Algebraic & Geometric Topology 19 (2019) 1711–1745

A dynamical characterization of acylindrically hyperbolic groups

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We give a dynamical characterization of acylindrically hyperbolic groups. As an application, we prove that nonelementary convergence groups are acylindrically hyperbolic.

20F65; 20F67

1 Introduction

The notion of an acylindrically hyperbolic group was introduced by Osin [16]. A group is called *acylindrically hyperbolic* if it admits a nonelementary acylindrical action on a Gromov hyperbolic space (for details, see Section 3.1). Nonelementary hyperbolic and nonelementary relatively hyperbolic groups are acylindrically hyperbolic. Other examples include all but finitely many mapping class groups of punctured closed surfaces, outer automorphism groups of nonabelian free groups, many of the fundamental groups of graphs of groups, groups of deficiency at least two, etc (see Osin [17] for details and other examples).

Not only do acylindrically hyperbolic groups form a rich class, but they also enjoy various nice algebraic, geometric and analytic properties. For example, every acylindrically hyperbolic group G has nontrivial $H_b^2(G, \ell^2(G))$, which allows one to apply the Monod–Shalom rigidity theory [15] for measure-preserving actions. Using methods from Dahmani, Guirardel and Osin [6], one can also find hyperbolically embedded subgroups in acylindrically hyperbolic groups and then use group-theoretic Dehn surgery to prove various algebraic results (eg SQ-universality). Yet there is also a version of the small cancellation theory for acylindrically hyperbolic groups (see Hull [12]). For a brief survey on those topics we refer to Osin [16; 17].

The work of Bowditch [3], Freden [8] and Tukia [18] provides a dynamical characterization of nonelementary hyperbolic groups by means of the notion of convergence groups. An action of a group G on a metrizable topological space M is called a

Published: 16 August 2019 DOI: 10.2140/agt.2019.19.1711

convergence action (or G is called a *convergence group* acting on M) if the induced diagonal action of G on the space of distinct triples

$$\Theta_3(M)=\{(x_1,x_2,x_3)\in M \mid x_1\neq x_2,\ x_2\neq x_3,\ x_1\neq x_3\}$$

is properly discontinuous. Convergence groups were introduced by Gehring and Martin [9] in order to capture the dynamical properties of Kleinian groups acting on the ideal spheres of real hyperbolic spaces. Although the original paper refers only to actions on spheres, the notion of convergence groups can be generalized to general compact metrizable topological spaces or even compact Hausdorff spaces. Bowditch [3; 4] and Tukia [18] proved that nonelementary hyperbolic groups are precisely uniform convergence groups acting on perfect compact metrizable topological spaces. Later, a characterization of relatively hyperbolic groups was given by Yaman [21].

Inspired by the result of Bowditch and Tukia, we introduce condition (C) for group actions on topological spaces and use it to characterize acylindrically hyperbolic groups.

Definition 1.1 Given a group G acting by homeomorphisms on a topological space M which has at least 3 points, we consider the following condition (see Figure 1):

(C) For every pair of distinct points $u,v\in\Delta=\{(x,x)\mid x\in M\}$, there exist open sets U and V of the product topological space M^2, containing u and v, respectively, such that for every pair of distinct points $a,b\in M^2\setminus\Delta$, there exist open sets A and B of the product topological space M^2 (A and B are permitted to intersect Δ), containing a and b, respectively, with

$$\left|\{g\in G\mid gA\cap U\neq\varnothing,\ gB\cap V\neq\varnothing\}\right|<\infty.$$

Theorem 1.2 *A group G that is not virtually cyclic is acylindrically hyperbolic if and only if G admits an action on some completely Hausdorff topological space M satisfying (C) with an element $g\in G$ having north–south dynamics on M.*

Recall that a topological space M is called *completely Hausdorff* if for any two distinct points $u,v\in M$, there exist open sets U and V containing u and v, respectively, such that $\overline{U}\cap\overline{V}=\varnothing$. Also recall that a element $g\in G$ is said to have north–south dynamics on M if g fixes exactly two points $x\neq y$ of M and "translates" everything outside of x towards y (see Definition 3.5 for details).

It was established earlier that nonelementary convergence groups are not virtually cyclic and contain elements with north–south dynamics. Thus, by proving that every convergence action satisfies (C), we obtain the following corollary:

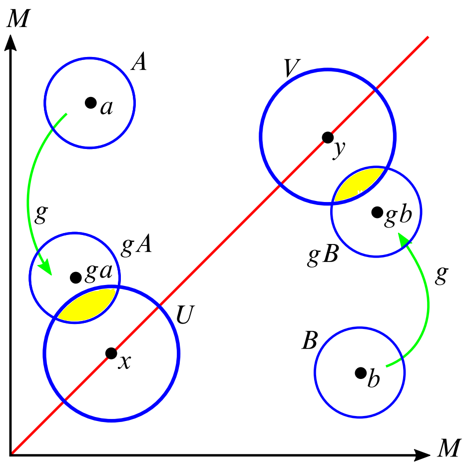

Figure 1: The (C) condition

Corollary 1.3 *Nonelementary convergence groups are acylindrically hyperbolic.*

Karlsson [13, Proposition 6] proved that if G is a finitely generated group whose Floyd boundary $\partial_F G$ has cardinality at least 3, then G acts on $\partial_F G$ by a nonelementary convergence action. Thus, as a further application of Theorem 1.2, we recover the following result:

Corollary 1.4 (Yang [22, Corollary 1]) *Every finitely generated group with Floyd boundary of cardinality at least* 3 *is acylindrically hyperbolic.*

The converse of Corollary 1.3 is not true, ie there exists an acylindrically hyperbolic group such that every convergence action of this group is elementary. In Section 7, we are going to prove that mapping class groups of closed orientable surfaces of genus at least 2 and noncyclic directly indecomposable right-angled Artin groups corresponding to connected graphs are examples of this kind.

For countable groups, applying a result of Balasubramanya [1], we show:

Theorem 1.5 *A countable group G that is not virtually cyclic is acylindrically hyperbolic if and only if G admits an action on the Baire space satisfying* (C) *and contains an element with north–south dynamics.*

Recall that the *Baire space* is the Cartesian product $\mathbb{N}^{\mathbb{N}}$ with the Tychonoff topology. Theorem 1.5 implies that acylindrical hyperbolicity of countable groups can be characterized by their actions on a particular space, the Baire space.

This paper is organized as follows. In Section 2–4, we survey some basic information about Gromov hyperbolic spaces, acylindrically hyperbolic groups and convergence groups. We introduce the notion of condition (C) in Section 5. In Section 6, we survey a construction due to Bowditch [3]. The proof of Theorem 1.2 is presented in Section 7 and separated into two parts. We first use geometric properties of Gromov hyperbolic spaces to prove that every acylindrically hyperbolic group is not virtually cyclic and admits an action satisfying (C) on a completely Hausdorff space with an element having north–south dynamics. The other direction of Theorem 1.2 is proved by using the construction of Bowditch. We also prove Theorem 1.5 and discuss Corollary 1.3 and its converse in Section 7.

Acknowledgements I would like to thank my supervisor, Denis Osin, for introducing me to the subject, for explaining his view on this topic, and for his proofreading for this paper. This paper would not have been written without his help. I would also like to thank the referee, who helped me make this article more precise and clear.

2 Gromov hyperbolic spaces

2.1 Definition

We start by recalling the well-known concept of a Gromov hyperbolic space. Suppose that (S,d) is a geodesic metric space with underlying space S and metric d. Let Δ be a geodesic triangle consisting of three geodesic segments γ_1, γ_2 and γ_3. For a number $\delta \geqslant 0$, Δ is called *δ–slim* if the distance between every point of γ_i and the union $\gamma_j \cup \gamma_k$ is less than δ, where $\{i,j,k\} = \{1,2,3\}$.

We say that (S,d) is a *δ–hyperbolic space* if geodesic triangles in S are all δ–slim. (S,d) is called a *Gromov hyperbolic space* if it is δ–hyperbolic for some $\delta \geqslant 0$. Gromov hyperbolic spaces generalize notions such as simplicial trees and complete simply connected Riemannian manifolds with constant negative sectional curvature while preserving most of their interesting properties (see Bridson and Haefliger [5] and Väisälä [19]).

