Algebra & Number Theory

Volume 2

2008

No. 4

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mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 2:4(2008)

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Bhargava proved a formula for counting, with certain weights, degree n étale extensions of a local field, or equivalently, local Galois representations to S_n. This formula is motivation for his conjectures about the density of discriminants of S_n-number fields. We prove there are analogous "mass formulas" that count local Galois representations to any group that can be formed from symmetric groups by wreath products and cross products, corresponding to counting towers and direct sums of étale extensions. We obtain as a corollary that the above mentioned groups have rational character tables. Our result implies that D_4 has a mass formula for certain weights, but we show that D_4 does not have a mass formula when the local Galois representations to D_4 are weighted in the same way as representations to S_4 are weighted in Bhargava's mass formula.

1. Introduction

Bhargava [2007] proved the following mass formula for counting isomorphism classes of étale extensions of degree n of a local field K:

$$\sum_{[L:K]=n \text{ étale}} \frac{1}{|\operatorname{Aut}(K)|} \cdot \frac{1}{\operatorname{Norm}(\operatorname{Disc}_K L)} = \sum_{k=0}^{n-1} p(k, n-k) q^{-k}, \tag{1-1}$$

where q is the cardinality of the residue field of K, and $p(k, n-k)$ denotes the number of partitions of k into at most $n-k$ parts. Equation (1-1) is proven using the beautiful mass formula of Serre [1978] which counts totally ramified degree n extensions of a local field. Equation (1-1) is at the heart of [Bhargava 2007, Conjecture 1] for the asymptotics of the number of S_n-number fields with discriminant $\leq X$, and also [Bhargava 2007, Conjectures 2–3] for the relative asymptotics of S_n-number fields with certain local behaviors specified. These conjectures are theorems for $n \leq 5$ [Davenport and Heilbronn 1971; Bhargava 2005; $\geq$ 2008].

MSC2000: primary 11S15; secondary 11R45.

Keywords: Local Field, Mass Formula, Counting Field Extension.

Kedlaya [2007, Section 3] has translated Bhargava's formula into the language of Galois representations so that the sum in (1-1) becomes a sum over Galois representations to S_n as follows:

$$\frac{1}{n!}\sum_{\rho:\mathrm{Gal}(K^{\mathrm{sep}}/K)\to S_n}\frac{1}{q^{c(\rho)}}=\sum_{k=0}^{n-1}p(k,n-k)q^{-k}, \tag{1-2}$$

where $c(\rho)$ denotes the Artin conductor of ρ composed with the standard representation $S_n \to \mathrm{GL}_n(\mathbb{C})$.

What is remarkable about the mass formulas in (1-1) and (1-2) is that the right hand side only depends on q and, in fact, is a polynomial (independent of q) evaluated at q^{-1}. A priori, the left hand sides could depend on the actual local field K, and even if they only depended on q, it is not clear there should be a uniform way to write them as a polynomial function of q^{-1}. This motivates the following definitions. Given a local field K and a finite group Γ, let $S_{K,\Gamma}$ denote the set of continuous homomorphisms $\mathrm{Gal}(K^{\mathrm{sep}}/K)\to\Gamma$ (for the discrete topology on Γ) and let q_K denote the size of the residue field of K. Given a function $c\colon S_{K,\Gamma}\to\mathbb{Z}_{\geq 0}$, we define the *total mass* of (K,Γ,c) to be

$$M(K,\Gamma,c):=\sum_{\rho\in S_{K,\Gamma}}\frac{1}{q_K^{c(\rho)}}.$$

(If the sum diverges, we could say the mass is ∞ by convention. In most interesting cases, for example see [Kedlaya 2007, Remark 2.3], and all cases we consider in this paper, the sum will be convergent.) Kedlaya gave a similar definition, but one should note that our definition of mass differs from that in [Kedlaya 2007] by a factor of $|\Gamma|$. In [Kedlaya 2007], $c(\rho)$ is always taken to be the Artin conductor of the composition of ρ and some $\Gamma\to\mathrm{GL}_n(\mathbb{C})$. We refer to such c as the *counting function attached to the representation* $\Gamma\to\mathrm{GL}_n(\mathbb{C})$. In this paper, we consider more general c.

Given a group Γ, a *counting function for* Γ is any function

$$c\colon\bigcup_K S_{K,\Gamma}\to\mathbb{Z}_{\geq 0}$$

where the union is over all isomorphism classes of local fields, such that

$$c(\rho)=c(\gamma\rho\gamma^{-1})$$

for every $\gamma\in\Gamma$. (Since an isomorphism of local fields only determines an isomorphism of their absolute Galois groups up to conjugation, we need this condition in order for the counting functions to be sensible.) Let c be a counting function for Γ and S be a class of local fields. We say that (Γ,c) has a *mass formula* for S if there exists a polynomial $f(x) \in \mathbb{Z}[x]$ such that for all local fields $K \in S$ we have

$$M(K, \Gamma, c) = f\Big(\frac{1}{q_K}\Big).$$

We also say that Γ has a mass formula for S if there is a c such that (Γ, c) has a mass formula for S.

