now we give a sufficient condition for a surface to be in some $Q_{\sigma,l}$. Note that an analogous construction for hyperelliptic flat surfaces has been done by Veech in [13].

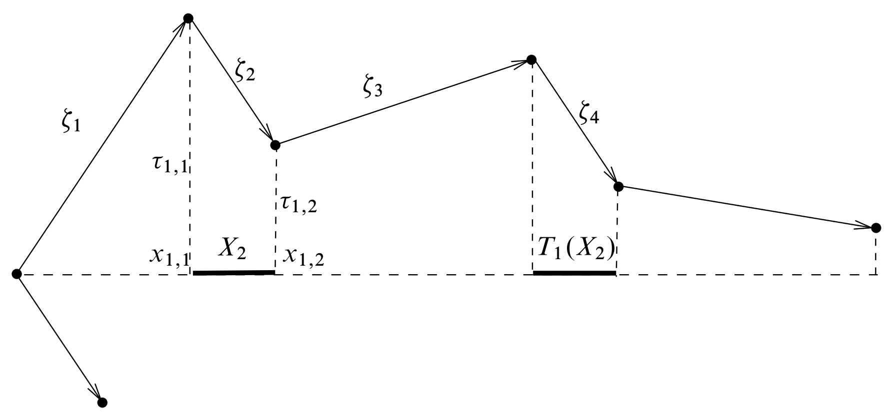

Figure 7: Construction of a polygon from a surface.

Proposition 2.2 *Let S be a flat surface with no vertical saddle connection. There exists an involution σ and an integer l such that $S \in ZR(Q_{\sigma,l})$.*

Proof Let X be a horizontal segment whose left end is a singularity. Up to cutting X on the right, we can assume that the vertical geodesic starting from its right end hits a singularity before meeting X again.

Let $x_{1,1} < \cdots < x_{1,l-1}$ be the points of discontinuity of T_1 and $(x_{1,0}, x_{1,l})$ be the endpoints of X. For each positive k, there exists $\tau_{1,k} > 0$ such that the vertical geodesic starting from $x_{1,k}$ in the positive direction stops at a singularity at time $\tau_{1,k}$ (here $\tau_{1,0} = 0$, since by convention $x_{1,0}$ is located at a singularity). Then for $k \geq 1$ we define $\zeta_k : (x_{1,k} - x_{1,k-1}) + i(\tau_{1,k} - \tau_{1,k-1})$. Now we perform a similar construction for geodesics that starts in the negative direction: let $x_{2,1} < \cdots < x_{2,m-1}$

be the points of discontinuity of T_2 and $(x_{2,0}, x_{2,m})$ be the extremities of X. For each $k \notin \{0, m\}$, the vertical geodesic starting from $x_{2,k}$ in the positive direction stops at a singularity at time $\tau_{2,k} < 0$ (here again $\tau_{2,0} = 0$ and $\tau_{2,l} > 0$). For $1 \le k \le m$, we define $\zeta_{k+l} : (x_{2,k} - x_{2,k-1}) + i(\tau_{2,k} - \tau_{2,k-1})$. So, we have a collection of complex numbers $\zeta_{l+1}, \dots, \zeta_{m+l}$ that defines a polygon $\mathcal{P}$.

We have always $\mathrm{Re}(\zeta_k) = \mathrm{Re}(\zeta_{\sigma_X(k)}) = |X_k|$. Let $1 \le k \le l$. If $\sigma_X(k) \le l$, then $\tau_{1,k-1} + \tau_{1,\sigma_X(k)} = \tau_{1,k} + \tau_{1,\sigma_X(k)-1} = h_k$ (with h_k the time of first return to X for the vertical geodesics starting from the subinterval X_k), otherwise there would exist a vertical saddle connection (see Figure 8). So $\mathrm{Im}(\zeta_k) = \mathrm{Im}(\zeta_{\sigma_X(k)})$. The other cases are analogous. Thus ζ is a suspension data, and $ZR(\zeta)$ is isometric to S. □

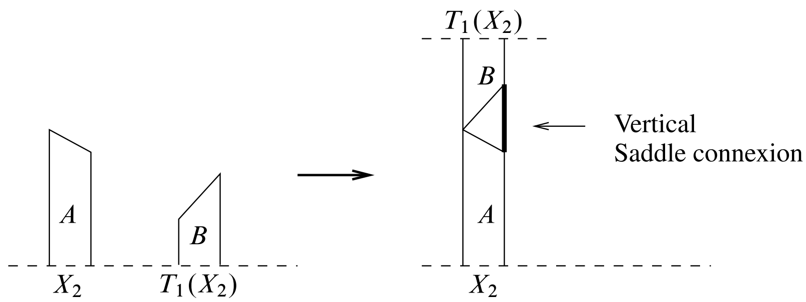

Figure 8: The complex numbers ζ_k and $\zeta_{\sigma_X(k)}$ are necessary equal.

Remark 2.3 In the previous construction, the suspension data constructed does not necessary give a suitable polygon. However, a sufficient condition to get a suitable polygon is to have $\tau_{1,l} = \min(\tau_{1,k},\ 0 < k \le l)$, were $\tau_{1,k}$ are as in the proof of the previous proposition. Up to choosing carefully a subinterval X' of X, this condition is satisfied and the construction will give a true polygon. Since for any surface, we can find a direction with no saddle connection, we can conclude that any surface can be unfolded into a polygon as in Figure 4, up to rotating that polygon.

2.3 A technical lemma

The following lemma is a technical lemma that will be needed in Section 4.2. It can be skipped in a first reading. We previously showed that a surface with no vertical saddle connection belongs to some $ZR(Q_{\sigma,l})$. Furthermore, the corresponding pair (σ, l) is completely defined by first return maps of the vertical foliation on a well chosen horizontal segment.

We define the set $Q'_{\sigma,l}$ defined in a similar way as $Q_{\sigma,l}$, but here we replace condition (2) by the following two conditions.

(2) $\forall i \notin \{1,\sigma(1)\}\quad \mathrm{Re}(\zeta_i) > 0$.

(2′) $\mathrm{Re}(\zeta_1) = \mathrm{Re}(\zeta_{\sigma(1)}) = 0$.

In other words, the first vector of the top broken line L_1 is now vertical and no other vector is vertical except the other one of the corresponding pair. Then we define in a very similar way a map ZR' from $Q'_{\sigma,l}$ to a stratum of the moduli space of quadratic differentials.

Note that the subset $Q'_{\sigma,l}$ is convex.

Lemma 2.4 *Let S be a flat surface with a unique vertical saddle connection joining two singularities P_1 and P_2. Let X be a horizontal segment whose left end is P_1, and such that the vertical geodesic starting from its left end is the unique vertical saddle connection joining P_1 to P_2. There exists (σ, l), that depends only on the first return maps on X of the vertical foliation and on the degree of P_2, such that $S \in ZR'(Q'_{\sigma,l})$.*

Proof We define as in Proposition 2.2 the $x_{i,j}$, $\tau_{i,j}$ and ζ_j, with the slight difference that now, $\tau_{1,0} > 0$. Now, because there exists only one vertical saddle connection, the same argument as before says that there exists at most one unordered pair $\{\zeta_{i_0}, \zeta_{\sigma(i_0)}\}$ such that $\zeta_{i_0} \neq \zeta_{\sigma(i_0)}$. If this pair doesn't exists, then the union of the vertical geodesics starting from X would be a strict subset of S, with boundary the unique vertical saddle connection. Therefore, we would have $P_1 = P_2$, contradicting the hypothesis.

Now we glue on the polygon $\mathcal{P}$ an Euclidean triangle of sides given by $\{\zeta_i, \zeta_{\sigma(i)}, i\tau_{1,0}\}$, and we get a new polygon. The sides of this polygon appear in pairs that are parallel and of the same length. We can therefore glue this pair and get a flat surface. By construction, we get a surface isometric to S, and so S belongs to some $ZR'(Q'_{\widetilde{\sigma},l})$. The permutation $\widetilde{\sigma}$ is easily constructed from σ as soon as we know i_0. This value is obtained by the following way: we start from the vertical saddle connection, close to the singularity P_2. Then, we turn around P_2 counterclockwise. Each half-turn is easily described in terms of the permutation σ. Then after performing $k_2 + 2$ half-turns, we must arrive again on the vertical saddle connection. This gives us the value of i_0. □

3 Hole transport

Hole transport is a surgery used by Masur and Zorich in [11] to show the existence of some configurations and especially to break an even singularity to a pair of odd ones.

It was defined along a simple path transverse to the vertical foliation. In this section, we generalize this construction to a larger class of paths and show that breaking a zero using that procedure does not depend on small perturbations of the path.

Hole transport also appears in the paper of Eskin, Masur and Zorich [4] for the computation of the Siegel–Veech constants for the moduli space of Abelian differentials. This improved surgery, and "dependence properties" that are Corollary 3.5 and Lemma 4.5 are a necessary toolkit for the future computation of the these Siegel–Veech constants for the case of quadratic differentials.

Definition 3.1 A hole is a connected component of the boundary of a flat surface given by a single saddle connection (loop). The saddle connection bounds a singularity. If this singularity has angle 3π, this hole is said to be simple.

Convention 1 We will always assume that the saddle connection defining the hole is vertical

A simple hole τ has a natural orientation given by the orientation of the underlying Riemann surface. In a neighborhood of the hole, the flat metric has trivial holonomy and therefore q is locally the square of an Abelian differential.

Convention 2 When defining the surgeries around a simple hole using flat coordinates, we will assume (unless explicit warning) that the flat coordinates come from a local square root ω of q, such that $\int_\tau \omega \in i\mathbb{R}^+$.