Some notation When one refers to a metric space (S,d), usually there is no ambiguity of the metric d once the underlying space S is clarified. Thus, we will omit the metric and just use a single letter S to indicate a metric space whenever there is no ambiguity of the metric. Also, for every $x \in S$ and $r > 0$, we will use $B_S(x,r)$ to denote the open ball in S with x as its center and r as its radius.

Remark 2.1 In the literature, properness is often part of the definition of a Gromov hyperbolic space. However, in this article, we do not assume that a Gromov hyperbolic space S is proper, ie some closed balls of S might not be compact.

We will use the notation $[s,t]$ to denote a geodesic segment between two points $s,t \in S$. Note that such a geodesic may not be unique. Thus, by $[s,t]$, we mean that we choose one geodesic between $s,t \in S$ and $[s,t]$ will only denote this chosen geodesic. We might specify our choice if necessary, but in most cases we will not do so and just choose an arbitrary geodesic implicitly.

2.2 Gromov product and Gromov boundary

We recall the notions of Gromov products and Gromov boundaries. Our main references are Bridson and Haefliger [5] and Väisälä [19]. We shall also prove certain properties of these objects which will be useful later in this article.

Let S be a δ–hyperbolic space. The Gromov product of x and y with respect to z, denoted by $(x \cdot y)_z$, where $x, y, z \in S$, is defined by

$$(x \cdot y)_z = \tfrac{1}{2}(d(x,z) + d(y,z) - d(x,y)).$$

One can reformulate Gromov hyperbolicity by using the Gromov product. In particular, we will use the following inequality many times later in this article:

$$(x \cdot y)_w \geqslant \min\{(x \cdot z)_w, (y \cdot z)_w\} - 4\delta \quad \text{for all } x, y, z, w \in S. \tag{1}$$

It can be easily extracted from the proofs of Propositions 1.17 and 1.22 in Chapter III.H of [5].

Define the Gromov boundary ∂S of S as follows: Pick a point $e \in S$. A sequence of points $\{s_n\}_{n\geqslant 1} \subset S$ is called *converging to* ∞ if $(s_i \cdot s_j)_e \to \infty$ as i and j tend to ∞. We say that two sequences $\{x_n\}_{n\geqslant 1}$ and $\{y_n\}_{n\geqslant 1}$ converging to ∞ are *equivalent* and write $\{x_n\}_{n\geqslant 1} \sim \{y_n\}_{n\geqslant 1}$ if $(x_n \cdot y_n)_e \to \infty$ as $n \to \infty$. It follows from (1) that $\sim$ is indeed an equivalence relation. The *Gromov boundary* ∂S is then defined as the set of all sequences in S converging to ∞ modulo the equivalence relation $\sim$. Elements of ∂S are equivalence classes of sequences in S converging to ∞ and we say that a sequence $\{x_n\}_{n\geqslant 1} \in S$ tends to a boundary point $x \in \partial S$ and write $x_n \to x$ as $n \to \infty$ if $\{x_n\}_{n\geqslant 1} \in x$.

The definition of the Gromov product can be extended to $S \cup \partial S$. Given $x, y \in S \cup \partial S$, if $x \in S$, $y \in \partial S$, define $(x \cdot y)_e$ by

$$(x \cdot y)_e = \inf\bigl\{\liminf_{n\to\infty} (x \cdot y_n)_e\bigr\},$$

where the infimum is taken over all sequences $\{y_n\}_{n\geqslant 1}$ tending to y; if $x \in \partial S$ and $y \in S$, then we define $(x \cdot y)_e$ by flipping the role of x and y in the last equality; finally, if $x, y \in \partial S$, define $(x \cdot y)_e$ by

$$(x \cdot y)_e = \inf\bigl\{\liminf_{i,j\to\infty} (x_i \cdot y_j)_e\bigr\},$$

where the infimum is taken over all sequences $\{x_n\}_{n\geqslant 1}$ tending to x and $\{y_n\}_{n\geqslant 1}$ tending to y.

Given a positive number ζ, for $s, t \in \partial S$, let

$$d'(s,t) = \exp(-\zeta (s \cdot t)_e), \quad \rho(s,t) = \inf \sum_{k=1}^{n} d'(s_k, s_{k+1}),$$

where the infimum is taken over all finite sequences $s = s_1, s_2, \dots, s_{n+1} = t$. By [19, Proposition 5.16], if ζ is small enough, ρ will be a metric for ∂S and d' and ρ will satisfy

$$\tfrac{1}{2} d'(s,t) \leqslant \rho(s,t) \leqslant d'(s,t) \quad \text{for all } s,t \in \partial S. \tag{2}$$

From now on we will fix a sufficiently small ζ such that ρ is a metric and that (2) holds.

Remark 2.2 We construct ∂S with the help of a chosen point e, but the Gromov boundary does not depend on the choice, ie we can pick another point $e' \in S$ and use the same procedure to produce a Gromov boundary of S with respect to e'. The two resulting boundaries can be naturally identified.

Note that ρ induces a topology τ on ∂S. While ρ does depend on the point e and the constant ζ we choose, τ is independent of those choices and thus we get a canonical topology on ∂S. In the sequel, the topological concepts of ∂S (for example, open sets) are the ones with respect to this canonical topology.

For $x \in S$ and $K \in \mathbb{R}$, we employ the notation

$$U_K(x) = \{s \in S \mid (x \cdot s)_e > K\}.$$

Also recall that $B_S(x,r)$ denotes the open ball in S centered at x with radius r and that $[u,v]$ denotes a geodesic segment between $u, v \in S$.

The following estimates, Lemmas 2.3–2.6, are well-known properties of hyperbolic spaces and Gromov products. For proofs, the readers are referred to [19].

Lemma 2.3 *Let x and y be two distinct points of ∂S. Then there exists $K > 0$ such that for every $u \in U_K(x)$ and $v \in U_K(y)$, we have $|(u \cdot v)_e - (x \cdot y)_e| < 12\delta$.*

Lemma 2.4 *Let u and v be two points of S. Then $d(e,[u,v]) - 8\delta \leqslant (u \cdot v)_e \leqslant d(e,[u,v])$.*

A direct consequence of Lemma 2.4 is:

Lemma 2.5 *Let u and v be two points of S and let $w \in [u,v]$; then $(u \cdot w)_e \geqslant (u \cdot v)_e - 8\delta$.*

Combine Lemmas 2.3 and 2.4:

Lemma 2.6 *Let x and y be two distinct points of ∂S. Then there exists $K > 0$ such that $|d(e,[u,v]) - (x \cdot y)_e| < 20\delta$ for every $u \in U_K(x)$ and $v \in U_K(y)$.*

Lemma 2.7 *Let x and y be two points of ∂S such that $(x \cdot y)_e > K$ for some number K. Suppose $\{x_n\}_{n \geq 1}$ is a sequence in S tending to x. Then there exists $N > 0$ such that $(x_n \cdot y)_e > K$ for all $n \geqslant N$.*

Proof Fix $\epsilon > 0$ such that $(x \cdot y)_e > K + \epsilon$. Let $\{y_n\}_{n \geq 1}$ be any sequence in S tending to y. By the definition of $(x \cdot y)_e$,

$$\liminf_{m,n \to \infty} (x_m \cdot y_n)_e \geqslant (x \cdot y)_e > K + \epsilon.$$

Thus, there exists $N > 0$ such that $(x_n \cdot y_m)_e > K + \epsilon$ for all $m, n \geqslant N$. In particular,

$$\liminf_{m \to \infty} (x_n \cdot y_m)_e \geqslant K + \epsilon$$

for all $n \geqslant N$.