Kedlaya [2007, Theorem 8.5] proved that $(W(B_n), c_{B_n})$ has a mass formula for all local fields, where $W(B_n)$ is the Weyl group of B_n and c_{B_n} is the counting function attached to the Weyl representation of B_n. This is in analogy with (1-2) which shows that $(W(A_n), c_{A_n})$ has a mass formula for all local fields, where $W(A_n) \cong S_n$ is the Weyl group of A_n and c_{A_n} is the counting function attached to the Weyl representation of A_n. Kedlaya's analogy is very attractive, but he found that it does not extend to the Weyl groups of D_4 or G_2 when the counting function is the one attached to the Weyl representation; he showed that mass formulas for all local fields do not exist for those groups and those particular counting functions.

The main result of this paper is the following.

Theorem 1.1. *Any permutation group that can be constructed from the symmetric groups S_n using wreath products and cross products has a mass formula for all local fields.*

The mass formula of Kedlaya [2007, Theorem 8.5] for $W(B_n) \cong S_2 \wr S_n$ was the inspiration for this result, and it is now a special case of Theorem 1.1.

Bhargava [2007, Section 8.2] asks whether his conjecture for S_n-extensions about the relative asymptotics of the number of global fields with specified local behaviors holds for other Galois groups. Ellenberg and Venkatesh [2005, Section 4.2] suggest that we can try to count extensions of global fields by quite general invariants of Galois representations. In [Wood 2008], it is shown that when counting by certain invariants of abelian global fields, such as conductor, Bhargava's question can be answered affirmatively. It is also shown in [Wood 2008] that when counting abelian global fields by discriminant, the analogous conjectures fail in at least some cases. In light of the fact that Bhargava's conjectures for the asymptotics of the number of S_n-number fields arise from his mass formula (1-1) for counting by discriminant, one naturally looks for mass formulas that use other ways of counting, such as Theorem 1.1, which might inspire conjectures for the asymptotics of counting global fields with other Galois groups.

In Section 2, we prove that if groups A and B have certain refined mass formulas, then $A \wr B$ and $A \times B$ also have such refined mass formulas, which inductively proves Theorem 1.1. Bhargava's mass formula for S_n, given in (1-2), is our base case. In Section 3, as a corollary of our main theorem, we see that any group formed from symmetric groups by taking wreath and cross products has a rational character table. This result, at least in such simple form, is not easily found in the literature.

In order to suggest what our results say in the language of field extensions, in Section 4 we mention the relationship between Galois representations to wreath products and towers of field extensions.

In Section 5, we discuss some situations in which groups have mass formulas for one way of counting but not another. In particular, we show that $D_4 \cong S_2 \wr S_2$ does not have a mass formula for all local fields when $c(\rho)$ is the counting function attached to the standard representation of S_4 restricted to $D_4 \subset S_4$. Consider quartic extensions M of K, whose Galois closure has group D_4, with quadratic subfield L. The counting function that gives the mass formula for D_4 of Theorem 1.1 corresponds to counting such extensions M weighted by

$$\left|\mathrm{Disc}(L|K)N_{L|K}(\mathrm{Disc}(M|L))\right|^{-1},$$

whereas the counting function attached to the standard representation of S_4 restricted to $D_4 \subset S_4$ corresponds to counting such extensions M weighted by

$$|\mathrm{Disc}(M|K)|^{-1} = \left|\mathrm{Disc}(L|K)^2 N_{L|K}(\mathrm{Disc}(M|L))\right|^{-1}.$$

So this change of exponent in the $\mathrm{Disc}(L|K)$ factor affects the existence of a mass formula for all local fields.

Notation. Throughout this paper, K is a local field and $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ is the absolute Galois group of K. All maps in this paper from G_K or subgroups of G_K are continuous homomorphisms, with the discrete topology on all finite groups. We let I_K denote the inertia subgroup of G_K. Recall that $S_{K,\Gamma}$ is the set of maps $G_K \to \Gamma$, and q_K is the size of residue field of K. Also, Γ will always be a permutation group acting on a finite set.