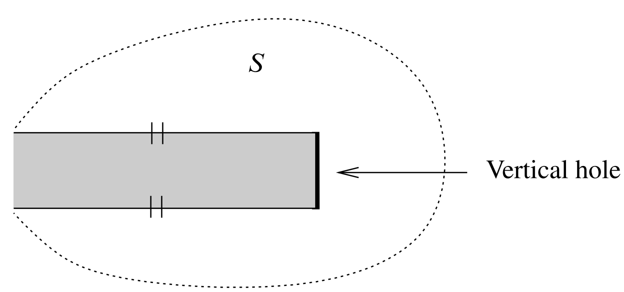

Figure 9: A hole in flat coordinates.

Remark 3.2 Under Convention 2, we may speak of the *left* or the *right* direction in a neighborhood of a simple hole. Note that there exists two horizontal geodesics starting from the singularity of and going to the right, and only one starting from the singularity and going to the left.

3.1 Parallelogram constructions

We first describe the three basic surgeries on the surface that allow us to transport a simple hole along a segment (see Figure 10). Consider a simple hole τ and chose flat coordinates in a neighborhood of the hole that satisfy Convention 2. We consider a vector v such that $\mathrm{Re}(dz(v)) > 0$ (ie the vector v goes "to the right" in our flat coordinates). Consider the domain Ω obtained as the union of geodesics of length $|v|$, starting at a point of τ with direction v. When Ω is an embedded parallelogram, we can remove it and glue together by translation the two sides parallel to v. Here we have transported the simple hole by the vector v. Note that the area changes under this construction.

When $\mathrm{Re}(dz(v)) < 0$, this construction (removing a parallelogram) cannot work. The singularity is the unique point of the boundary that can be the starting point of a geodesic of direction v. Now from the corresponding geodesic, we perform the reverse construction with respect to the previous one: we cut the surface along a segment of length v and paste in a parallelogram. By means of this construction we transport the hole along the vector v.

When $\mathrm{Re}(dz(v)) = 0$, we consider a geodesic segment of direction v starting from the singularity, and cut the surface along the segment, then glue it with a shift ("Earthquake construction").

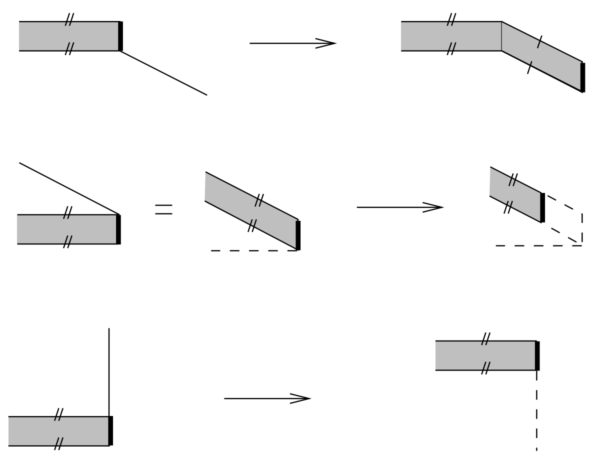

Figure 10: Parallelogram constructions.

There is an easy way to create a pair of holes in a compact flat surface: we consider a geodesic segment embedded in the surface, we cut the surface along that segment and paste in a parallelogram as in the previous construction. We get parallel holes of the same length (but with opposite orientation). Note that we can assume that the length of these holes is arbitrary small. In a similar way, we can create a pair of holes by removing a parallelogram.

3.2 Transport along a piecewise geodesic path

Now we consider a piecewise geodesic simple path $\gamma=\gamma_1\dots\gamma_n$ with edges represented by the vectors $v_1,v_2,\dots,v_n$. We assume for simplicity that none of the v_i is vertical. The spirit is to transport the hole by iterating the previous constructions. We make the hole to "follow the path" γ in the following way (under Convention 2).

- At step number i, we ask that the geodesic γ_i starts from the singularity of the hole.
- When $\mathrm{Re}(dz(v_i))>0$, we ask γ_i to be the bottom of the parallelogram Ω defined in the previous construction.

Naive iteration does not necessary preserve these conditions. The surgery can indeed disconnect the path but then we can always reconnect γ by adding a geodesic segment. If the first condition is satisfied, but not the second, we can add a surgery along a vertical segment of the size of the hole to fulfill it. We just have to check that each iteration between two consecutive segments of the initial path can be done in a finite number of steps, see Figure 11.

(1) If $\mathrm{Re}(dz(v_i))$ and $\mathrm{Re}(dz(v_{i+1}))$ have the same sign, then as soon as both transports are successively possible, our two conditions keep being fulfilled.

(2) If $\mathrm{Re}(dz(v_i))>0$ and $\mathrm{Re}(dz(v_{i+1}))<0$, and if (v_i,v_{i+1}) is positively oriented, the surgery with v_i disconnect the path, and we must add a new segment $\widetilde{v}$, but then $\mathrm{Re}(\widetilde{v})$ and $\mathrm{Re}(v_{i+1})$ are both negative, therefore, we can iterate the surgery keeping the two conditions fulfilled.

(3) If $\mathrm{Re}(dz(v_i))<0$ and $\mathrm{Re}(dz(v_{i+1}))>0$, and if (v_i,v_{i+1}) is negatively oriented, we must add a surgery along a vertical segment to fulfill the second condition.

(4) It is an easy exercise to check that for any other configuration of (v_i,v_{i+1}), the direct iteration of the elementary surgeries works.

Of course, in the process we have just described, we implicitly assumed that at each step, the condition imposed for the basic surgeries (ie the parallelogram must be imbedded in the surface) is fulfilled. But considering any compact piecewise geodesic path, the process will be well defined as soon as the hole is small enough.

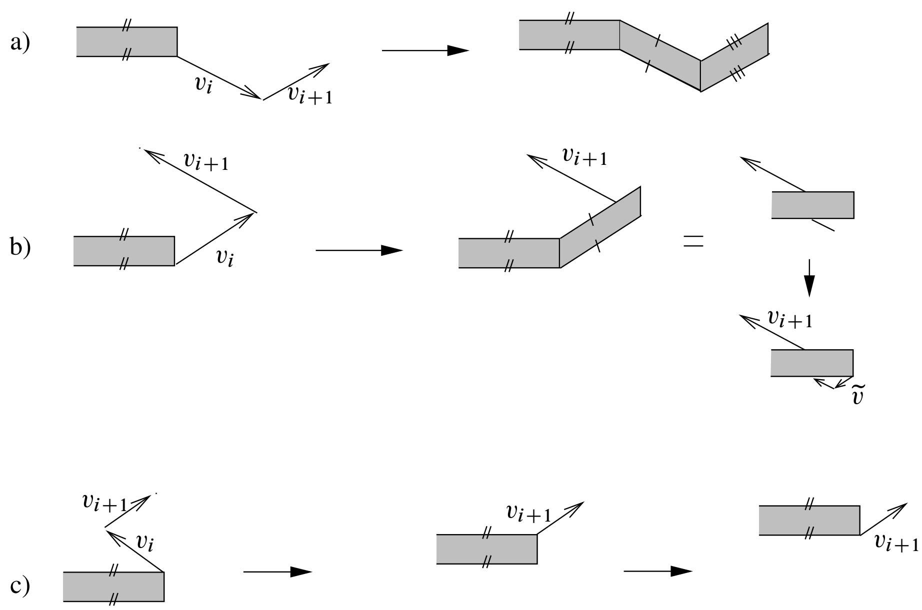

Figure 11: Hole transport along a piecewise geodesic curve.

Remark 3.3 We can also define hole transport along a piecewise geodesic path that have self intersections. Here hole transport will disconnect the path at each intersections, but we can easily reconnect it and hole transport also ends in a finite number of steps. We will not need hole transport along such paths.

3.3 Application: breaking up an even singularity

We consider a singularity P of order $k = k_1 + k_2$. When k_1 and k_2 are not both odd, there is a local surgery that continuously break this singularity into pair of singularities of order k_1 and k_2 (see Section 4.1.1). When k_1 and k_2 are both odd, this local surgery fails. Following Masur and Zorich [11] we use hole transport instead.

Consider a pair (I, II) of sectors of angle π in a small neighborhood of P, and such that the image of the first one by a rotation of $(k_2+1)\pi$ is the second sector. Now let γ be a simple broken line that starts and ends at P, and such that its first segment belongs to sector I and its last segment belongs to sector II. We require parallel transport along γ to be $\mathbb{Z}/2\mathbb{Z}$ (this has sense because k is even, so P admits a parallel vector field in its neighborhood).

Then, we create a pair of holes by cutting the first segment and pasting in a parallelogram. Denote by ε the length of these holes. One hole is attached to the singularity. The

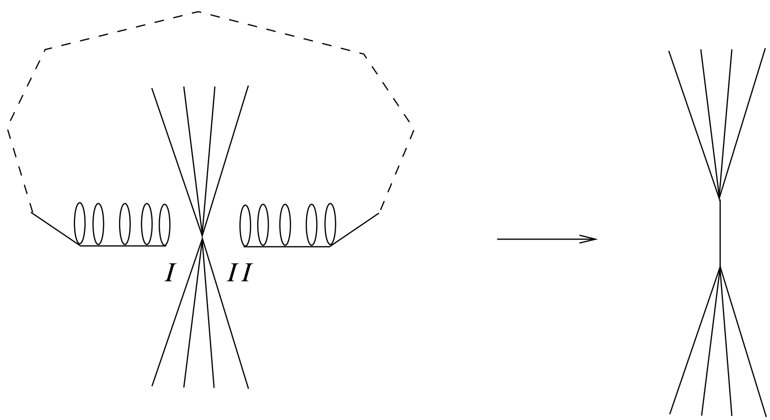

Figure 12: Breaking a singularity.

other one is a simple hole. We can transport it along γ, to the sector II. Then gluing the holes together, we get a singular surface with a pair of conical singularities that are glued together. If we desingularise the surface, we get a flat surface with a pair of singularities of order k_1 and k_2 and a vertical saddle connection of length ε. We will denote by $\Psi(S,\gamma,\varepsilon)$ this surface. The construction is continuous with respect to the variations of ε.