As the above inequality holds for any sequence $\{y_n\}_{n \geq 1}$ tending to y, we have $(x_n \cdot y) \geqslant K + \epsilon > K$ for all $n \geqslant N$. □

Lemma 2.8 *Let x and y be two distinct points of ∂S. Then there exist $D, K > 0$ such that for every $u \in U_K(x)$ and $v \in U_K(y)$, we have $d(e,[u,v]) < D$.*

Proof Since $x \neq y$, there exists $D > 0$ such that $(x \cdot y)_e < D - 20\delta$. By Lemma 2.6, we can pick $K > 0$ large enough that $d(e, [u, v]) < D$ for every $u \in U_K(x), v \in U_K(y)$. □

Lemma 2.9 *Let x be a point of ∂S. Then, for every $R > 0$, there exists $K > 0$ such that $d(e, U_K(x)) > R$.*

Proof We only need to prove that for every $R > 0$, $(x \cdot z)_e < R$ for all $z \in B_S(e, R)$. Fix any $z \in B_S(e, R)$. Let $\{x_n\}_{n\geqslant 1}$ be any sequence in S tending to x as $n \to \infty$. By Lemma 2.4, $\liminf_{n\to\infty}(x_n \cdot z)_e \leqslant \liminf_{n\to\infty} d(e, [x_n, z]) \leqslant d(e, z) < R$. As $\{x_n\}_{n\geqslant 1}$ is arbitrary, we obtain $(x \cdot z)_e < R$. □

Lemma 2.10 *Let x be a point of ∂S. Then, for every $R > 0$, there exists $K > 0$ such that for every $u_1, u_2 \in U_K(x)$, we have $[u_1, u_2] \subset U_R(x)$.*

Proof Let $K = R + 17\delta$ and let u_1 and u_2 be two points of $U_K(x)$. We first prove that $(u_1 \cdot u_2)_e > R + 13\delta$. Let $\{x_n\}_{n\geqslant 1}$ be any sequence in S tending to x. By (1),

$$(u_1 \cdot u_2)_e \geqslant \min\{(u_1 \cdot x_n)_e, (u_2 \cdot x_n)_e\} - 4\delta$$

for all n. Pass to a limit and we obtain $(u_1 \cdot u_2) > K - 4\delta = R + 13\delta$.

Let t be any point of $[u_1, u_2]$. As $(u_1 \cdot u_2)_e > R + 13\delta$, we have $(u_1 \cdot t)_e > R + 5\delta$ by Lemma 2.5. By (1) again,

$$(t \cdot x_n)_e \geqslant \min\{(t \cdot u_1)_e, (u_1 \cdot x_n)_e\} - 4\delta$$

for all n. By passing to a limit and by the arbitrariness of $\{x_n\}_{n\geqslant 1}$, we obtain $(t \cdot x)_e \geqslant R + \delta > R$ and thus $t \in U_R$. □

Lemma 2.11 *Let x and y be two distinct points of ∂S. Then, for every $R > 0$, there exists $K > 0$ such that for every $u \in U_K(x)$ and $v \in U_K(y)$, we have $d(u, v) > R$.*

Proof Given any $R > 0$, by Lemmas 2.8 and 2.9, if K is large enough, we will have $d(e, [u, v]) < D$ and that $d(e, u) > R + D$ for every $u \in U_K(x), v \in U_K(y)$. Fix one such K and let $u \in U_K(x)$ and $v \in U_K(y)$. Select $t \in [u, v]$ such that $d(e, t) = d(e, [u, v])$ by the compactness of $[u, v]$. Then $d(u, v) \geqslant d(u, t) \geqslant d(u, e) - d(e, t) > R$, as desired. □

Proposition 2.12 *Let x and y be two distinct points of ∂S. Then, for every $R>0$, there exists $K>0$ such that for every $u_1,u_2\in U_K(x)$ and every $v_1,v_2\in U_K(y)$, we have $d([u_1,u_2],[v_1,v_2])>R$.*

Proof Given any $R>0$, by Lemma 2.11, there exists $K'>0$ such that for every $u\in U_{K'}(x)$ and $v\in U_{K'}(y)$, we have $d(u,v)>R$. By Lemma 2.10, there exists $K>0$ such that $[u_1,u_2]\subset U_{K'}(x)$ and $[v_1,v_2]\subset U_{K'}(y)$ for every $u_1,u_2\in U_K(x)$ and $v_1,v_2\in U_K(y)$. It follows that $d([u_1,u_2],[v_1,v_2])>R$ for every $u_1,u_2\in U_K(x)$ and $v_1,v_2\in U_K(y)$. □

Lemma 2.13 *Let x and y be two distinct points of ∂S. Then there exists $D>0$ with the following property*:

For every $K>D$, there exists $R>0$ such that for every $u\in U_R(x)$, $v\in U_R(y)$ and every $t\in[u,v]\setminus B_S(e,K)$, we have

$$\max\{(t\cdot x)_e,(t\cdot y)_e\}>K-D-12\delta.$$

Proof Use Lemma 2.8 and pick $D>0$ and $R>K$ such that $d(e,[u,v])<D$ for every $u\in U_R(x)$ and $v\in U_R(y)$. Fix $u\in U_R(x)$ and $v\in U_R(y)$. Let $t\in[u,v]\setminus B_S(e,K)$, let $[t,u]$ (resp. $[t,v]$) be the subgeodesic of $[u,v]$ from u to t (resp. from t to v) and pick $s\in[u,v]$ such that $d(e,s)=d(e,[u,v])$ by the compactness of $[u,v]$. Without loss of generality, we may assume that $s\notin[t,u]$.

We prove that $[t,u]\cap B_S(e,K-D)=\varnothing$ by contradiction. Suppose there is some z belonging to $[t,u]\cap B_S(e,K-D)$. As $t\notin B_S(e,K)$, we have

$$d(t,z)\geqslant d(t,e)-d(e,z)>D.$$

As $d(e,t)>K$ and $d(e,s)<D$, we have $d(s,t)\geqslant d(e,t)-d(e,s)=K-D$. Thus,

$$d(s,z)=d(s,t)+d(t,z)>K-D+D=K.$$

But $d(s,z)\leqslant d(s,e)+d(e,z)<D+K-D=K$, a contradiction.

Apply Lemma 2.4 and we see that $(t\cdot u)_e>K-D-8\delta$. Let $\{x_n\}_{n\geqslant 1}$ be a sequence in S tending to x. By (1), $(t\cdot x_n)_e\geqslant\min\{(u\cdot x_n)_e,(t\cdot u)_e\}-4\delta$ for all n. Pass to a limit and we obtain $(t\cdot x)_e>\min\{R,K-D-8\delta\}-4\delta=K-D-12\delta$. □

Lemma 2.14 *Let x and y be two points of ∂S. Then, for $K>0$, $u_1,u_2\in U_{K+6\delta}(x)$ and $v_1,v_2\in U_{K+6\delta}(y)$, $[u_1,v_1]\cap B_S(e,K)$ lies inside the 2δ–neighborhood of $[u_2,v_2]$.*

Proof Given any $K>0$, fix $u_1,u_2\in U_{K+6\delta}(x)$ and $v_1,v_2\in U_{K+6\delta}(y)$. Let $\{x_n\}_{n\geqslant 1}$ be a sequence in S tending to x. By (1), $(u_1\cdot u_2)_e\geqslant\{(u_1\cdot x_n)_e,(u_2\cdot x_n)_e\}-4\delta$ for all n. Pass to a limit and we obtain $(u_1\cdot u_2)>K+2\delta$. By Lemma 2.4, $d(e,[u_1,u_2])>K+2\delta$. Similarly, $d(e,[v_1,v_2])>K+2\delta$.

Consider the geodesic quadrilateral $[u_1,u_2]$, $[u_2,v_2]$, $[v_2,v_1]$, $[v_1,u_1]$. By hyperbolicity, $B_S(e,K)\cap[u_1,v_1]$ lies inside the 2δ–neighborhood of $[u_1,u_2]\cup[u_2,v_2]\cup[v_2,v_1]$. Since $d(e,[u_1,u_2])>K+2\delta$, we have $d([u_1,u_2],B_S(e,K))>2\delta$ by the triangle inequality. Likewise, $d([v_1,v_2],B_S(e,K))>2\delta$. It follows that $[u_1,v_1]\cap B_S(e,K)$ lies inside the 2δ–neighborhood of $[u_2,v_2]$. □

Lemma 2.15 *Let x, y and z be three distinct points of ∂S. Then, for every $K>0$, there exists $R>0$ such that for every $u\in U_R(x)$, $v\in U_R(y)$ and $w\in U_R(z)$, we have $d(w,[u,v])>K$.*

Proof By Lemmas 2.8 and 2.11, there exists $D>0$ with the following property: Given any $K>0$, there exists $R'>0$ such that

$$\max\{d(e,[u,w]),d(e,[v,w])\}<D,\quad \min\{d(u,w),d(v,w)\}>K$$

for all $u\in U_{R'}(x)$, $v\in U_{R'}(y)$ and $w\in U_{R'}(z)$. By Lemmas 2.9 and 2.13, there exists $R>R'$ such that

$$[u,v]\setminus B_S(e,R'+D+12\delta)\subset U_{R'}(x)\cup U_{R'}(y),\quad d(w,B_S(e,R'+D+12\delta))>K$$

for all $u\in U_R(x)$, $v\in U_R(y)$ and $w\in U_R(z)$.