2. Proof of Theorem 1.1

In order to prove Theorem 1.1, we prove finer mass formulas first. Instead of summing over all representations of G_K, we stratify the representations by *type* and prove mass formulas for the sum of representations of each type. Let $\rho\colon G_K \to \Gamma$ be a representation such that the action of G_K has r orbits $m_1, \dots, m_r$. If, under restriction to the representation $\rho : I_K \to \Gamma$, orbit m_i breaks up into f_i orbits of size e_i, then we say that ρ is of *type* $(f_1^{e_1} f_2^{e_2} \cdots f_r^{e_r})$ (where the terms $f_i^{e_i}$ are unordered formal symbols, as in [Bhargava 2007, Section 2]). Let L_i be the fixed field of the stabilizer of an element in m_i. So, $[L_i\colon K] = |m_i|$. Since $I_{L_i} = G_{L_i} \cap I_K$ is the stabilizer in I_K of an element in m_i, we conclude that $e_i = [I_K : I_{L_i}]$, which is the ramification index of L_i/K. Thus, f_i is the inertial degree of L_i/K.

Given Γ, a counting function c for Γ, and a type

$$\sigma = (f_1^{e_1} f_2^{e_2} \cdots f_r^{e_r}),$$

we define the *total mass* of (K, Γ, c, σ) to be

$$M(K, \Gamma, c, \sigma) := \sum_{\substack{\rho \in S_{K,\Gamma} \\ \text{type } \sigma}} \frac{1}{q_K^{c(\rho)}}.$$

We say that (Γ, c) has *mass formulas for S by type* if for every type σ there exists a polynomial $f_{(\Gamma,c,\sigma)}(x) \in \mathbb{Z}[x]$ such that for all local fields $K \in S$ we have

$$M(K, \Gamma, c, \sigma) = f_{(\Gamma,c,\sigma)}\Big(\frac{1}{q_K}\Big).$$

Bhargava [2007, Proposition 1] actually proved that S_n has mass formulas for all local fields by type. Of course, if (Γ, c) has mass formulas by type, then we can sum over all types to obtain a mass formula for (Γ, c).

The key step in the proof of Theorem 1.1 is the following.

Theorem 2.1. *If A and B are finite permutation groups, S is some class of local fields, and (A, c_A) and (B, c_B) have mass formulas for S by type, then there exists a counting function c (given in (2-3)) such that $(A \wr B, c)$ has mass formulas for S by type.*

Proof. Let K be a local field in S. Let A act on the left on the set $\mathscr{A}$ and B act on the left on the set $\mathscr{B}$. We take the natural permutation action of $A \wr B$ acting on a disjoint union of copies of $\mathscr{A}$ indexed by elements of $\mathscr{B}$. Fix an ordering on $\mathscr{B}$ so that we have canonical orbit representatives in $\mathscr{B}$. Given $\rho : G_K \to A \wr B$, there is a natural quotient $\bar{\rho} : G_K \to B$. Throughout this proof, we use j as an indexing variable for the set $\mathscr{B}$ and i as an indexing variable for the r canonical orbit representatives in $\mathscr{B}$ of the $\rho(G_K)$ action. Let i_j be the index of the orbit representative of j's orbit. Let $S_j \subset G_K$ be the stabilizer of j, and let S_j have fixed field L_j. We define $\rho_j : G_{L_j} \to A$ to be the given action of G_{L_j} on the j-th copy of $\mathscr{A}$. We say that ρ has *wreath type*

$$\Sigma = (f_1^{e_1}(\sigma_1) \cdots f_r^{e_r}(\sigma_r)) \tag{2-1}$$

if $\bar{\rho}$ has type $\sigma = (f_1^{e_1} \cdots f_r^{e_r})$ (where $f_i^{e_i}$ corresponds to the orbit of i) and ρ_i has type σ_i. Note that type is a function of wreath type; if ρ has wreath type Σ as above where

$$\sigma_i = \big(f_{i,1}^{e_{i,1}} \cdots f_{i,r_i}^{e_{i,r_i}}\big),$$

then ρ has type $((f_i f_{i,k})^{e_i e_{i,k}})_{1 \le i \le r,\, 1 \le k \le r_i}$.

We consider the function c defined as follows:

$$c(\rho) = c_B(\bar{\rho}) + \sum_{j \in \mathscr{B}} \frac{c_A(\rho_j)}{|\{\bar{\rho}(I_K) j\}|}. \tag{2-2}$$

Since $c_B(\bar{\rho})$ only depends on the B-conjugacy class of $\bar{\rho}$ and $c_A(\rho_j)$ depends only on the A-conjugacy class of ρ_j, we see that conjugation by elements of $A \wr B$ does not affect the right hand side of (2-3) except by reordering the terms in the sum. Thus c is a counting function.

Since ρ_j and ρ_{i_j} are representations of conjugate subfields of G_K and since c_A is invariant under A-conjugation, $c_A(\rho_j) = c_A(\rho_{i_j})$. There are $f_i e_i$ elements in the orbit of i under $\bar{\rho}(G_K)$ and e_{i_j} elements in the orbit of j under $\bar{\rho}(I_K)$, so

$$c(\rho) = c_B(\bar{\rho}) + \sum_{i=1}^{r} \frac{f_i e_i}{e_i} c_A(\rho_i)$$

and thus

$$c(\rho) = c_B(\bar{\rho}) + \sum_{i=1}^{r} f_i c_A(\rho_i). \tag{2-3}$$

Using this expression for $c(\rho)$, we will prove that $(A \wr B, c)$ has mass formulas by wreath type. Then, summing over wreath types that give the same type, we will prove that $(A \wr B, c)$ has mass formulas by type.