3.4 Dependence on small variations of the path

The previous construction might depend on the choice of the broken line. We show the following proposition.

Proposition 3.4 *Let γ and γ' be two broken lines that both start from P, sector I and end to P, sector II. Let ε be a positive real number. We assume that there exists an open subset U of S, such that the following hold.*

- *U contains $\gamma\backslash\{P\}$ and $\gamma'\backslash\{P\}$.*
- *U is homeomorphic to a disc and have no conical singularities.*
- *The surgery described in Section 3.3, with parameters (γ,ε) or (γ',ε) does not affect $\partial U\backslash P$.*

Then $\Psi(S,\gamma,\varepsilon)$ and $\Psi(S,\gamma',\varepsilon)$ are isometric.

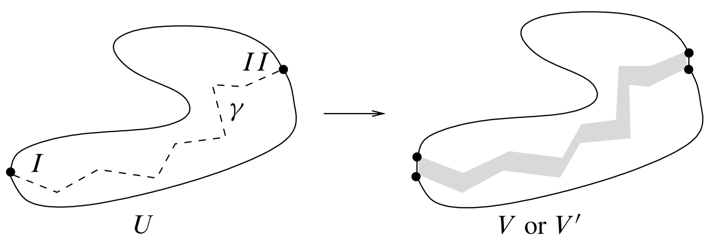

Figure 13: The boundary of U and V (or V').

Proof We denote by ∂U the boundary of the natural compactification of U (that differ from the closure of U in S, see Section 1.2). We denote by $\widetilde{P}$ and $\widetilde{P}'$ the ends of γ in ∂U (that are also the ends of γ' by assumption). We denote by V (resp. V') the flat discs obtained from U after the hole surgery along γ (resp. γ'). Our goal is to prove that V and V' are isometric.

The hole surgery along γ (resp. γ') does not change the metric in a neighborhood of $\partial U \backslash \{\widetilde{P}, \widetilde{P}'\}$. Furthermore, the fact that both γ and γ' start and end at sectors I and II correspondingly implies that V and V' are isometric in a neighborhood of their boundary. We denote by f this isometry. Surprisingly, we can find two flat discs that are isometric in a neighborhood of their boundary but not globally isometric (see Figure 14).

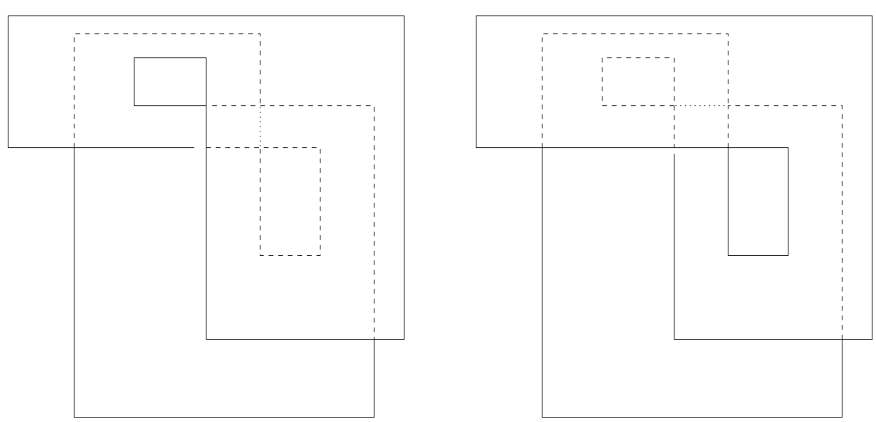

Figure 14: Immersion in $\mathbb{R}^2$ of two non isometric flat discs with isometric boundaries.

In our case, we have an additional piece of information that will make the proof possible: hole transport does not change the vertical foliation (recall that the hole is always assumed to be vertical). Therefore, for each vertical geodesics in V with endpoints $\{x,y\} \subset \partial V$, then $\{f(x), f(y)\}$ are the endpoints of a vertical geodesic of V'.

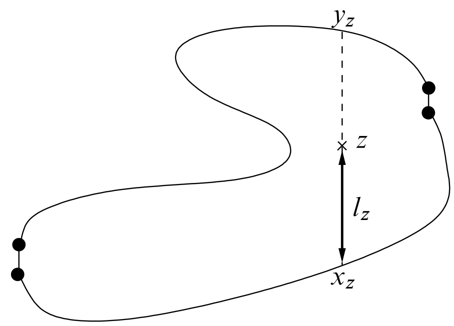

Figure 15: Parameters on a flat disc.

For each $z \in V$ we define $x_z \in \partial V$ (resp. y_z) the intersection of the vertical geodesic starting from z in the negative direction (resp. positive direction) and the boundary of V (see Figure 15). We also call l_z the length of this geodesic. We can assume that ∂V is piecewise smooth. So we can restrict ourself to the open dense subset $V_1 \subset V$ of z such that x_z and y_z are regular and nonvertical points.

Then we define Φ: $V_1 \to V'$ that send z to $\phi_{l_z}(f(x_z))$, where ϕ is the vertical geodesic flow. Because V and V' are (noncompact) translation surfaces, the length of the vertical segment $[x_z, y_z]$ is obtained by integrating the corresponding 1–form along any path between x_z and y_z. Such a path can be chosen in a neighborhood of the boundary of V. Then, the isometry f implies that this length is the same as the length of the vertical segment $[f(x_z), f(y_z)]$. Therefore Φ is well defined and coincides to f in a neighborhood of the boundary of V. This map is also smooth because $z \mapsto (x_z, l_z)$ are smooth on V_1. It's easy to check that $D\Phi(z) \equiv Id$ and that Φ continuously extends to an isometry from V to V'. □

Corollary 3.5 *Let γ' be close enough to γ and such that γ and γ' intersect the same sectors of a neighborhood of P. Then $\Psi(S,\gamma,\varepsilon)$ and $\Psi(S,\gamma',\varepsilon)$ are isomorphic for ε small enough.*

Proof If γ' is close enough to γ (and intersect the same sectors in a neighborhood of P), then there exists a open flat disk that contains γ and γ', and for ε small enough, the last condition of Proposition 3.4 is fulfilled. □

Remark 3.6 using Proposition 3.4, one can also extend hole transport along a differentiable curve.

4 Simple configuration domains

Recall the following notation: if $\mathcal{Q}(k_1,k_2,\dots,k_r)$ is a stratum of meromorphic quadratic differentials with at most simple poles, then $\mathcal{Q}_1(k_1,k_2,\dots,k_r)$ is the subset of area 1 flat surfaces in $\mathcal{Q}(k_1,k_2,\dots,k_r)$, and $\mathcal{Q}_{1,\delta}(k_1,k_2,\dots,k_r)$ is the subset of flat surfaces in $\mathcal{Q}_1(k_1,k_2,\dots,k_r)$ that have at least a saddle connection of length less than δ.

Definition 4.1 A configuration domain is said to be *simple* if the corresponding configuration is realized by a simple and non closed saddle connection.

The goal of this section is to prove the following theorem, which proves the Main Theorem for the case of simple configuration domains (but for a larger class of strata).

Theorem 4.2 *Let $\mathcal{Q}(k_1,k_2,\dots,k_r)$ be a stratum of quadratic differentials with $(k_1,k_2)\neq(-1,-1)$ and such that the stratum $\mathcal{Q}(k_1+k_2,k_3,\dots,k_r)$ is connected. Let $\mathcal{C}$ be the subset of flat surfaces S in $\mathcal{Q}^N(k_1,\dots,k_r)$ such that the shortest saddle connection of S is simple and joins a singularity of order k_1 to a distinct singularity of order k_2. For any pair $N\geq 1$ and $\delta>0$, the sets $\mathcal{C}$, $\mathcal{C}\cap\mathcal{Q}_1(k_1,k_2,\dots,k_r)$ and $\mathcal{C}\cap\mathcal{Q}_{1,\delta}(k_1,k_2,\dots,k_r)$ are non empty and connected.*

In this section we denote by P_1 and P_2 the two zeros of order k_1 and k_2 respectively and by γ the simple saddle connection between them. There are two different cases.

- When k_1 and k_2 are not both odd, then there exists a canonical way of shrinking the saddle connection γ if it is small enough. Furthermore, this surgery doesn't change the metric outside a neighborhood of γ. This is the local case.
- When k_1 and k_2 are both odd, then we still can shrink γ, to get a surface in the stratum $\mathcal{Q}(k_1+k_2,k_3,\dots,k_r)$, but this changes the metric outside a neighborhood of γ and this is not canonical. This is done by reversing the procedure of Section 3.3.

4.1 Local case

4.1.1 Breaking up a singularity Here we follow Eskin, Masur and Zorich [4; 11]. Consider a singularity P of order $k \geq 0$, and a partition $k = k_1 + k_2$ with $k_1, k_2 \geq -1$. We assume that k_1 and k_2 are not both odd. If ρ is small enough, then the set $\{x \in S, d(x, P) < \rho\}$ is a metric disc embedded in S. It is obtained by gluing $k+2$ standards Euclidean half-disks of radius ρ.