Fix arbitrary $u\in U_R(x)$, $v\in U_R(y)$ and $w\in U_R(z)$. We verify that $d(w,[u,v])>K$. Pick $t\in[u,v]$ such that $d(w,t)=d(w,[u,v])$ by the compactness of $[u,v]$. By our choice of R, either $t\in U_{R'}(x)\cup U_{R'}(y)$ or $t\in B_S(e,R'+D+12\delta)$. If $t\in U_{R'}(x)$ or $U_{R'}(y)$, then $d(w,[u,v])=d(w,t)>K$ by our choice of R'. If $t\in B_S(e,R'+D+12\delta)$, we will still have $d(w,[u,v])=d(w,t)>K$ by our choice of R. □

Proposition 2.16 *Let x, y and z be three distinct points of ∂S. Then for every $K>0$, there exists $R>0$ such that for every $u\in U_R(x)$, $v\in U_R(y)$ and $w_1,w_2\in U_R(z)$, we have $d([u,v],[w_1,w_2])>K$.*

Proof Given any $K>0$, by Lemma 2.15, there exists $R'>0$ such that $d(w,[u,v])>K$ for every $u\in U_{R'}(x)$, $v\in U_{R'}(y)$ and $w\in U_{R'}(z)$. By Lemma 2.10, there exists

$R > 0$ such that $[w_1, w_2] \subset U_{R'}(z)$ for every $w_1, w_2 \in U_R(z)$. It follows that $d([u,v],[w_1,w_2]) > K$ for every $u \in U_R(x)$, $v \in U_R(y)$ and $w_1, w_2 \in U_R(z)$. $\square$

Lemma 2.17 *Let u and v be two points of S. Select a geodesic $[u,v]$ connecting u and v and let $T = \{z \in [u,v] \mid d(e,z) \leqslant d(e,[u,v]) + 42\delta\}$. Then the diameter of T is at most 88δ.*

Proof Suppose, to the contrary, that there exists $x, y \in T$ such that $d(x,y) > 88\delta$. Let $[x,y]$ be the subgeodesic of $[u,v]$ between x and y. Let t be the midpoint of $[x,y]$. Obviously, both $d(x,t)$ and $d(y,t)$ are strictly greater than 44δ.

Consider the geodesic triangle $[x,e]$, $[e,y]$, $[x,y]$. There is a point $w \in [x,e] \cup [e,y]$ such that $d(t,w) < \delta$. If $w \in [x,e]$, then, since $d(x,t) > 44\delta$, $d(x,w) > 44\delta - \delta > 43\delta$ by the triangle inequality, hence

$$d(t,e) \leqslant d(t,w) + d(w,e) < \delta + d(e,[u,v]) + 42\delta - 43\delta < d(e,[u,v]).$$

Similarly, if $w \in [e,y]$, then the same argument with y in place of x shows that $d(y,e) < d(e,[u,v])$. Either case contradicts the definition of $d(e,[u,v])$. $\square$

Proposition 2.18 *Let $\{p_n\}_{n \geqslant 1}$, $\{q_n\}_{n \geqslant 1}$, $\{r_n\}_{n \geqslant 1}$ and $\{s_n\}_{n \geqslant 1}$ be sequences in S tending to four distinct boundary points p, q, r and s, respectively. For each n, choose a point a_n (resp. b_n) in $[p_n, q_n]$ (resp. $[r_n, s_n]$) such that $d(e, a_n) = d(e, [p_n, q_n])$ (resp. $d(e, b_n) = d(e, [r_n, s_n])$) by the compactness of $[p_n, q_n]$ (resp. $[r_n, s_n]$).*

If m and n are large enough, $[a_m, b_m]$ will be in the 92δ–neighborhood of $[a_n, b_n]$.

Proof By Lemma 2.6, there exists N_1 such that if $n > N_1$,

$$|d(e,[p_n,q_n]) - (p \cdot q)_e| < 20\delta.$$

There exists N_2 such that if $m, n > N_2$, both $d(e,[p_m,p_n])$ and $d(e,[q_m,q_n])$ will be strictly greater than $(p \cdot q)_e + 22\delta$, by the fact that $\{p_n\}_{n \geqslant 1}$ and $\{q_n\}_{n \geqslant 1}$ are sequences tending to ∞ and Lemma 2.4. Let $m, n > \max\{N_1, N_2\}$ and consider the geodesic quadrilateral consisting of the four sides $[p_m, q_m]$, $[q_m, q_n]$, $[q_n, p_n]$ and $[p_n, p_m]$. There is a point $a_{m,n} \in [p_m, q_m] \cup [q_m, q_n] \cup [p_n, p_m]$ such that $d(a_{m,n}, a_n) < 2\delta$. Since

$$d(e, a_{m,n}) \leqslant d(e, a_n) + 2\delta \leqslant (p \cdot q)_e + 22\delta,$$

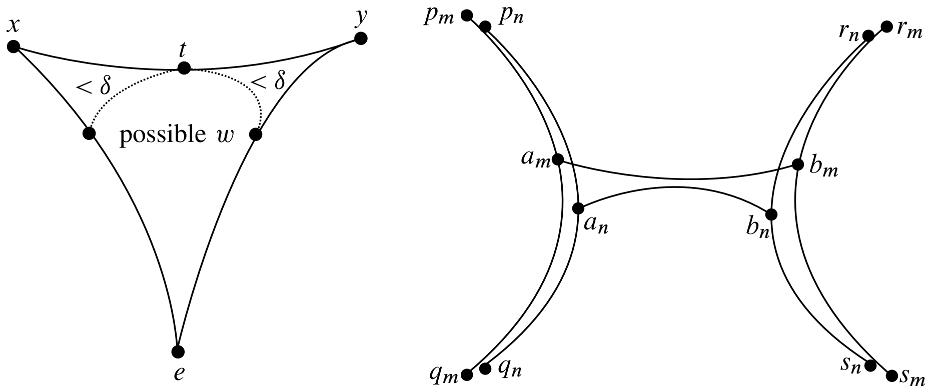

Figure 2: Ideas behind Lemma 2.17 and Proposition 2.18. Left: the estimate of T. Right: as n increases, $[a_n, b_n]$ is stabilized.

we have $a_{m,n} \in [p_m, q_m]$. We already know that both $|d(e, [p_n, q_n]) - (p \cdot q)_e|$ and $|d(e, [p_m, q_m]) - (p \cdot q)_e|$ are less than or equal to 20δ. Therefore,

$$|d(a_n, e) - d(e, [p_m, q_m])| = |d(e, [p_n, q_n]) - d(e, [p_m, q_m])| \leqslant 40\delta.$$

The triangle inequality implies that

$$d(a_{m,n}, e) - d(e, [p_m, q_m]) \leqslant d(a_{m,n}, a_n) + d(a_n, e) - d(e, [p_m, q_m]) \leqslant 42\delta.$$

By Lemma 2.17, $d(a_{m,n}, a_m) \leqslant 88\delta$; thus, $d(a_m, a_n) \leqslant (88 + 2)\delta = 90\delta$. Similarly, there exists $N_3 > 0$ such that if $m, n > N_3$, then $d(b_m, b_n) \leqslant 90\delta$. Now let

$$m, n > \max\{N_1, N_2, N_3\}$$

and consider the geodesic quadrilateral $[a_m, a_n]$, $[a_n, b_n]$, $[b_n, b_m]$, $[b_m, a_m]$. Every point of $[a_m, b_m]$ is 2δ–close to a point in the union of the other three sides, which is 90δ–close to $[a_n, b_n]$, thus $[a_m, b_m]$ is in the 92δ–neighborhood of $[a_n, b_n]$. □

3 Group actions on Gromov hyperbolic spaces

3.1 Acylindrically hyperbolic groups

Let (S, d) be a Gromov hyperbolic space and let G be a group acting on S by isometries. The action of G is called *acylindrical* if for every $\epsilon > 0$ there exist $R, N > 0$ such that for every two points x and y with $d(x, y) \geqslant R$, there are at most

N elements $g \in G$ satisfying both $d(x, gx) \leqslant \epsilon$ and $d(y, gy) \leqslant \epsilon$. The *limit set* $\Lambda(G)$ of G on ∂S is the set of limit points in ∂S of a G–orbit in S, ie

$$\Lambda(G) = \{x \in \partial S \mid \text{there exists a sequence in } Gs \text{ tending to } x \text{ for some } s \in S\}.$$

If $\Lambda(G)$ contains at least three points, we say the action of G is *nonelementary*. Acylindrically hyperbolic groups are defined by Osin [16]:

Definition 3.1 A group G is called *acylindrically hyperbolic* if it admits a nonelementary acylindrical action by isometries on a Gromov hyperbolic space.