Remark 2.2. For a permutation group Γ, let d_Γ be the counting function attached to the permutation representation of Γ (which is the discriminant exponent of the associated étale extension). Then we can compute

$$d_{A\wr B} = |\mathcal{A}|\, d_B(\bar{\rho}) + \sum_{i=1}^{r} f_i\, d_A(\rho_i),$$

which is similar to the expression given in (2-3) but differs by the presence of $|\mathcal{A}|$ in the first term. In particular, when we have mass formulas for (A, d_A) and (B, d_B), the mass formula for $A \wr B$ that we find in this paper is not with the counting function $d_{A\wr B}$. We will see in Section 5, when A and B are both S_2, that $S_2 \wr S_2 \cong D_4$ does not have a mass formula with $d_{A\wr B}$.

Lemma 2.3. *The correspondence $\rho \mapsto (\bar{\rho}, \rho_1, \dots, \rho_r)$ described above gives a function Ψ from $S_{K,A\wr B}$ to tuples $(\phi, \phi_1, \dots, \phi_r)$ where $\phi : G_K \to B$, the groups S_i are the stabilizers of canonical orbit representatives of the action of ϕ on B, and $\phi_i : S_i \to A$. The map Ψ is $(|A|^{|\mathcal{B}|-r})$-to-one and surjective.*

Proof. Lemma 2.3 holds when G_K is replaced by any group. It suffices to prove the lemma when $\bar{\rho}$ and ϕ are transitive because the general statement follows by multiplication. Let $b \in \mathcal{B}$ be the canonical orbit representative. Given a

$$\phi\colon G_K \to B \quad (\text{or a } \bar{\rho}\colon G_K \to B)$$

for all $j \in \mathcal{B}$, choose a $\sigma_j \in G_K$ such that $\phi(\sigma_j)$ takes b to j. Given a $\rho\colon G_K \to A \wr B$, let α_j be the element of A such that $\rho(\sigma_j)$ acts on the b-th copy of $\mathcal{A}$ by α_j and then moves the b-th copy of $\mathscr{A}$ to the j-th copy. Then for $g \in G_K$, the map ρ is given by

$$\rho(g) = \bar\rho(g)(a_j)_{j\in\mathscr{B}} \in BA^{|\mathscr{B}|} = A \wr B, \tag{2-4}$$

where

$$a_j = \alpha_{\bar\rho(g)(j)}\rho_1\bigl(\sigma^{-1}_{\bar\rho(g)(j)} g \sigma_j\bigr)\,\alpha_j^{-1},$$

and $a_j \in A$ acts on the j-th copy of $\mathscr{A}$. For any transitive maps $\phi\colon G_K \to B$ and $\phi_b : S_b \to A$ and for any choices of $\alpha_j \in A$ for all $j \in \mathscr{B}$ such that $\alpha_b = \phi_b(\sigma_b)$, we can check that (2-4) for $\bar\rho = \phi$ and $\rho_1 = \phi_b$ gives a homomorphism $\rho\colon G_K \to A \wr B$ with $(\bar\rho, \rho_1) = (\phi, \phi_b)$, which proves the lemma. □

If Σ is as in (2-1), then

$$\sum_{\substack{\rho:G_K\to A\wr B\\ \text{wreath type } \Sigma}} \frac{1}{q_K^{c(\rho)}} = |A|^{|\mathscr{B}|-r} \sum_{\substack{\phi:G_K\to B\\ \text{type } \sigma}} \sum_{\substack{\phi_1:S_1\to A\\ \text{type } \sigma_1}} \sum_{\substack{\phi_2:S_2\to A\\ \text{type } \sigma_2}} \cdots \sum_{\substack{\phi_r:S_r\to A\\ \text{type } \sigma_r}} \frac{1}{q_K^{c_B(\phi)+\sum_{i=1}^r f_i c_A(\phi_i)}} \tag{2-5}$$

where S_i is the stabilizer under ϕ of a canonical orbit representative of the action of ϕ on $\mathscr{B}$. The right hand side of (2-5) factors, and $S_i \subset G_K$ has fixed field L_i with residue field of size $q_K^{f_i}$. We conclude that

$$\begin{aligned}\sum_{\substack{\rho:G_K\to A\wr B\\ \text{wreath type } \Sigma}} \frac{1}{q_K^{c(\rho)}} &= |A|^{|\mathscr{B}|-r} \sum_{\substack{\phi:G_K\to B\\ \text{type } \sigma}} \frac{1}{q_K^{c_A(\phi)}} \sum_{\substack{\phi_1:G_{L_1}\to A\\ \text{type } \sigma_1}} \frac{1}{q_K^{f_1 c_B(\phi_1)}} \cdots \sum_{\substack{\phi_r:G_{L_r}\to A\\ \text{type } \sigma_r}} \frac{1}{q_K^{f_r c_B(\phi_r)}}\\ &= |A|^{|\mathscr{B}|-r} f_{(B,c_B,\sigma)}\Bigl(\frac{1}{q_K}\Bigr) \prod_{i=1}^r f_{(A,c_A,\sigma_i)}\Bigl(\frac{1}{q_K^{f_i}}\Bigr).\end{aligned}$$