There is a well known local construction that breaks the singularity P into two singularities of order k_1 and k_2, and which is obtained by changing continuously the way of gluing the half-discs together (see Figure 16, or [4; 11]). This construction is area preserving.

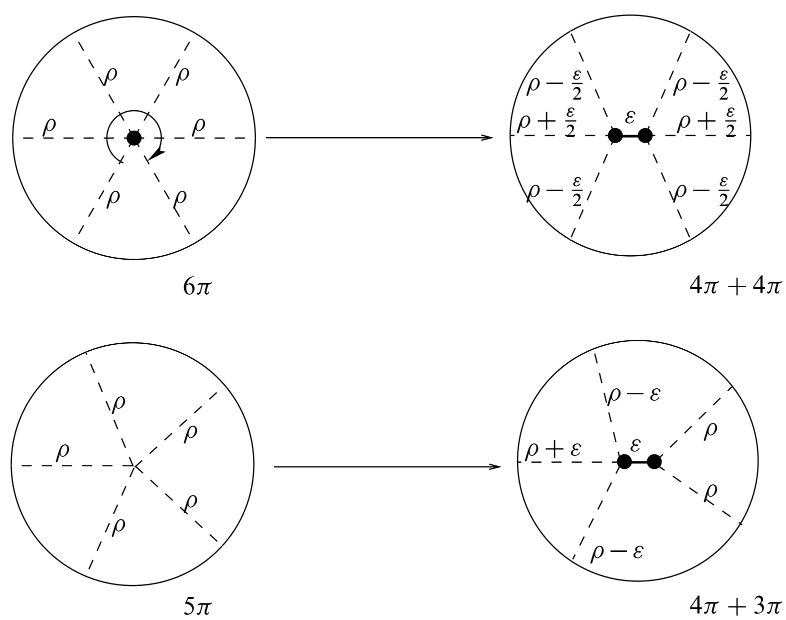

Figure 16: Breaking up a zero into two zeroes (after [4; 11]).

4.1.2 Structure of the neighborhood of the principal boundary When γ is small enough, (for example $|\gamma| \leq |\gamma'|/10$, for any other saddle connection γ'), then we can perform the reverse construction because a neighborhood of γ is precisely obtained from a collection of half-discs glued as before. This defines a canonical map $\Phi \colon V \to \mathcal{Q}(k_1 + k_2, k_3, \dots, k_r)$, where V is a subset of $\mathcal{Q}(k_1, k_2, k_3, \dots, k_r)$. We can choose $U^N \subset V$ such that $\Phi^{-1}(\{\widetilde{S}\}) \cap U^N$ is the set of surfaces such that the shrinking process leads to $\widetilde{S}$, and whose smallest saddle connection is of length smaller than $\min\left(\frac{|\tilde\gamma|}{100}, \frac{|\tilde\gamma|}{2N}\right)$ with $\tilde\gamma$ the smallest saddle connection of $\widetilde{S}$. From the proof of [4, Lemma 8.1], this map gives to U^N a structure of a topological orbifold bundle over $\mathcal{Q}(k_1+k_2,k_3,\dots,k_r)$, with the punctured disc as a fiber. By assumption, $\mathcal{Q}(k_1+k_2,k_3,\dots,k_r)$ is connected, and therefore U^N is connected, so the proof will be completed after the following three steps.

- $U^N \subset \mathcal{C}$.
- There exists $L>0$ such that $\mathcal{Q}^L(k_1,\dots,k_r)\cap\mathcal{C}\subset U^N$.
- For any $S\in\mathcal{C}$, there exists a continuous path $(S_t)_t$ in $\mathcal{C}$ that joins S to $\mathcal{Q}^L(k_1,\dots,k_r)$.

4.1.3 Proof of Theorem 4.2: local case To prove the first step, it is enough to show that U^N is a subset of $\mathcal{Q}^N(k_1,k_2,\dots,k_r)$: let S be a flat surface in U^N and let $\widetilde{S}=\Phi(S)$. We denote by γ the smallest saddle connection of S. The surgery doesn't change the surface outside a small neighborhood of the corresponding singularity of $\widetilde{S}$. If $|\tilde\gamma|$ is the length of the smallest saddle connection of $\widetilde{S}$, then S has no saddle connections of length smaller than $|\tilde\gamma|-|\gamma|$ except γ, which has length smaller than $\frac{|\tilde\gamma|}{2N}$ by construction. We have $\frac{|\tilde\gamma|-|\gamma|}{|\gamma|}=\frac{|\tilde\gamma|}{|\gamma|}-1>2N-1\geq N$, so S belongs to $\mathcal{Q}^N(k_1,k_2,\dots,k_r)$. Hence we have proved that $U^N\subset\mathcal{C}$.

To prove the second step, we remark that if $S\in\mathcal{Q}^L(k_1,\dots,k_r)\cap\mathcal{C}$, for $L\geq 10$, then the smallest saddle connection of $\Phi(S)$ is of length at least $L|\gamma|-|\gamma|$, where γ is the smallest saddle connection of S. Hence if $|\gamma|\leq\min\left(\frac{(L-1)|\gamma|}{100}, \frac{(L-1)|\gamma|}{2N}\right)$ then $S\in U^N$. So we have proved that $\mathcal{Q}^L(k_1,\dots,k_r)\cap\mathcal{C}\subset U^N$ for $L\geq\max(101,2N+1)$.

The last step is given by the following lemma.

Lemma 4.3 *Let S be a surface in $\mathcal{Q}^N(k_1,\dots,k_r)$ whose smallest saddle connection S is simple and joins a singularity of order k_1 to a singularity of order k_2, and let L be a positive number. Then we can find a continuous path in $\mathcal{Q}^N(k_1,\dots,k_r)$, that joins S to a surface whose second smallest saddle connection is at least L times greater than the smallest one.*

Proof The set $\mathcal{Q}^N(k_1,\dots,k_r)$ is open, so up to a small continuous perturbation of S, and up to changing S by $r_\theta.S$ for some suitable θ, we can assume that S has no vertical saddle connection except the smallest one.

Now we use the geodesic flow g_t on S. There is a natural bijection from the saddle connections of S to the saddle connections of $g_t.S$. The holonomy vector $v=(v_1,v_2)$

of a saddle connection becomes $v_t = (e^{-t}v_1, e^t v_2)$. This imply that the quotient of the length of a given saddle connection to the length of the smallest one increases and goes to infinity.

The set of holonomy vectors of saddle connections is discrete, and therefore, any other saddle connection of $g_t.S$ has length greater than L times the length of the smallest one, as soon as t is large enough. $\square$

Note that the previous proof is the same if we restrict ourselves to area 1 surfaces. The case when restricted to the δ–neighborhood of the boundary is also analogous, since $U^N \cap \mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$ is still a bundle over $\mathcal{Q}_1(k_1,\dots,k_r)$ with the punctured disc as a fiber.

Hence the theorem is proven when k_1 and k_2 are non both odd.

4.2 Proof of Theorem 4.2: non-local case

We first show that the two surfaces that are close enough (in a certain sense that will be specified below) to the stratum $\mathcal{Q}(k_1+k_2,k_3,\dots,k_r)$ belong to the same configuration domain. Then we show that we can always continuously reach that neighborhood.

4.2.1 Neighborhood of the principal boundary Contrary to the local case, we do not have a canonical map from a subset of $\mathcal{Q}(k_1,k_2,\dots,k_r)$ to $\mathcal{Q}(k_1+k_2,\dots,k_r)$ that gives to this subset a structure of a bundle.

Let $S \in \mathcal{Q}(k_1+k_2,\dots,k_r)$, and let ν be a path in S, we will say that ν is *admissible* if it satisfies the hypothesis of the singularity breaking procedure of Section 3.3. Let ν be an admissible closed path whose endpoint is a singularity P of degree k_1+k_2 and let $\varepsilon > 0$ be small enough for the breaking procedure. Recall that $\Psi(S,\nu,\varepsilon)$ denotes the surface in $\mathcal{Q}(k_1,k_2,\dots,k_r)$ obtained after breaking the singularity P, using the procedure of Section 3.3 along the path ν, with a vertical hole of length ε.

Proposition 4.4 *Let (S,S') be a pair of surfaces in $\mathcal{Q}(k_1+k_2,\dots,k_r)$ and ν (resp. ν') be an admissible broken line in S (resp. S'). Then $\Psi(S,\gamma,\varepsilon)$ and $\Psi(S',\gamma',\varepsilon)$ belong to the same configuration domain for any sufficiently small ε.*

Proof By assumption, $\mathcal{Q}(k_1+k_2,\dots,k_r)$ is connected, so there exists a path $(S_t)_{t\in[0,1]}$, that joins S and S'. We can find a family of broken lines γ_t of S_t such that, for ε small enough, the map $t \mapsto \Psi(S_t,\gamma_t,\varepsilon)$ is well defined and continuous for $t \in [0,1]$. The surface $\Psi(S',\gamma_1,\varepsilon)$ might differ from $\Psi(S',\gamma',\varepsilon)$ for two reasons.