Theorem 3.2 (Osin [16, Theorem 1.2]) *For a group G, the following are equivalent:*

(AH_1) *G admits a nonelementary acylindrical and isometric action on a Gromov hyperbolic space.*

(AH_2) *G is not virtually cyclic and admits an isometric action on a Gromov hyperbolic space such that at least one element of G is loxodromic and satisfies the WPD condition.*

Recall that an element $g \in G$ is called *loxodromic* if the map $\mathbb{Z} \to S$, $n \mapsto g^n s$, is a quasi-isometric embedding for some (equivalently, any) $s \in S$. The WPD condition, due to Bestvina and Fujiwara [2], is defined as follows:

Definition 3.3 Let G be a group acting isometrically on a Gromov hyperbolic space (S, d) and let g be an element of G. One says that g satisfies the *weak proper discontinuity* condition (or g is a *WPD* element) if for every $\epsilon > 0$ and every $s \in S$, there exists $K \in \mathbb{N}$ such that

$$\left|\{h \in G \mid d(s, hs) < \epsilon,\ d(g^K s, hg^K s) < \epsilon\}\right| < \infty.$$

In fact, g satisfies the WPD condition for every s if and only if g satisfies the same condition for just one $s \in S$. More precisely, let us consider the following condition:

($\star$) There is a point $s \in S$ such that for every $\epsilon > 0$, there exists $K \in \mathbb{N}$ with

$$\left|\{h \in G \mid d(s, gs) < \epsilon, d(g^K s, hg^K s) < \epsilon\}\right| < \infty.$$

Lemma 3.4 *Let G be a group acting isometrically on a Gromov hyperbolic space (S, d) and let g be an element of G; then g satisfies the WPD condition if and only if g satisfies ($\star$).*

Proof Clearly, WPD implies $(\star)$. On the other hand, suppose that g satisfies $(\star)$ for some point $s_0 \in S$, but g does not satisfy the WPD condition. Thus, there is some $\epsilon_1 > 0$ and $s_1 \in S$ such that for every $K \in \mathbb{N}$, we have

$$\left|\{h \in G \mid d(s_1, hs_1) < \epsilon_1,\ d(g^K s_1, hg^K s_1) < \epsilon_1\}\right| = \infty.$$

Let $\epsilon = 2d(s_0, s_1) + \epsilon_1$ and let K_0 be an integer such that

$$\left|\{h \in G \mid d(s_0, hs_0) < \epsilon,\ d(g^{K_0} s_0, hg^{K_0} s_0) < \epsilon\}\right| < \infty. \tag{3}$$

For any element $h \in G$, if $d(s_1, hs_1) < \epsilon_1$, then

$$d(s_0, hs_0) \leqslant d(s_0, s_1) + d(s_1, hs_1) + d(hs_1, hs_0) < \epsilon.$$

Similarly, if h is an element in G such that $d(g^{K_0} s_1, hg^{K_0} s_1) < \epsilon_1$, then

$$d(g^{K_0} s_0, hg^{K_0} s_0) < \epsilon.$$

As $\left|\{h \in G \mid d(s_1, hs_1) < \epsilon_1,\ d(g^{K_0} s_1, hg^{K_0} s_1) < \epsilon_1\}\right| = \infty$, it follows that

$$\left|\{h \in G \mid d(s_0, hs_0) < \epsilon,\ d(g^{K_0} s_0, hg^{K_0} s_0) < \epsilon\}\right| = \infty.$$

This contradicts inequality (3). □

3.2 Induced actions on Gromov boundaries

Let G be a group acting isometrically on a Gromov hyperbolic space (S, d). As mentioned in Section 2.2, the Gromov boundary ∂S of S is defined via sequences of points in S tending to ∞ and there is a canonical topology for ∂S. Note that G maps one sequence tending to ∞ to another such sequence, so it naturally acts on ∂S and this action is by homeomorphisms (see Väisälä [19] for details).

If an element $g \in G$ is loxodromic, then $\{g^{-n} e\}_{n \geqslant 1}$ and $\{g^n e\}_{n \geqslant 1}$ are two sequences in S tending to different boundary points $x, y \in \partial S$, respectively, and g fixes these boundary points. Moreover, g actually has the so-called north–south dynamics on ∂S. This is well known when the space S is proper. Nevertheless, the original idea of Gromov [10] works even for nonproper spaces. The readers are referred to Hamann [11] for a detailed proof.

Definition 3.5 Let G be a group acting by homeomorphisms on a topological space M. We say an element $g \in G$ has *north–south dynamics* on M if the following two conditions are satisfied:

(1) g fixes exactly two distinct points $x, y \in M$.

(2) For every pair of open sets U and V containing x and y, respectively, there exists $N > 0$ such that $g^n(M \setminus U) \subset V$ for all $n > N$.

Lemma 3.6 (Hamann [11, Proposition 3.4]) *Suppose that a group G acts isometrically on a Gromov hyperbolic space S and has a loxodromic element g. Let ∂S be the Gromov boundary of S with the topology defined in Section 2.2. Then, with respect to the action of G on ∂S (induced by the action of G on S), g has north–south dynamics.*

4 Convergence groups

Let G be a group acting on a compact metrizable topological space M by homeomorphisms (with respect to the topology induced by the metric d). We assume that both G and M are infinite sets since otherwise the notion of convergence groups will be trivial. G is called a *discrete convergence group* if for every infinite sequence $\{g_n\}_{n \geqslant 1}$ of distinct elements of G, there exists a subsequence $\{g_{n_k}\}$ and points $a, b \in M$ such that $g_{n_k}|_{M \setminus \{a\}}$ converges to b locally uniformly, that is, for every compact set $K \subset M \setminus \{a\}$ and every open neighborhood U of b, there is an $N \in \mathbb{N}$ such that $g_{n_k}(K) \subset U$ whenever $n_k > N$. In what follows, when we say a group G is a convergence group, we always mean that G is a discrete convergence group, and we will call the action of G on M a convergence action.

An equivalent definition of a convergence action can be formulated in terms of the action on the space of distinct triples. Let

$$\Theta_3(M) = \{(x_1, x_2, x_3) \in M \mid x_1 \neq x_2,\ x_2 \neq x_3,\ x_1 \neq x_3\}$$

be the set of distinct triples of points in M, endowed with the subspace topology induced by the product topology of M^3. Notice that $\Theta_3(M)$ is noncompact with respect to this topology. Clearly, the action of G on M naturally induces an action of G on $\Theta_3(M)$ given by $(x_1, x_2, x_3) \to (gx_1, gx_2, gx_3)$ for all $g \in G$.

Proposition 4.1 (Bowditch [4, Proposition 1.1]) *The action of G on M is a convergence action if and only if the action of G on $\Theta_3(M)$ is properly discontinuous, that is, for every compact set $K \subset \Theta_3(M)$, there are only finitely many elements $g \in G$ such that $gK \cap K \neq \varnothing$.*

Remark 4.2 Let G be a convergence group acting on a compact metrizable topological space M. Elements of G can be classified into the following three types:

- **Elliptic** Having finite order.
- **Parabolic** Having infinite order and fixing a unique point of M.
- **Loxodromic** Having infinite order and fixing exactly two points of M.