So, $(A \wr B, c)$ has mass formulas by wreath type, and thus by type. □

Kedlaya [2007, Lemma 2.6] noted that if (Γ, c) and (Γ', c') have mass formulas f and f', then $(\Gamma\times\Gamma', c'')$ has mass formula ff', where $c''(\rho\times\rho') = c(\rho) + c'(\rho')$. We can strengthen this statement to mass formulas by type using a much easier version of our argument for wreath products. We define the *product type* of a representation $\rho\times\rho' : G_K \to \Gamma\times\Gamma'$ to be (σ, σ'), where σ and σ' are the types of ρ and ρ' respectively. Then

$$\sum_{\substack{\rho\times\rho':G_K\to\Gamma\times\Gamma'\\ \text{product type } (\sigma,\sigma')}} \frac{1}{q_K^{c''(\rho\times\rho')}} = \sum_{\substack{\phi:G_K\to\Gamma\\ \text{type } \sigma}} \frac{1}{q_K^{c(\rho)}} \sum_{\substack{\phi_1:G_{L_1}\to\Gamma'\\ \text{type } \sigma'}} \frac{1}{q_K^{c'(\rho')}}.$$

If Γ and Γ' have mass formulas by type, then the above gives mass formulas of $\Gamma\times\Gamma'$ by product type. Since type is a function of product type, we can sum the mass formulas by product type to obtain mass formulas by type for $\Gamma\times\Gamma'$.

This, combined with Theorem 2.1 and Bhargava's mass formula for S_n by type [Bhargava 2007, Proposition 1], proves Theorem 1.1.

3. Groups with rational character tables

Kedlaya [2007, Proposition 5.3, Corollary 5.4, Corollary 5.5] showed that if $c(\rho)$ is the counting function attached to $\Gamma \to \mathrm{GL}_n(\mathbb{C})$, then the following statement holds: (Γ, c) has a mass formula for all local fields K with q_K relatively prime to $|\Gamma|$ if and only if the character table of Γ has all rational entries. The proofs of [Kedlaya 2007, Proposition 5.3, Corollary 5.4, Corollary 5.5] hold for any counting function c that is determined by $\rho(I_K)$. This suggests that we define a *proper* counting function to be a counting function c that satisfies the following: if we have

$$\rho : G_K \to \Gamma \quad \text{and} \quad \rho' : G_{K'} \to \Gamma$$

with $q_K, q_{K'}$ relatively prime to $|\Gamma|$, and if $\rho(I_K) = \rho'(I_{K'})$, then $c(\rho) = c(\rho')$.

For proper counting functions, we always have partial mass formulas proven as in [Kedlaya 2007, Corollary 5.4].

Proposition 3.1. *Let a be an invertible residue class mod $|\Gamma|$ and c be a proper counting function. Then (Γ, c) has a mass formula for all local fields K with $q_K \in a$.*

The following proposition says exactly when these partial mass formulas agree, again proven as in [Kedlaya 2007, Corollary 5.5].

Proposition 3.2. *Let c be a proper counting function for Γ. Then (Γ, c) has a mass formula for all local fields K with q_K relatively prime to $|\Gamma|$ if and only if Γ has a rational character table.*

So, when looking for a group and a proper counting function with mass formulas for all local fields, we should look among groups with rational character tables (which are relatively rare, for example including only 14 of the 93 groups of order < 32 [Conway 2006]). All specific counting functions that have been so far considered in the literature are proper. It is not clear if there are any interesting nonproper counting functions.

Our proof of Theorem 2.1 has the following corollary.

Corollary 3.3. *Any permutation group that can be constructed from the symmetric groups using wreath products and cross products has a rational character table.*

Proof. We first show that the counting function c defined in (2-2) is proper if c_A and c_B are proper. We consider only fields K with q_K relatively prime to $|\Gamma|$. Since $c_B(\bar{\rho})$ only depends on $\bar{\rho}(I_K)$, it is clear that the $c_B(\bar{\rho})$ term only depends on $\rho(I_K)$.