- The paths γ_1 and γ', that both start from the same singularity P, might not start and end at the same sectors. In that case, we consider the path $r_\theta S'$ obtained by rotating the surface S' by an angle of θ. We find as before a family of broken lines $\gamma_{1,\theta} \in r_\theta S'$. Then, for some θ_k an integer multiple of π, we will have $r_{\theta_k} S' = S'$ and γ_{1,θ_k} that starts and ends on the same sectors than γ'.
- Even if the paths γ_1 and γ' start and end in the same sectors of the singularity P, they might be very different (for example in a different homotopy class of $S'\backslash\{\text{sing}\}$, where $\{\text{sing}\}$ denotes the set of conical singularities of S), so Proposition 3.4 does not apply. This case is solved by the following lemma, which says that the resulting surfaces are in the same configuration domain. □

Lemma 4.5 *For any surface $S \in \mathcal{Q}(k_1+k_2, k_3, \dots, k_r)$, the configuration domain that contains a surface obtained by the non-local singularity breaking construction does not depend on the choice of the admissible path, once sector I is chosen, and the hole is small enough.*

Proof We consider a surface S in $\mathcal{Q}(k_1+k_2, \dots, k_r)$ and perform the breaking procedure. We do not change the resulting configuration domain if we perform some small perturbation of S. Therefore, we can assume that S has no vertical saddle connections (this is the case for almost all surface). Now we consider an admissible path and perform the corresponding singularity breaking procedure and get a surface S_1. Then we choose a horizontal segment X_1 in sector I adjacent to the singularity k_1. Then we perform the same construction for another admissible path (and get a surface S_2) and consider a horizontal segment X_2 of the same length as before (see Figure 17).

Because the hole transport preserves the vertical foliation, the first return maps on X_1 and X_2, of the vertical flow in the two surfaces are isomorphic as soon as the hole is small enough.

Now from Lemma 2.4, there exists (σ, l) such that S_1 and S_2 belong to $ZR'(Q'_{\sigma,l})$, with parameters $\zeta_1^1, \dots, \zeta_{l+m}^1$ and $\zeta_1^2, \dots, \zeta_{l+m}^2$. Note that $\mathrm{Re}(\zeta_i^1) = \mathrm{Re}(\zeta_i^2)$, because these depends only on the first returns maps of the vertical foliation (and they coincide). The family of polygons with parameters $t\zeta_i^1 + (1-t)\zeta_i^2$ gives a path in $MZ'(Q'_{\sigma,l})$ that joins S_1 and S_2. Furthermore, the singularity breaking procedure is continuous with respect to ε. Hence, for all i, ζ_i^1 and ζ_i^2 are arbitrary close as soon as ε is small enough. Consequently, the constructed path in $MZ'(Q'_{\sigma,l})$ keeps being in a configuration domain.

□

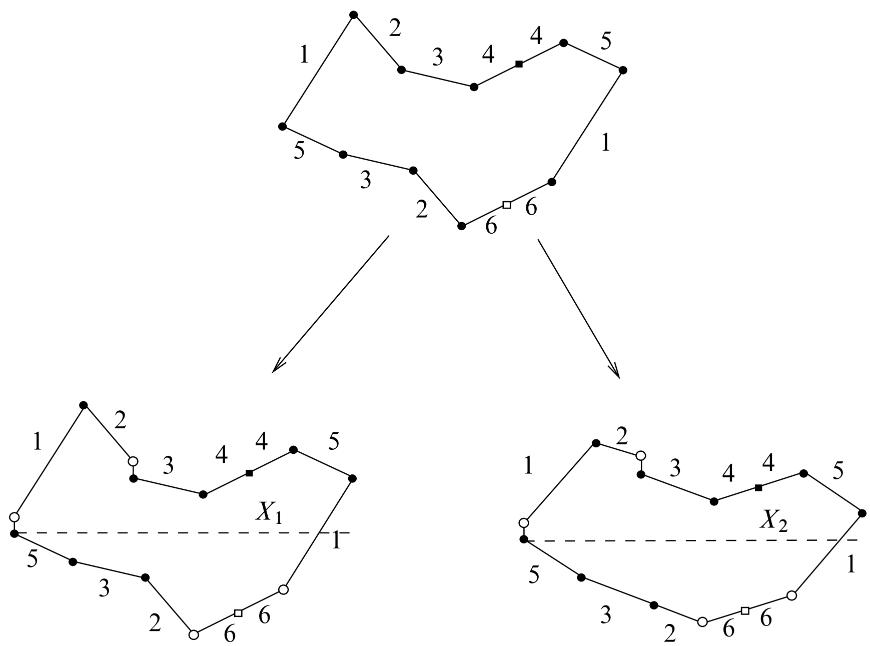

Figure 17: Breaking a singularity with two different paths.

Now for each $S \in \mathcal{Q}(k_1 + k_2, \dots, k_r)$ and each admissible path γ, we can find $\varepsilon_{S,\gamma}$ maximal such that $\Psi(S, \gamma, \varepsilon) \in \mathcal{Q}^N(k_1, \dots, k_r)$ for all $\varepsilon < \varepsilon_{S,\gamma}$. Now we consider the set

$$U^N = \bigcup_{\theta \in [0,2\pi]} \bigcup_{S,\gamma} \bigcup_{0<\varepsilon<\varepsilon_{S,\gamma}} r_\theta(\Psi(S,\gamma,\varepsilon)).$$

This set is in a connected subset of $\mathcal{Q}^N(k_1, \dots, k_r)$ from Proposition 4.4.

4.2.2 Reaching a neighborhood of the principal boundary Now we consider a surface in $\mathcal{Q}^N(k_1, \dots, k_r)$ whose unique smallest saddle connection joins a singularity of order k_1 to a singularity of order k_2. As in the local case, we can assume that its smallest saddle connection is vertical and that there are no other vertical saddle connections. Then we make use of the Teichmüller geodesic flow. This allows us to assume that the smallest saddle connection is arbitrary small compared to any other saddle connection.

We then want to contract the saddle connection using the reverse procedure of Section 3.3.

Proposition 4.6 *Let N be greater than or equal to 1. There exists $L > N$ such that $\mathcal{Q}^L(k_1,\dots,k_r)\cap\mathcal{C}\subset U^N$.*

Proof We choose L large enough such that we can find L' satisfying $2N < L'$, and $1 \ll L' \ll L$. Denote by γ the smallest saddle connection and by ε its length. We want to find a path suitable for reversing the construction of Section 3.3. When contracting γ in such way, we must insure that the surface stay in $\mathcal{Q}^N(k_1,\dots,k_r)$, by keeping a lower bound of the length of the saddle connections different from the shortest one.

Let B be the open $L'\varepsilon$–neighborhood of γ, and $\{B_i\}_{i\in\{3,\dots,r\}}$ the open $L'\varepsilon$–neighborhoods of the singularities that are not endpoints of γ. Note that each of these neighborhoods is naturally isometric to a collection of half-disk glued along their boundary. We denote by S' the closed subset of S obtained by removing to S the set $\cup_i B_i\cup B$.

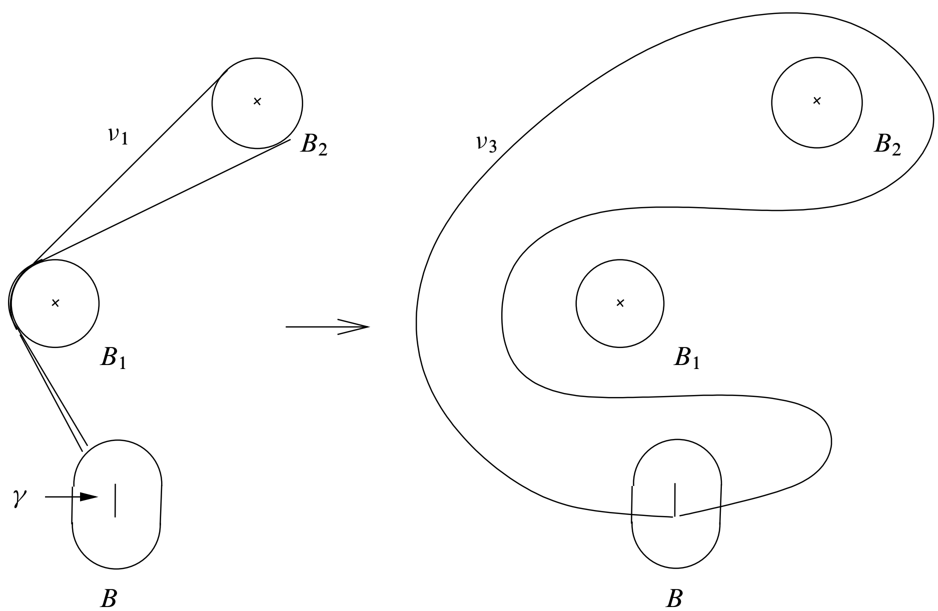

Figure 18: Constructing a suitable path.

Now we consider the set of paths of S' whose endpoints are on ∂B and with nontrivial holonomy (which makes senses in a neighborhood of ∂B), and we choose a path ν_1 of minimal length with this property. Note that, we do not change the holonomy of a path by "uncrossing" generic self intersections (see Figure 19). Therefore, we can choose our path such that, after a small perturbation, it has no self intersections.

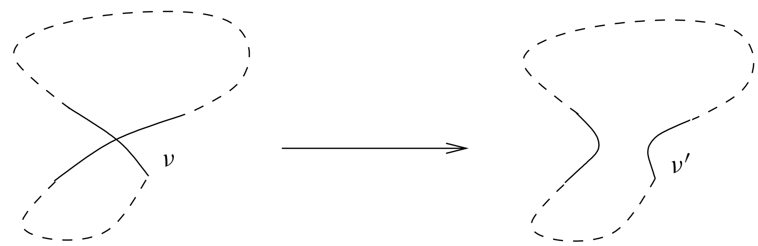

Figure 19: Uncrossing an intersection does not change the holonomy.

Now the condition $L' \ll L$ implies that we can find a path ν_2 in the same homotopy class, such that the ε–neighborhood of ν_2 is homeomorphic to a disk. Now joining carefully the endpoints of ν_2 to each sides of γ, we get a path ν_3. By construction, we can use this path to contract the saddle connection γ. The surgery doesn't touch the εN–neigborhoods of the singularities, except for the endpoints of γ, hence any saddle connection that starts from such singularity will have a length greater than $N\varepsilon$ during the shrinking process. A saddle connection starting from an endpoint a γ, and different from γ will leave B. choosing properly ν_3, then the length of such saddle connection will have a length greater than $(L'-1)\varepsilon$ during the shrinking process, and $L'-1 \geq N + (N-1) \geq N$.