Moreover, a parabolic element cannot share its fixed point with a loxodromic element (see Tukia [18, Theorem 2G]).

A convergence group G is called *elementary* if it preserves setwise a nonempty subset of M with at most two elements. The next theorem is a combination of [18, Theorems 2S, 2U and 2T]:

Theorem 4.3 (Tukia, 1994) *If G is a nonelementary convergence group acting on a compact metrizable topological space M, the following statements hold:*

(1) *G contains a nonabelian free group as its subgroup and thus cannot be virtually abelian.*

(2) *There is an element $g \in G$ having north–south dynamics on M.*

For more information on convergence groups, the readers are referred to Bowditch [4] and Tukia [18].

5 The (C) condition

In this section, we prove some properties of condition (C) (see Definition 1.1).

Lemma 5.1 *Let G be a convergence group acting on a compact metrizable topological space M. Then this action satisfies (C).*

Proof Let $u = (x, x)$ and $v = (y, y)$ be two distinct points on the diagonal Δ of M^2 (hence $x, y \in M$ and $x \neq y$), let d be a metric on M compatible with its topology, and let

$$U = B_M\big(x, \tfrac{1}{3}d(x, y)\big) \times B_M\big(x, \tfrac{1}{3}d(x, y)\big),$$
$$V = B_M\big(y, \tfrac{1}{3}d(x, y)\big) \times B_M\big(y, \tfrac{1}{3}d(x, y)\big).$$

Then U and V are open sets in M^2 containing u and v, respectively. Let us check (C) for U and V.

Let $a=(p,q)$ and $b=(r,s)$ be two distinct points of $M^2\setminus\Delta$ (hence $p,q,r,s\in M$, $p\neq q$ and $r\neq s$) and let

$$A=B_M\big(p,\tfrac{1}{3}d(p,q)\big)\times B_M\big(q,\tfrac{1}{3}d(p,q)\big),\quad B=B_M\big(r,\tfrac{1}{3}d(r,s)\big)\times B_M\big(s,\tfrac{1}{3}d(r,s)\big).$$

Then A and B are open sets in M^2 containing a and b, respectively. Suppose that (C) does not hold for U and V. Then there exists an infinite sequence $\{g_n\}_{n=1}^{\infty}$ of distinct elements of G such that $g_nA\cap U\neq\varnothing$ and $g_nB\cap V\neq\varnothing$ for all $n\geqslant 1$. In other words,

$$g_n^{-1}U\cap A\neq\varnothing,\quad g_n^{-1}V\cap B\neq\varnothing$$

for all $n\geqslant 1$.

Consider the infinite sequence $\{g_n^{-1}\}_{n=1}^{\infty}$ of distinct elements of G. By the convergence property and passing to a subsequence, one may assume that there exist two points $z,t\in M$ such that $g_n^{-1}|_{M\setminus\{z\}}$ converges to t locally uniformly. By the triangle inequality, we have $d(z,x)+d(z,y)\geqslant d(x,y)$, and thus at least one of $d(z,x)$ and $d(z,y)$ is strictly greater than $\frac{1}{3}d(x,y)$. Without loss of generality, we may assume that $d(z,x)>\frac{1}{3}d(x,y)$. As $g_n^{-1}|_{M\setminus\{z\}}$ converges to t locally uniformly, there exists a positive integer N such that

$$g_N^{-1}\big(B_M\big(x,\tfrac{1}{3}d(x,y)\big)\big)\subset B_M\big(t,\tfrac{1}{6}d(p,q)\big).$$

Note that $g_N^{-1}U\cap A\neq\varnothing$. As a consequence, one has

$$g_N^{-1}\big(B_M\big(x,\tfrac{1}{3}d(x,y)\big)\big)\cap B_M\big(p,\tfrac{1}{3}d(p,q)\big)\neq\varnothing,$$
$$g_N^{-1}\big(B_M\big(x,\tfrac{1}{3}d(x,y)\big)\big)\cap B_M\big(q,\tfrac{1}{3}d(p,q)\big)\neq\varnothing,$$

and thus

$$B_M\big(t,\tfrac{1}{6}d(p,q)\big)\cap B_M\big(p,\tfrac{1}{3}d(p,q)\big)\neq\varnothing, \tag{4}$$
$$B_M\big(t,\tfrac{1}{6}d(p,q)\big)\cap B_M\big(q,\tfrac{1}{3}d(p,q)\big)\neq\varnothing. \tag{5}$$

Now, (4) (resp. (5)) implies that $d(t,p)<\frac{1}{6}d(p,q)+\frac{1}{3}d(p,q)=\frac{1}{2}d(p,q)$ $\big($resp. $d(t,q)<\frac{1}{6}d(p,q)+\frac{1}{3}d(p,q)=\frac{1}{2}d(p,q)\big)$. Thus $d(p,q)\leqslant d(t,p)+d(t,q)<\frac{1}{2}d(p,q)+\frac{1}{2}d(p,q)=d(p,q)$, a contradiction. □

Remark 5.2 Let G be a group acting on a topological space M satisfying condition (C). In order to prepare for the proof of Theorem 1.2, let us reformulate Definition 1.1 in terms of the action of G on M instead of M^2. Let $u=(x,x)$ and $v=(y,y)$ be two distinct points on the diagonal Δ of M^2 (hence $x,y\in M$ and $x\neq y$). Condition (C) requires the existence of open sets U and V in M^2

containing u and v, respectively, with certain properties. By shrinking U and V if necessary, let us assume that $U = X \times X$ and $V = Y \times Y$, where X and Y are open sets in M containing x and y, respectively. Suppose that a and b are two distinct points of $M^2 \setminus \Delta$. There are several cases to consider.

Case 1 The coordinates of a and b involve only two distinct points of M, ie $a = (p, q)$ and $b = (q, p)$, where p and q are distinct points of M (note that a and b are different points of M^2 as M^2 is the set of ordered pairs of M).

Condition (C) asserts the existence of open sets A and B in M^2 containing a and b, respectively, with certain properties. By shrinking A and B if necessary, we may assume that $A = A_1 \times A_2$ and $B = A_2 \times A_1$, where A_1 and A_2 are open sets in M containing p and q, respectively. Then (C) can be rephrased as

$$\left|\{g \in G \mid gA_1 \cap X,\ gA_1 \cap Y,\ gA_2 \cap X,\ gA_2 \cap Y \text{ are all nonempty}\}\right| < \infty.$$

Case 2 The coordinates of a and b involve only three distinct points of M. For example, $a = (p, q)$ and $b = (p, r)$, where p, q and r are three distinct points of M.

Again, condition (C) asserts the existence of certain open sets A and B, and one can assume that $A = A_1 \times A_2$ and $B = A_1 \times B_2$, where A_1, A_2 and B_2 are open sets in M containing p, q and r, respectively. In this case, (C) can be rephrased as

$$\left|\{g \in G \mid gA_1 \cap X,\ gA_1 \cap Y,\ gA_2 \cap X,\ gB_2 \cap Y \text{ are all nonempty}\}\right| < \infty.$$

The other cases where the coordinates of a and b involve only three distinct points of M can be treated in the same way.

Case 3 The coordinates of a and b involve four distinct points of M, ie and $b = (r, s)$, where p, q, r and s are four distinct points of M.

Once again, condition (C) asserts the existence of certain open sets A and B, and one can assume that $A = A_1 \times A_2$ and $B = B_1 \times B_2$, where A_1, A_2, B_1 and B_2 are open sets in M containing p, q, r and s, respectively. In this case, (C) can be rephrased as

$$\left|\{g \in G \mid gA_1 \cap X,\ gA_2 \cap X,\ gB_1 \cap Y,\ gB_2 \cap Y \text{ are all nonempty}\}\right| < \infty.$$

For further reference, let us sum up the above discussion.