Since $I_{L_j} = I_K \cap S_j$, we have

$$\rho_j(I_{L_j}) = \rho(I_{L_j}) = \rho(I_K) \cap \mathrm{Stab}(j).$$

Since $c_A(\rho_j)$ depends only on $\rho_j(I_{L_j})$, we see that it depends only on $\rho(I_K)$. The sum in (2-2) then depends only on $\rho(I_K)$. So the c defined in (2-2) is proper. Clearly the $c''(\rho \times \rho')$ defined for cross products is proper if c and c' are proper. The counting function in Bhargava's mass formula for S_n (see (1-2)) is an Artin conductor and thus is proper. So we can prove Theorem 1.1 with a proper counting function and conclude the corollary.

One can show in a similar way that even in wild characteristics, the counting function c defined in (2-3) depends only on the images of the higher ramification groups G_K^m, that is, if

$$\rho : G_K \to A \wr B \quad \text{and} \quad \rho' : G'_K \to A \wr B$$

have $\rho(G_K^m) = \rho'(G_{K'}^m)$ for all $m \in [0, \infty)$, then $c(\rho) = c(\rho')$, as long as the same is true for c_A and c_B. □

So, for example, $((S_7 \wr S_4) \times S_3) \wr S_8$ has a rational character table. Corollary 3.3 does not seem to be a well-reported fact in the literature; the corollary shows that all Sylow 2-subgroups of symmetric groups (which are cross products of wreath products of S_2's) have rational character table, which was posed as an open problem in [Mazurov and Khukhro 1999, Problem 15.25] and solved in [Revin 2004; Kolesnikov 2005]. However, since

$$A \wr (B \wr C) = (A \wr B) \wr C \quad \text{and} \quad A \wr (B \times C) = (A \wr B) \times (A \wr C),$$

any of the groups of Corollary 3.3 can be constructed only using the cross product and $\wr S_n$ operations. It is well known that the cross product of two groups with rational character tables has a rational character table. Furthermore, Pfeiffer [1994] explains how GAP computes the character table of $G \wr S_n$ from the character table of G, and one can check that if G has rational character table then all of the values constructed in the character table of $G \wr S_n$ are rational, which implies Corollary 3.3.

One might hope that all groups with rational character tables have mass formulas by type, but this is not necessarily the case. For example, considering

$$(C_3 \times C_3) \rtimes C_2$$

(where C_2 acts nontrivially on each factor separately) in the tame case in type $(1^3\, 2^1\, 1^1)$, one can check that for $q \equiv 1 \pmod 3$ the mass is zero and for $q \equiv 2 \pmod 3$ the mass is nonzero.

4. Towers and direct sums of field extensions

Kedlaya explains the correspondence between Galois permutation representations and étale extensions in [Kedlaya 2007, Lemma 3.1]. We have seen this correspondence already in other terms. If we have a representation $\rho : G_K \to \Gamma$ with r orbits, S_i is the stabilizer of an element in the i-th orbit, and L_i is the fixed field of S_i, then ρ corresponds to $L = \bigoplus_{i=1}^{r} L_i$. For a local field F, let $\wp_F$ be the prime of F. In this correspondence, if c is the counting function attached to the permutation representation of Γ, then c is the discriminant exponent of the extension L/K [Kedlaya 2007, Lemma 3.4]. In other words,

$$\wp_K^{c(\rho)} = \mathrm{Disc}(L|K).$$

We can interpret the representations $\rho : G_K \to A \wr B$ as towers of étale extensions $M/L/K$. If we take $\bar{\rho} : G_K \to B$, then $L = \bigoplus_{i=1}^{r} L_i$ is just the étale extension of K corresponding to $\bar{\rho}$. Then if M is the étale extension of K corresponding to ρ, we see that $M = \bigoplus_{i=1}^{r} M_i$, where M_i is the étale extension of L_i corresponding to $\rho_i : G_{L_i} \to A$. So we see that M is an étale extension of L, though L might not be a field.

Let c be the counting function of our mass formula for wreath products, given by (2-3). From (2-3), we obtain

$$\wp_K^{c(\rho)} = \wp_K^{c_B(\bar{\rho})} \prod_{i=1}^{r} N_{L_i|K}\big(\wp_{L_i}^{c_A(\rho_i)}\big).$$

For example, if c_A and c_B are both given by the discriminant exponent (or equivalently, attached to the permutation representation), then

$$\wp_K^{c(\rho)} = \mathrm{Disc}(L|K) \prod_{i=1}^{r} N_{L_i|K}(\mathrm{Disc}(M_i|L_i)). \tag{4-1}$$

For comparison,

$$\mathrm{Disc}(M|K) = \mathrm{Disc}(L|K)^{[M:L]} \prod_{i=1}^{r} N_{L_i|K}(\mathrm{Disc}(M_i|L_i)).$$

As we will see for $\Gamma = D_4$ in the next section, representations $\rho : G_K \to \Gamma$ can give not only field extensions of K whose Galois closure has Galois group Γ, but also field extensions whose Galois closure has Galois group a proper subgroup of Γ, as well as direct sums of field extensions. One could say that representations $\rho : G_K \to A \wr B$ correspond to towers of "A-extensions" over "B-extensions" and further relate iterated wreath products to iterated towers. Similarly, one could say that a representation $\rho : G_K \to A \times B$ corresponds to a direct sum of an "A-extension" and a "B-extension." The quotes indicate that the extensions do not necessarily have Galois closure with group A or B. In fact, it seems the most convenient way to define "A-extensions" or isomorphisms of "A-extensions" is simply to use the language of Galois representations as we have in this paper.