Therefore, when contracting γ, there is no saddle connection except γ that is of length smaller than $N|\gamma| \leq N\varepsilon$, were ε is the initial length of the saddle connection γ. Up to rescaling the surface, we can assume that the area of the surface is constant under the deformation process. □

Now let $\mathcal{C}$ be the open subset of surfaces in $\mathcal{Q}^N(k_1,\dots,k_r)$ whose unique smallest saddle connection joins a singularity of order k_1 to a singularity of order k_2. The previous proposition shows that there exists a path from any $S \in \mathcal{C}$ to U^N, which is pathwise connected. Therefore $\mathcal{C}$ is pathwise connected and hence, connected. This also implies the connectedness of $\mathcal{C} \cap \mathcal{Q}_1(k_1,\dots,k_r)$.

Now let $\delta > 0$ and let S_1 and S_2 be two surfaces in $\mathcal{C} \cap \mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$. There exists a path $(S_t)_{t\in\{1,2\}}$ in $\mathcal{C} \cap \mathcal{Q}_1(k_1,\dots,k_r)$ that joins S_1 to S_2. We can easily deduce from $(S_t)_t$ a path in $\mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$ that join S_1 and S_2. Indeed, denote by $l(t)$ the length of the shortest saddle connection of S_t. The function l is continuous, and there exists a continuous function l' with $l'(j) = l(j)$, for $j = 1,2$, and such that $l'(t)$ is always smaller than δ. Then we apply to S_t the matrix $A(t)$ in $\mathrm{SL}_2(\mathbb{R})$ that multiply the length of this saddle connection by the factor $\lambda(t) = \frac{l'(t)}{l(t)}$ and multiply by$\lambda(t)^{-1}$

the distances in the orthogonal direction. By construction, we get a continuous path in $\mathcal{C}\cap\mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$ that joins S_1 and S_2.

Hence we have proven the theorem for the case when k_1 and k_2 are odd.

5 Configuration domains in strata of quadratics differentials on the Riemann sphere

Theorem 5.1 describes all the configurations of $\hat{\text{h}}$omologous saddle connections that exist on a given stratum of quadratic differential on $\mathbb{CP}^1$. It was proved by the author in [1]. We now show that they are in bijections with the configuration domains. In this section, we denote by γ a collection $\{\gamma_i\}$ of saddle connections.

Theorem 5.1 *Let $\mathcal{Q}(k_1,\dots,k_r)$ be a stratum of quadratic differentials on $\mathbb{CP}^1$ different from $\mathcal{Q}(-1,-1,-1,-1)$, and let γ be a maximal collection of $\hat{\text{h}}$omologous saddle connections on a generic surface in that stratum. Then the possible configurations for γ are given in the list below (see Figure 20).*

a) *Let $\{k,k'\}\subset\{k_1,\dots,k_r\}$ be an unordered pair of integers such that $(k,k')\neq(-1,-1)$. The set γ consists of a single saddle connection joining a singularity of order k to a distinct singularity of order k'.*

b) *Let (a_1,a_2) be a pair of positive integers such that $a_1+a_2=k\in\{k_1,\dots,k_r\}$ (with $k\geq 2$), and let $A_1\sqcup A_2$ be a partition of the set $\{k_1,\dots,k_r\}\backslash\{k\}$ such that $(\sum_{a\in A_i}a)+a_i\equiv 2 \mod 4$ for each i. The set γ consists of a simple saddle connection that decomposes the sphere into two 1–holed spheres S_1 and S_2, such that each S_i has interior singularities of order given by A_i, and has a single boundary singularity of order a_i.*

c) *Let $\{a_1,a_2\}\subset\{k_1,\dots,k_r\}$ be a pair of positive integers. Let $A_1\sqcup A_2$ be a partition of $\{k_1,\dots,k_r\}\backslash\{a_1,a_2\}$ such that for each i, we have $(\sum_{a\in A_i}a)+a_i\equiv 2 \mod 4$. The set γ consists of two closed saddle connections that decompose the sphere into two 1–holed spheres S_1 and S_2 and a cylinder, and such that each S_i has interior singularities of orders given by A_i and has a boundary singularity of order a_i.*

d) *Let $k\in\{k_1,\dots,k_r\}$ be a positive integer. The set γ is a pair of saddle connections of different lengths, and such that the largest one starts and ends from a singularity of order k and decompose the surface into a 1– holed sphere and a half-pillowcase, while the shortest one joins a pair of poles and is on the other end of the half pillowcase.*

When the stratum is $\mathcal{Q}(-1,-1,-1,-1)$*, there is only one configuration, which corresponds to two saddle connections that are the two boundary components of a cylinder (the surface is a "pillowcase", see Figure 1).*

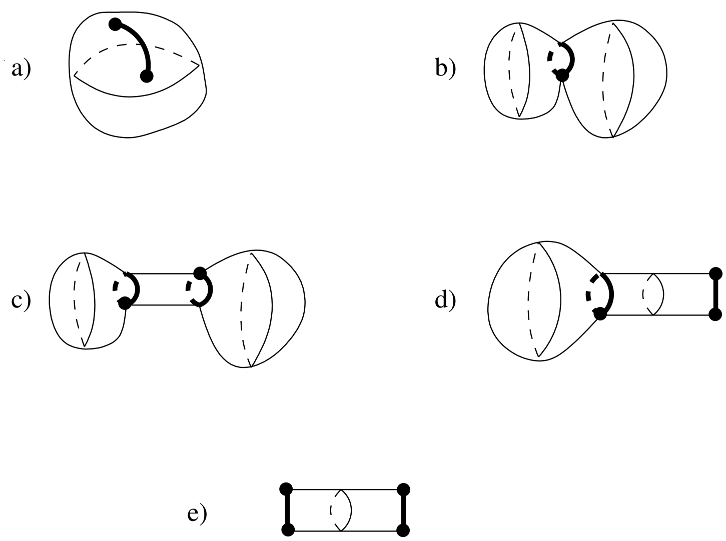

Figure 20: "Topological picture" of configurations for $\mathbb{CP}^1$.

Now let $S \in \mathcal{Q}^N(k_1,\dots,k_r)$. We can define $\mathcal{F}_S$ to be the maximal collection of ĥomologous saddle connections that contains the smallest one. We have the following lemma.

Lemma 5.2 *The configuration associated to* $\mathcal{F}_S$ *is locally constant with respect to* S*.*

Proof Any saddle connection in $\mathcal{F}_S$ persists under any small continuous deformation. This lemma is obvious as soon the number of elements of $\mathcal{F}_S$ is locally constant.

Let γ_1 be a saddle connection of minimal length. We assume that after a small perturbation S' of S, we get a bigger collection of saddle connections. That means that a new saddle connection γ_2 appears. Therefore there was another saddle connection γ_3 nonĥomologous to γ_1, of length less than or equal to $|\gamma_2/2|$ (see Figure 21). But this is impossible since it would therefore be of length less than or equal to the length of γ_1, contradicting the hypothesis. □

The following lemma (due to Kontsevich) implies that Theorem 4.2 can be used for any stratum of quadratic differentials on $\mathbb{CP}^1$ (see also Kontsevich and Zorich [7]).

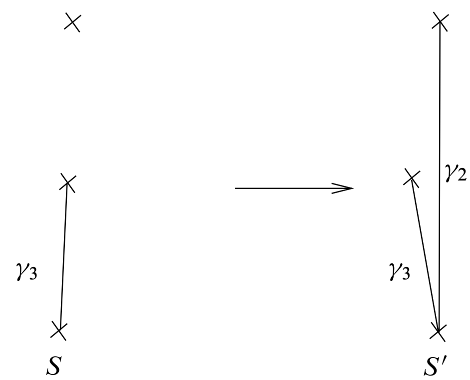

Figure 21: The configuration associated to $\mathcal{F}_S$ is locally constant.

Lemma (Kontsevich) *Any stratum of quadratic differentials on $\mathbb{CP}^1$ is non empty and connected.*

Proof There is only one complex structure on $\mathbb{CP}^1$. Therefore, we can work on the standard atlas $\mathbb{C} \cup (\mathbb{C}^* \cup \infty)$ of the Riemann sphere.

Now we remark that if we fix $(z_1, \dots, z_r) \in \mathbb{C}^r$ that are pairwise distinct, and $k_1, \dots, k_r$ some integers greater than or equal to -1, then the quadratic differential on $\mathbb{C}$, $q(z) = \prod (z - z_i)^{k_i} dz^2$, extends to a quadratic differential on $\mathbb{CP}^1$ with possibly a singularity of order $-4 - \sum_i k_i$ over the point ∞. Now two quadratic differentials on a compact Riemann surface with the same singularities are equal up to a multiplicative constant (because they differ by a holomorphic function).