Lemma 5.3 *Let G be a group acting on a topological space M which has at least 3 points. Then this action of G satisfies (C) if and only if for every pair of distinct points $x, y \in M$, there exist open sets U and V, in the topological space M, containing x and y, respectively, and satisfying the following conditions (C_1)–(C_3) simultaneously:*

(C_1) *For every pair of distinct points $p, q \in M$, there exist open sets A and B, of the topological space M, containing p and q, respectively, with*

$$\left|\{g \in G \mid gA \cap U,\ gA \cap V,\ gB \cap U,\ gB \cap V \text{ are all nonempty}\}\right| < \infty.$$

(C_2) *For every three distinct points $p, q, r \in M$, there exist open sets A, B and C of the topological space M, containing p, q and r, respectively, with*

$$\left|\{g \in G \mid gA \cap U,\ gB \cap V,\ gC \cap U,\ gC \cap V \text{ are all nonempty}\}\right| < \infty.$$

(C_3) *For every four distinct points $p, q, r, s \in M$, there exist open sets A, B, C and D of the topological space M, containing p, q, r and s, respectively, with*

$$\left|\{g \in G \mid gA \cap U,\ gB \cap U,\ gC \cap V,\ gD \cap V \text{ are all nonempty}\}\right| < \infty.$$

In the rest of this paper, we say that a pair of distinct points $x, y \in M$ *satisfy* (C_1) (resp. (C_2), (C_3)) if there exist open sets U and V, in the topological space M, containing x and y, respectively, and satisfying (C_1) (resp. (C_2), (C_3)).

6 Annulus system and hyperbolicity

Throughout this section, let G be a group acting on a topological space M. In Section 7, we are going to prove that if the action $G \curvearrowright M$ satisfies condition (C) and there exists $g \in G$ such that g has north–south dynamics on M, then G admits an isometric action on some Gromov hyperbolic space with g being a loxodromic WPD element (which implies that G is either acylindrically hyperbolic or virtually cyclic, by Theorem 3.2). The proof relies on a construction of Bowditch [3] called an annulus system, which is surveyed below.

Definition 6.1 An *annulus*, A, is an ordered pair, (A^-, A^+), of disjoint closed subsets of M such that $M \setminus (A^- \cup A^+) \neq \varnothing$.

For an annulus A and $g \in G$, we write gA for the annulus (gA^-, gA^+).

An *annulus system* on M is a set of annuli. The system is called *symmetric* if $-A := (A^+, A^-) \in \mathcal{A}$ whenever $A \in \mathcal{A}$.

Let A be an annulus. Given any subset $K \subset M$, we write $K < A$ if $K \subset \operatorname{int} A^-$ and write $A < K$ if $K \subset \operatorname{int} A^+$, where $\operatorname{int} A^-$ (resp. $\operatorname{int} A^+$) denotes the interior of A^- (resp. A^+). Thus $A < K$ if and only if $K < -A$. If B is another annulus, we write $A < B$ if $\operatorname{int} A^+ \cup \operatorname{int} B^- = M$.

Given an annulus system $\mathcal{A}$ on M and $K, L \subset M$, define $(K \mid L) = n \in \{0, 1, \dots, \infty\}$, where n is the supremum of all positive integers m such that there exist m annuli $A_1, \dots, A_m$ in $\mathcal{A}$ with $K < A_1 < A_2 < \cdots < A_m < L$ (if no such m exists, set $(K|L) = 0$). For finite sets we drop braces and write $(a, b|c, d)$ to mean $(\{a, b\}|\{c, d\})$. This gives us a well-defined function $M^4 \to [0, +\infty]$. Note that this function is G–invariant, ie $(gx, gy \mid gz, gw) = (x, y \mid z, w)$ for all $g \in G$ provided that the annulus system $\mathcal{A}$ is G–invariant.

Definition 6.2 The function from M^4 to $[0, +\infty]$, defined as above, is called the *crossratio* associated with $\mathcal{A}$.

Recall the definition of a quasimetric on a set Q:

Definition 6.3 Given $r \geqslant 0$, an r*–quasimetric* ρ on a set Q is a function $\rho\colon Q^2 \to [0, +\infty)$ satisfying $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y) + r$ for all $x, y, z \in Q$.

A *quasimetric* is an r–quasimetric for some $r \geqslant 0$. Given $s \geqslant 0$ and a quasimetric space (Q, ρ), an s*–geodesic segment* is a finite sequence of points $x_0, x_1, \dots, x_n$ such that $-s \leqslant \rho(x_i, x_j) - |i - j| \leqslant s$ for all $0 \leqslant i, j \leqslant n$. A quasimetric is a *path quasimetric* if there exists $s \geqslant 0$ such that every pair of points can be connected by an s–geodesic segment. A quasimetric is called a *hyperbolic quasimetric* if there is some $k \geqslant 0$ such that the 4–point definition of k–hyperbolicity holds via the Gromov product (see Bridson and Haefliger [5, Chapter III.H, Definition 1.20]).

Given an annulus system $\mathcal{A}$ on M, one can construct a quasimetric on $\Theta_3(M)$ from the crossratio associated with $\mathcal{A}$, where

$$\Theta_3(M) = \{(x_1, x_2, x_3) \in M^3 \mid x_1 \neq x_2,\ x_2 \neq x_3,\ x_3 \neq x_1\}$$

is the set of distinct triples of M. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two points of $\Theta_3(M)$. Define the function $\rho\colon (\Theta_3(M))^2 \to [0, +\infty]$ by

$$\rho(x, y) = \max(x_i, x_j \mid y_k, y_l),$$

where $(\cdot, \cdot \mid \cdot, \cdot)$ denotes the crossratio associated with $\mathcal{A}$ and the maximum is taken over all $i, j, k, l \in \{1, 2, 3\}$ with $i \neq j$ and $k \neq l$.

Consider two axioms on the crossratio $(\cdot, \cdot|\cdot, \cdot)$ (and hence on the annulus system $\mathcal{A}$):

(A1) If $x \neq y$ and $z \neq w$, then $(x, y \mid z, w) < \infty$.

(A2) There is some $k \geqslant 0$ such that there are no four points $x, y, z, w \in M$ with $(x, y \mid z, w) > k$ and $(x, z \mid y, w) > k$.

Proposition 6.4 *Suppose that G is a group acting on a topological space M, and that $\mathcal{A}$ is a symmetric, G–invariant annulus system on M satisfying (A1) and (A2). Then the map ρ defined as above is a hyperbolic G–invariant path quasimetric on $\Theta_3(M)$.*

By ρ being G–invariant, we mean $\rho(gx, gy) = \rho(x, y)$ for all $x, y \in \Theta_3(M)$ and $g \in G$.

Proof The fact that ρ is a hyperbolic path quasimetric follows from Bowditch [3, Propositions 4.2 and 6.5 and Lemma 4.3]. Note that Bowditch assumes that M is compact, but he does not use this assumption in the proofs of loc. cit. The fact that ρ is G–invariant follows from the fact that $\mathcal{A}$ is G–invariant and the relationship between ρ and $\mathcal{A}$. □

Note that Proposition 6.4 only produces a space $\Theta_3(M)$ with a G–invariant hyperbolic quasimetric ρ, but, as mentioned in the beginning of this section, we need to construct an isometric action of G on some Gromov hyperbolic space, which is a geodesic metric space. This can be easily achieved by passing to a geodesic metric space quasi-isometric with $\Theta_3(M)$.

Definition 6.5 Let (Q, d) and (Q', d') be two quasimetric spaces. A map $f\colon Q \to Q'$ is called a *quasi-isometry* from Q to Q' if there exist $\lambda, C, D > 0$ such that

(1) the inequality $d(x, y)/\lambda - C < d'(f(x), f(y)) < \lambda d(x, y) + C$ holds for all $x, y \in Q$;

(2) every point of Q' is within distance D from the image of f.