5. Masses for D_4

By Proposition 3.2 we know, at least for proper counting functions, that the existence of a mass formula for a group Γ for fields with q_K relatively prime to $|\Gamma|$ does not depend on the choice of the counting function. However, in wild characteristic this is not the case. For example, D_4, the dihedral group with 8 elements, is isomorphic to $S_2 \wr S_2$, so by Theorem 1.1 there is a c (given in (2-3)) for which D_4 has a mass formula for all local fields. An expression for c in terms of étale extensions can be read off from (4-1). In particular, for a surjective representation $\rho : G_K \to D_4$ corresponding to a quartic field extension M of K with a quadratic subextension L,

$$\wp_K^{c(\rho)} = \mathrm{Disc}(L|K) N_{L|K}(\mathrm{Disc}(M|L)). \tag{5-1}$$

For this c, for all local fields K, we have that

$$M(K, D_4, c) := \sum_{\rho \in S_{K,D_4}} \frac{1}{q_K^{c(\rho)}} = 8 + \frac{16}{q_K} + \frac{16}{q_K^2}.$$

From the definition of c given in (2-2) and the description of the absolute tame Galois group of a local field, we can compute $M(K, D_4, c)$ for a field K with q_K odd. By Theorem 2.1 we know the formula holds for all K.

However, the counting function for D_4 that has been considered when counting global extensions (for example in [Cohen et al. 2002]) is the one attached the faithful permutation representation of D_4 on a four element set (equivalently the discriminant exponent of the corresponding étale extension). We call this counting function d, and in comparison with (5-1) we have

$$\wp_K^{d(\rho)} = \mathrm{Disc}(M|K) = \mathrm{Disc}(L|K)^2 N_{L|K}(\mathrm{Disc}(M|L)).$$

With d, we now show that D_4 does not have a mass formula for all local fields.

Using the correspondence of Section 4, we can analyze the representations

$$\rho : G_K \to D_4 \subset S_4$$

in Table 1, where

$$I = \mathrm{image}(\rho), \quad j = \left|\left\{ s \in S_4 \;\middle|\; sIs^{-1} \subset D_4 \right\}\right| \quad \text{and} \quad k = |\,\mathrm{Centralizer}_{S_4}(I)|.$$

We take the D_4 in S_4 generated by $(1\,2\,3\,4)$ and $(1\,3)$.

I	j	k	L
D_4	8	2	degree 4 field whose Galois-closure$/K$ has group D_4
C_4	8	4	degree 4 field Galois$/K$ with group $C_4 \cong \mathbb{Z}/4$
$\langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle$	24	4	degree 4 field Galois$/K$ with group $V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$
$\langle (1\,3), (2\,4) \rangle$	8	4	$L_1 \oplus L_2$ with $[L_i : K]$=2 and L_i distinct fields
$\langle (1\,3)(2\,4) \rangle$, $\langle (1\,2)(3\,4) \rangle$ or $\langle (1\,4)(2\,3) \rangle$	24	8	$L_1 \oplus L_2$ with $[L_i : K]$=2 and $L_1 \cong L_2$ fields
$\langle (2\,4) \rangle$ or $\langle (1\,3) \rangle$	8	4	$L_1 \oplus K \oplus K$ with $[L_1 : K]$=2 and L_1 a field
1	24	24	$K \oplus K \oplus K \oplus K$

Table 1

Each isomorphism class of algebras appears $\frac{j}{k}$ times from a representation $\rho : G_K \to D_4$ (see [Kedlaya 2007, Lemma 3.1]). Let $S(K, G, m)$ be the set of isomorphism classes of degree m field extensions of K whose Galois closure over K has group G. Then from the above table we see that

$$M(K, D_4, d) = \sum_{F \in S(K, D_4, 4)} \frac{4}{|\operatorname{Disc} F|} + \sum_{F \in S(K, C_4, 4)} \frac{2}{|\operatorname{Disc} F|} + \sum_{F \in S(K, V_4, 4)} \frac{6}{|\operatorname{Disc} F|} + \sum_{\substack{F_1, F_2 \in S(K, C_2, 2) \\ F_1 \ncong F_2}} \frac{2}{|\operatorname{Disc} F_1||\operatorname{Disc} F_2|} + \sum_{F \in S(K, C_2, 2)} \frac{3}{|\operatorname{Disc} F|^2} + \sum_{F \in S(K, C_2, 2)} \frac{2}{|\operatorname{Disc} F|} + 1,$$