Therefore, any stratum of quadratic differentials on $\mathbb{CP}^1$ is a quotient of $\mathbb{C}$ times a space of configurations of points on a sphere, which is connected. □

Main Theorem *Let $\mathcal{Q}(k_1, \dots, k_r)$ be a stratum of quadratic differentials with at most simple poles. Let N be greater than or equal to 1. There is a natural bijection between the configurations of $\hat{\text{h}}$omologous saddle connections on $\mathcal{Q}(k_1, \dots, k_r)$ described in Theorem 5.1 and the connected components of $\mathcal{Q}^N(k_1, \dots, k_r)$.*

Proof Lemma 5.2 implies that there is a well defined map Ψ from the set of connected components of $\mathcal{Q}^N(k_1, \dots, k_r)$ to the set of existing configurations for the stratum. This map is surjective because if we choose a generic surface S with a maximal collection of $\hat{\text{h}}$omologous saddle connections γ that realizes the given configuration $\mathcal{C}$, then after a small continuous perturbation of the surface, we can assume that there are no other saddle connections on S parallel to an element of γ. Then we use the Teichmüller geodesic flow to contract the elements of γ, until γ contains the smallest saddle connection of the surface. Then by construction, this surface belongs to $\Psi^{-1}(\mathcal{C})$.

Now we prove that Ψ is injective. We keep the notations of Theorem 5.1, and consider $U = \Psi^{-1}(\{\mathcal{C}\})$, for $\mathcal{C}$ any existing configuration.

- If $\mathcal{C}$ belongs to the a) case, then U is connected from Theorem 4.2 and the lemma of Kontsevich.
- If $\mathcal{C}$ belongs to the b) case, then we consider a surface S in U. Its smallest saddle connection γ_0 is closed and separates the surface in a pair (S_1', S_2') of 1–holed spheres with boundary singularities of orders a_1 and a_2 correspondingly. Now for each S_i' we decompose the boundary saddle connection of S_i' in two segments starting from the boundary singularity, and glue together these two segments, then we get a pair of closed flat spheres $S_i \in \mathcal{Q}(A_i, a_i - 1, -1)$, $i = 1, 2$. For each of the sphere, the smallest saddle connection γ_i' is simple and joins a singularity Q_i of order $(a_i - 1)$ to a newborn pole P_i, and is of length $|\gamma_0|/2$, where $|\gamma_0|$ is the length of γ_0. Let η_i be the smallest saddle connection of S_i except γ_i'.
  - If η_i intersects the interior of γ_i', then it is easy to find another saddle connection on S_i, smaller than η_i and different from γ_i'.
  - If η_i does not intersect γ_i', or intersect it in Q_i, then η_i was a saddle connection on S, hence $|\eta_i| > 2N|\gamma_i'|$.
  - If η_i intersects P_i, then we can find a saddle connection in S of length smaller than $|\eta_i| + |\gamma_0|/2$.

These remarks imply that S_i is in $\mathcal{Q}^{2N-1}(A_i, a_i - 1, -1)$ which is a subset of $\mathcal{Q}^N(A_i, a_i - 1, -1)$. Hence we have defined a map f from U to $U_1 \times U_2$, with U_i a simple configuration domain of $\mathcal{Q}^N(A_i, a_i - 1, -1)$.

Conversely, let $\{S_i\}_{i \in \{1,2\}}$ be two surfaces in $\mathcal{Q}^{2N}(A_i, a_i - 1, -1)$, such that for each S_i, the smallest saddle connection γ_i is simple and joins a pole to a singularity of order $a_i - 1$. If γ_1 and γ_2 are in the same direction and have the same length, then we can reconstruct a surface $S = f^{-1}(S_1, S_2)$ in $\mathcal{Q}(k_1, \dots, k_r)$ by cutting S_i along γ_i, and gluing together the two resulting surfaces by an appropriate isometry. The surface S belongs to $\mathcal{Q}^N(k_1, \dots, k_r)$. Note that in the reconstruction of the surface, the length of smallest saddle connection is doubled, hence we must start from $\mathcal{Q}^{2N}(A_i, a_i - 1, -1)$, and not $\mathcal{Q}^N(A_i, a_i - 1, -1)$.

Now we prove the connectedness of U: let X^1, X^2 be two flat surfaces in U. After a small perturbation and after using the geodesic flow, we get a surface S^1 (resp.

S^2) in the same connected component of U as X^1 (resp. X^2), with S^1 and S^2 in $\mathcal{Q}^{2N}(k_1,\dots,k_r)$.

There exist continuous paths $(S_{i,t})_{t\in[1,2]}\in\mathcal{Q}^{2N}(A_i,a_i-1,-1)$ such that $(S_{1,j},S_{2,j})= f(S^j)$ for $j=1,2$. The pair $(S_{1,t},S_{2,t})$ belongs to $f(U)$ if and only if their smallest saddle connections are parallel and have the same length. This condition is not necessary satisfied, but rotating and rescaling $S_{2,t}$ gives a continuous path A_t in $GL_2(\mathbb{R})$ such that $S_{1,t}$ and $A_t.S_{2,t}$ satisfy that condition. Note that we necessary have $A_2.S_{2,2}=S_{2,2}$. Therefore $f^{-1}\big(S_{1,t},A_2.S_{2,t}\big)$ is a continuous path in U that joins S^1 to S^2. So the subset U is connected. Note that the connectedness of U clearly implies the connectedness of $U\cap\mathcal{Q}_1(k_1,\dots,k_r)$.

The cases c) and d) are analogous and left to the reader, hence the Main Theorem is proven.

Note that the connectedness of U also implies the connectedness of $U\cap\mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$ by using the same argument as in the end of Section 4.2. Hence Proposition 1.2 is proven too. □

Definition 5.3 A configuration is said to be *symmetric* if there exists a nontrivial isomorphism f of the corresponding graph of connected component Γ, such that:

- f commutes with the permutations of the edges associated to the configuration,
- f preserves the order of the boundary singularities,
- f preserves the order of the interior singularities.

Corollary 5.4 *Let $\mathcal{Q}(k_1,\dots,k_r)$ be a stratum of quadratic differentials on $\mathbb{CP}^1$, and let $N\geq 1$. If a connected component of $\mathcal{Q}^N(k_1,\dots,k_r)$ admits orbifoldic points, then the corresponding configuration is symmetric and the locus of orbifoldic points are a finite union of copies (or coverings) of open subset of configuration domains, which are manifolds, of smaller strata.*

Proof Recall that S corresponds to an orbifoldic point if and only if S admits a nontrivial orientation preserving isometry. Now let U be a connected component of $\mathcal{Q}^N(k_1,\dots,k_r)$, $S\in U$ an orbifoldic point, and let τ be an orientation preserving isometry of S.

Suppose that U corresponds to the $a)$ case of Theorem 5.1. Then τ must preserve the smallest saddle connection γ_0 of S. Either τ fixes the endpoints of S, either it interchanges them. In the first case, $\tau=Id$, in the other case it is uniquely determined and is an involution that fixes the middle of γ_0. In that case the endpoints of γ_0 have the same order $k \geq 0$. Then S/τ is a half-translation surface whose smallest saddle connection is of length $|\gamma_0|/2$ and joins a singularity of order $k \geq 0$ to a pole. Any other saddle connection in S/τ is of length l or $l/2$ for l the length of a saddle connection (different from γ_0) on S. Therefore, S/τ belongs to a configuration domain of $a)$ type in the corresponding stratum. The flat surface S/τ does not have a nontrivial orientation preserving isometry because $k \neq -1$. Therefore the configuration domain that contains S/τ is a manifold. The involution τ induces an involution on the set of zeros of S and the stratum and configuration domain corresponding to S/τ depends only on that involution. This induces a covering from the locus of orbifoldic points whose corresponding involution share the same combinatorial data to an open subset of a manifold.

If U corresponds to the $b)$ case, then similarly, a nontrivial isometric involution τ interchanges the two 1–holed spheres of the decomposition. We have $A_1 = A_2$ and $a_1 = a_2 > 0$ (see notations of Theorem 5.1), hence the configuration is symmetric. The set of orbifoldic points is isomorphic to the configuration domain of $a)$ type with data $\{a_1, -1\}$ which is a manifold.

If U corresponds to the $c)$ case then similarly, τ interchanges the two 1–holed sphere of the decomposition. We must have $A_1 = A_2$ and $a_1 = a_2 > 0$. The set of orbifoldic points is isomorphic to an open subset of a configuration domain of $d)$ type, which is a manifold (see next).

In the $d)$ case, any isometry τ fix the saddle connection γ_1 that separates the surface in a 1–holed sphere and a half-pillowcase, which are nonisometric. Hence they are fixed by τ. Now since τ is orientation preserving, it is easy to check that necessary, τ is trivial. □

Here we use Theorem 4.2 and the description of configurations to show that any stratum of quadratic differentials on $\mathbb{CP}^1$ admits only one topological end.

Corollary 5.5 *Let $\mathcal{Q}_1(k_1, \dots, k_r)$ be any stratum of quadratic differential on $\mathbb{CP}^1$. Then the subset $\mathcal{Q}_{1,\delta}(k_1, \dots, k_r)$ is connected for any $\delta > 0$.*

Proof Let $S \in \mathcal{Q}_{1,\delta}(k_1, \dots, k_r)$. We first describe a path from S to a simple configuration domain with corresponding singularities of orders $\{-1, k\}$. Then we show that all of these configuration domains are in the same connected component of $\mathcal{Q}_{1,\delta}(k_1, \dots, k_r)$.

Let γ_1 be a saddle connection of S of length less than δ (we can assume that γ_1 is vertical). Up to the Teichmüller geodesic flow action, we can assume that γ_1 is of length less than δ^2. Now let P be a pole. There exists a saddle connection γ_2 of length less than 1 starting from P, otherwise the 1–neighborhood of P would be an embedded half-disk of radius 1 in the surface, and would be of area $\frac{\pi}{2} > 1$. Then up to a slight deformation, we can assume that there are no other saddle connections parallel to γ_1 or γ_2 (except the ones that are $\hat{\text{h}}$omologous to γ_1 or γ_2). Now we contract γ_2 using the Teichmüller geodesic flow. This gives a path $(g_t.S)_{t\geq 0}$ in $\mathcal{Q}_1(k_1,\dots,k_r)$. For each $t \geq 0$ the saddle connections corresponding to γ_1 and γ_2 in $g_t.S$ are of length at most $\delta^2 e^{t/2}$ and $e^{-t/2}$ respectively. Hence the first one is smaller than or equal to δ for $0 \leq t \leq -2\ln(\delta)$, and the second one is smaller than δ for $t > -2\ln(\delta)$. Hence the path $g_t.S$ is in the δ–neigborhood of the boundary, and we now can assume that γ_2 is of length smaller than δ.