where if $\wp_F$ is the prime of F and $\operatorname{Disc} F = \wp_F^m$, then $|\operatorname{Disc} F| = q_F^m$. Using the Database of Local Fields [Jones and Roberts 2006] we can compute that $M(\mathbb{Q}_2, D_4, d) = \frac{121}{8}$. For fields with $2 \nmid q_K$, the structure of the tame quotient of the absolute Galois group of a local field allows us to compute the mass to be

$$8 + \frac{8}{q_K} + \frac{16}{q_K^2} + \frac{8}{q_K^3}$$

(also see [Kedlaya 2007, Corollary 5.4]) which evaluates to 17 for $q_K = 2$. Thus (D_4, d) does not have a mass formula for all local fields.

As another example, Kedlaya [2007, Proposition 9.3] found that $W(G_2)$ does not have a mass formula for all local fields of residual characteristic 2 when c is the Artin conductor of the Weyl representation. However, $W(G_2) \cong S_2 \times S_3$ and thus it has a mass formula for all local fields with counting function the sum of the Artin conductors of the standard representations of S_2 and S_3.

It would be interesting to study what the presence or absence of mass formulas tells us about a counting function, in particular with respect to how global fields can be counted asymptotically with that counting function. As in Bhargava [2007, Section 8.2], we can form an Euler series

$$M_c(\Gamma, s) = C(\Gamma)\Bigg(\sum_{\rho\in S_{\mathbb{R},\Gamma}} \frac{1}{|\Gamma|}\Bigg)\prod_p\Bigg(\frac{1}{|\Gamma|}\sum_{\rho\in S_{\mathbb{Q}_p,\Gamma}} \frac{1}{p^{c(\rho)s}}\Bigg) = \sum_{n\geq 1} m_n n^{-s},$$

where $C(\Gamma)$ is some simple, yet to be explained, rational constant. (We work over $\mathbb{Q}$ for simplicity, and the product is over rational primes.) For a representation $\rho : G_{\mathbb{Q}} \to \Gamma$, let ρ_p be the restriction of ρ to $G_{\mathbb{Q}_p}$. The idea is that m_n should be a heuristic of the number of Γ-extensions of $\mathbb{Q}$ (that is, surjective $\rho : G_{\mathbb{Q}} \to \Gamma$) with

$$\prod_p p^{c(\rho_p)} = n,$$

though m_n is not necessarily an integer.

Bhargava [2007, Section 8.2] asks the following question.

Question 5.1. Does

$$\lim_{X\to\infty} \frac{\sum_{n=1}^{X} m_n}{\left|\left\{\text{isom. classes of surjective } \rho : G_{\mathbb{Q}} \to \Gamma \text{ with } \prod_p p^{c(\rho_p)} \leq X\right\}\right|} = 1?$$

Bhargava in fact asks more refined questions in which some local behaviors are fixed. With the counting function d for D_4 attached to the permutation representation (that is, the discriminant exponent), we can form $M_d(D_4, s)$ and compute numerically the above limit. We use the work of Cohen, Diaz y Diaz, and Oliver on counting D_4-extensions by discriminant (see [Cohen et al. 2006] for a recent value of the relevant constants) to calculate the limit of the denominator, and we use standard Tauberian theorems (see [Narkiewicz 1983, Corollary, p. 121]) and PARI/GP [2006] to calculate the limit of the numerator. Of course, $C(D_4)$ has not been decided, but it does not appear (by using the `algdep` function in PARI/GP) that any simple rational $C(D_4)$ will give an affirmative answer to the above question.

In light of our mass formula for a different counting function c for D_4, we naturally wonder about Question 5.1 in the case of D_4 and that c. Answering this question would require counting D_4 extensions M with quadratic subfield L by

$$\operatorname{Disc}(L|\,\mathbb{Q})\, N_{L|\mathbb{Q}}(\operatorname{Disc}(M|L))$$

instead of by discriminant, which is

$$\operatorname{Disc}(L|\mathbb{Q})^2 N_{L|\mathbb{Q}}(\operatorname{Disc}(M|L)).$$

Acknowledgements

I would like to thank Manjul Bhargava for his guidance while I was doing this research and thorough and helpful comments on earlier versions of this manuscript. I would also like to thank Kiran Kedlaya for providing me an early version of [Kedlaya 2007] and for answers to my questions about that paper. The referee gave suggestions for additions and improvements to the paper that were incorporated and much appreciated.

This work was supported by a National Defense Science and Engineering Graduate Fellowship.

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Communicated by Hendrik W. Lenstra
Received 2007-11-28 Revised 2008-03-31 Accepted 2008-04-28

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