The other end of γ_2 is a singularity of order k. If $k \geq 0$, then from the list of configurations given in Theorem 5.1, the saddle connection γ_2 is simple.

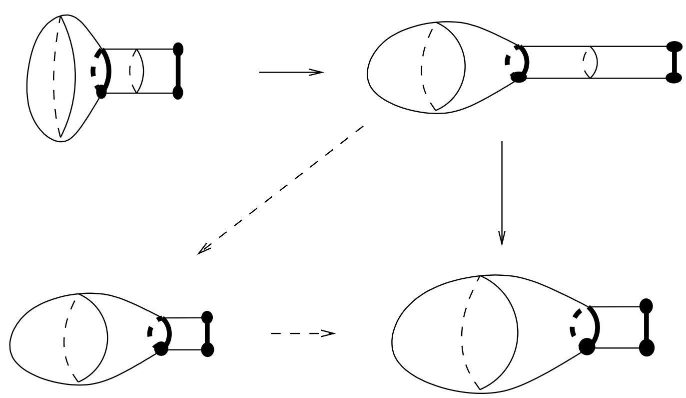

Figure 22: Deformation of a surface in $\mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$.

We assume that $k = -1$, then the surface is a 1–holed sphere glued with a cylinder, one end of this cylinder is γ_2 (we have a half-pillowcase), and the other end of that cylinder is a closed saddle connection whose endpoint is a singularity P' of order $k' > 0$. We can assume, up to using the Teichmüller geodesic flow, that γ_2 is of length at most $(1-c)\delta$, where c is the area of the cylinder. Now we consider γ_3 to be the shortest path from P to P'. It is clear that γ_3 is a simple saddle connection. Now up to twisting and shrinking the cylinder, we can make this saddle connection as small as possible (see Figure 22). However, this transformation, is not area preserving and we must rescale the surfaces to keep area one surfaces . This rescalling increase the length of γ_2 by a factor which is at most $\frac{1}{1-c}$, and therefore the length of γ_2 is always smaller than δ during this last deformation, and the resulting surface is in a simple configuration domain with corresponding singularities of orders $\{-1,k'\}$.

Now let $(U_i)_{i=1,2}$ be simple configuration domains. Up to renumbering, we can assume that their corresponding configurations are represented by simple paths that joins a pole to a singularity of order $k_i>0$, for $i=1,2$ (here we assume that there exists two distinct singularities of positive order, the complementary case is trivial). From Theorem 4.2, for each $i=1,2$, the set $U_i\cap\mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$ is connected. So, it is enough to find a path between two specific surfaces in U_i that stays in $\mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$. We have $r\geq 4$, so we can assume that $k_{r-1}=k_r=-1$. We start from a surface in $\mathcal{Q}(k_1-1,k_2-1,k_3,\dots,k_{r-2})$ and for $i=1,2$, we successively break a singularity of order k_i-1 into two singularities of order k_i and -1. We get a surface in $\mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$ with two arbitrary small saddle connections. We can assume that one of these short saddle connections is vertical, and the other not. Then action on this surface by the Teichmüller geodesic flow easily gives a path between U_1 and U_2 that keeps being in $\mathcal{Q}_{1,\delta}(k_1,\dots,k_r)$. □

Remark 5.6 As was seen previously, one can see more or less a stratum of quadratic differentials as a space of configuration of points in a sphere, hence one could use it to prove Corollary 1.3. However, Corollary 5.5 is stated in terms of flat metrics, and it is not clear how to relate precisely the degenerations we have described in terms of configurations of $\hat{\text{h}}$omologuous saddle connections and the corresponding degenerations in the space of configurations of points. Moreover, the previous proof could be more easily extended to other strata.

Appendix. A geometric criterion for $\hat{\text{h}}$omologous saddle connections

Here we give a proof of the following theorem.

Theorem (H Masur, A Zorich) *Consider two distinct saddle connections γ,γ' on a half-translation surface. The following assertions are equivalent.*

a) *The two saddle connections γ and γ' are $\hat{\text{h}}$omologous.*
b) *The ratio of their length is constant under any small deformation of the surface inside the ambient stratum.*
c) *They have no interior intersection and one of the connected component of $S\backslash\{\gamma\cup\gamma'\}$ has trivial linear holonomy.*

Proof The proofs of the statements $a \Leftrightarrow b$ and $c \Rightarrow b$ are the same as in Masur and Zorich [11]. We will write them for completeness. Our proof of $b \Rightarrow c$ is new and more geometric than the initial proof.

We first show that statement $a)$ is equivalent to statement $b)$. We have defined $[\hat{\gamma}]$ and $[\hat{\gamma}']$ in $H_1^-(\widehat{S}, \widehat{P}, \mathbb{Z})$. We claim that they are primitive cycles. Let γ_1 and γ_2 be the two preimages of γ in $\widehat{S}$. If $[\gamma_1] = -[\gamma_2]$, then $[\hat{\gamma}] = [\gamma_1]$ is primitive since it is realized by a simple curve. Otherwise $[\gamma_1]$ and $[\gamma_2]$ are independent in $H_1(\widehat{S}, \widehat{P}, \mathbb{Z})$, since they cannot be equal and are primitive. We assume first that γ_1 and γ_2 are closed paths. If they have no intersection point, then by choosing suitably a path joining γ_1 and γ_2, one can realize $[\hat{\gamma}] = [\gamma_1] - [\gamma_2]$ by a simple curve, and hence it is a primitive cycle. If they have an intersection point $\widehat{P}$, then it is the preimage of the adjacent singularity P of γ, which is therefore a ramification point. Since the natural involution on $\widehat{S}$ is a rotation in a neigborhood of $\widehat{P}$, one can always deform γ_1 and γ_2 to get two simple closed curves with no intersection point.

Now we assume that γ_1 and γ_2 are not closed, then we can find a basis of $H_1(\widehat{S}, \widehat{P}, \mathbb{Z})$ that contains $[\gamma_1]$ and $[\gamma_2]$. Hence we can find one that contains $[\gamma_1] - [\gamma_2]$ and $[\gamma_2]$, hence $[\gamma_1] - [\gamma_2]$ is primitive. So we have proved that $[\hat{\gamma}]$ and $[\hat{\gamma}']$ are primitive.

If γ and γ' are $\hat{\text{h}}$omologous, then integrating ω along the cycles $[\hat{\gamma}]$ and $[\widehat{\gamma'}]$, we see that the ratio of their length belongs to $\{-1/2, 1, 2\}$, and this ratio is obviously constant under any small deformations of the surface. Conversely, if they are not $\hat{\text{h}}$omologous, then (γ, γ') is a free family on $H_1^-(\widehat{S}, \widehat{P}, \mathbb{C})$ (since they are primitive elements of $H_1^-(\widehat{S}, \widehat{P}, \mathbb{Z})$) and so $\int_{\hat{\gamma}} \omega$ and $\int_{\hat{\gamma}'} \omega$ correspond to two independent coordinates in a neighborhood of S. Therefore the ratio of their length is not locally constant.

Now assume $c)$. We denote by S^+ a connected component of $S \backslash \{\gamma, \gamma'\}$ that has trivial holonomy. Its boundary is a union of components homeomorphic to $\mathbb{S}^1$. The saddle connections have no interior intersections, so this boundary is a union of copies of γ and γ' and it is easy to check that both γ and γ' appears in that boundary. The flat structure on S^+ is defined by an Abelian differential ω. Now we have $\int_{\partial S^+} \omega = 0$, which impose a relation on $|\gamma|$ and $|\gamma'|$. This relation is preserved in a neighborhood of S, and therefore, the ratio is locally constant and belongs to $\{1/2, 1, 2\}$ depending on the number of copies of each saddle connections on the boundary of S^+.

Now assume $b)$. We can assume that the saddle connection γ is vertical. Then using the Teichmüller geodesic flow g_t on S, for some small t, induce a small deformation of S. The hypothesis implies that the saddle connection γ' is necessary vertical too, and so the two saddle connections are parallel and hence have no interior intersections. Let S_1 and S_2 the connected components of $S \backslash \{\gamma, \gamma'\}$ that bounds γ (we may have

$S_1 = S_2$), and assume that S_1 has nontrivial linear holonomy. That implies there exists a simple broken line ν with nontrivial linear holonomy that starts and ends on the boundary of S_1 that correspond to γ. Now, we create an small hole by adding a parallelogram on the first segment of the path ν. This creates only one hole τ in the interior of S_1 because the other one is sent to the boundary (this procedure adds the length of the hole to the length of the boundary). If we directly move the hole τ to the boundary, we obtain a flat surface isometric to the initial surface S_1. But if we first transport τ along ν, then this will change its orientation, and its length will be added again to the length of the boundary. So the resulting surface has a boundary component corresponding to γ bigger than the initial surface S_1. The surgery did not affect the boundary corresponding to γ'. Assume now that S_2 has also nontrivial holonomy, then performing the same surgery on S_2, and gluing back S_1 and S_2, this gives a slight deformation of S that change the length of γ and not the length of γ'. This contradicts the hypothesis $b)$. □

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