Proposition 6.6 *Let G be a group acting on a topological space M and let ρ be a G–invariant hyperbolic path quasimetric on $\Theta_3(M)$. Then there is a Gromov hyperbolic space (S, ρ') such that G acts isometrically on S and that there is a G–equivariant quasi-isometry $f\colon \Theta_3(M) \to S$.*

Proof The proof can be easily extracted from [3]. We provide it for convenience of the reader. Let s be a number such that every pair of points in $\Theta_3(M)$ can be connected by an s–geodesic. Construct the undirected graph S whose vertex set is just

$\Theta_3(M)$ and two vertices x and y are connected by an edge if $\rho(x,y) \leqslant s+1$. Define a path metric, ρ', on S by deeming every edge to have unit length. We see that S is connected and that the inclusion $f\colon \Theta_3(M) \hookrightarrow S$ is a quasi-isometry. Since $\rho'(x,y)$ is an integer for every pair of vertices $x, y \in \Theta_3(M)$, S is a geodesic metric space. Now, ρ' is a hyperbolic metric since ρ is hyperbolic and f is a quasi-isometry. Hence, S is a Gromov hyperbolic space. Moreover, the action of G on $\Theta_3(M)$ induces an action of G on S: for every $g \in G$, g maps a vertex x to the vertex gx, and this action uniquely extends to an isometric action on S since our definition of edges is G–equivariant. In particular, the action of G on S is isometric. Clearly, f is G–equivariant. $\square$

Let G be a group acting by isometries on a hyperbolic quasimetric space Q and let $g \in G$. Define that g is loxodromic (resp. satisfying condition ($\star$)) with respect to the action $G \curvearrowright Q$ in exactly the same manner as for actions of G on Gromov hyperbolic spaces (see Section 3). The following lemma reduces the proof in Section 7.

Lemma 6.7 *Let G be a group acting by isometries on a hyperbolic quasimetric space Q and let $g \in G$ be a loxodromic element satisfying ($\star$) with respect to the action $G \curvearrowright Q$. Suppose that G also admits an action on a Gromov hyperbolic space Q' and there is a G–equivariant quasi-isometry $f\colon Q \to Q'$. Then $g \in G$ is a loxodromic WPD element with respect to the action $G \curvearrowright Q'$.*

To prove Lemma 6.7, one checks that g is loxodromic and satisfies ($\star$) with respect to the action $G \curvearrowright Q'$ and then applies Lemma 3.4. We leave the details to the reader.

7 Proof of Theorem 1.2

Throughout this section, let (S,d) be a δ–hyperbolic space and let ∂S be the Gromov boundary of S. As in Section 2, pick some point $e \in S$ and define the Gromov product with the aid of e. Fix a sufficiently small number ζ and then define ρ on ∂S so that ρ is a metric and thus induces the topology τ. We will use the notation

$$U_K(x) = \{s \in S \mid (x \cdot s)_e > K\}, \quad \partial U_K(x) = \{s \in \partial S \mid (x \cdot S)_e > K\}$$

for $x \in \partial S$ and $K \in \mathbb{R}$. Recall that $B_M(x,r)$ denotes the open ball in a metric space M centered at a point $x \in M$ with radius r, that $[u,v]$ denotes a geodesic segment between $u, v \in S$, and that in Remark 5.2 and Lemma 5.3, we have reformulated Definition 1.1 as the combination of (C_1), (C_2) and (C_3).

Lemma 7.1 *Let G be a group acting on S by isometries. Then every pair of distinct points of ∂S satisfies (C$_1$).*

Proof Let x and y be two distinct points of ∂S. By Lemma 2.8, there exist $D, R > 0$ such that $d(e, [u, v]) < D$ for every $u \in U_R(x)$ and $v \in U_R(y)$. By (2), there exist open subsets U and V of ∂S containing x and y, respectively, such that $U \subset \partial U_R(x)$ and $V \subset \partial U_R(y)$. We examine (C$_1$) for U and V.

Let p and q be two distinct points of ∂S. Using Proposition 2.12, we can find $K > 0$ such that $d([a_1, a_2], [b_1, b_2]) > 2D$ for all $a_1, a_2 \in U_K(p)$ and $b \in U_K(q)$. By (2), there exist open subsets A and B of ∂S containing p and q, respectively, such that $A \subset \partial U_K(p)$ and $B \subset \partial U_K(q)$.

Suppose that there exists $g \in G$ such that $gA \cap U$, $gA \cap V$, $gB \cap U$ and $gB \cap V$ are all nonempty. Let $p' \in gA \cap U$, $p'' \in gA \cap V$, $q' \in gB \cap U$ and $q'' \in gB \cap V$ and let $\{p'_n\}_{n\geqslant 1}$, $\{p''_n\}_{n\geqslant 1}$, $\{q'_n\}_{n\geqslant 1}$ and $\{q''_n\}_{n\geqslant 1}$ be sequences in S tending to p', p'', q' and q'', respectively. Then $\{gp'_n\}_{n\geqslant 1}$, $\{gp''_n\}_{n\geqslant 1}$, $\{gq'_n\}_{n\geqslant 1}$ and $\{gq''_n\}_{n\geqslant 1}$ are sequences tending to gp', gp'', gq' and gq'', respectively. As

$$\min\{(p \cdot p')_e, (p \cdot p'')_e, (q \cdot q')_e, (q \cdot q'')_e\} > K,$$
$$\min\{(x \cdot gp')_e, (x \cdot gq')_e, (y \cdot gp'')_e, (y \cdot gq'')_e\} > R,$$

by Lemma 2.7 there exists $N > 0$ such that

$$\min\{(p \cdot p'_N)_e, (p \cdot p''_N)_e, (q \cdot q'_N)_e, (q \cdot q''_N)_e\} > K$$

and

$$\min\{(x \cdot gp'_N)_e, (x \cdot gq'_N)_e, (y \cdot gp''_N)_e, (y \cdot gq''_N)_e\} > R.$$

By our choice of R, the geodesics $[gp'_N, gp''_N]$ and $[gq'_N, gq''_N]$ intersect $B_S(e, D)$ nontrivially and thus $d([p'_N, p''_N], [q'_N, q''_N]) = d([gp'_N, gp''_N], [gq'_N, gq''_N]) < 2D$. But by our choice of K, $d([p'_N, p''_N], [q'_N, q''_N]) > 2D$, a contradiction. □

Lemma 7.2 *Let G be a group acting on S by isometries. Then every pair of distinct points of ∂S satisfies (C$_2$).*

Proof Let x and y be two distinct points of ∂S. By Lemma 2.8, there exist $D, R > 0$ such that $d(e, [u, v]) < D$ for every $u \in U_R(x)$, $v \in U_R(y)$. By (2), there exist open subsets U and V of ∂S containing x and y, respectively, such that $U \subset \partial U_R(x)$ and $V \subset \partial U_R(y)$. We examine (C$_2$) for U and V.

Let p, q and r be three distinct points of ∂S. By Proposition 2.16, there exists $K > 0$ such that $d([a,b],[c_1,c_2]) > 2D$ for every $a \in U_K(p)$ and $b \in U_K(q)$ and every $c_1, c_2 \in U_K(r)$. By (2), there exist open subsets A, B and C of ∂S containing p, q and r, respectively, such that $A \subset \partial U_K(p)$, $B \subset \partial U_K(q)$ and $C \subset \partial U_K(r)$.

Suppose that there exists $g \in G$ such that $gA \cap U$, $gB \cap V$, $gC \cap U$ and $gC \cap V$ are all nonempty. Thus, $A \cap g^{-1}U$, $B \cap g^{-1}V$, $C \cap g^{-1}U$ and $C \cap g^{-1}V$ are all nonempty. Pick

$$p' \in A \cap g^{-1}U, \quad q' \in B \cap g^{-1}V, \quad r' \in C \cap g^{-1}U, \quad r'' \in C \cap g^{-1}V$$

and let $\{p'_n\}_{n\geqslant 1}$, $\{q'_n\}_{n\geqslant 1}$, $\{r'_n\}_{n\geqslant 1}$ and $\{r''_n\}_{n\geqslant 1}$ be sequences in S tending to p', q', r' and r'', respectively. Then $\{gp'_n\}_{n\geqslant 1}$, $\{gq'_n\}_{n\geqslant 1}$, $\{gr'_n\}_{n\geqslant 1}$ and $\{gr''_n\}_{n\geqslant 1}$ are sequences in S tending to gp', gq', gr' and gr'', respectively. As

$$\min\{(p\cdot p')_e, (q\cdot q')_e, (r\cdot r')_e, (r\cdot r'')_e\} > K,$$
$$\min\{(x\cdot gp')_e, (y\cdot gq')_e, (x\cdot gr')_e, (y\cdot gr'')_e\} > R,$$

by Lemma 2.7 there exists $N > 0$ such that

$$\min\{(p\cdot p'_N)_e, (q\cdot q'_N)_e, (r\cdot r'_N)_e, (r\cdot r''_N)_e\} > K$$

and

$$\min\{(x\cdot gp'_N)_e, (y\cdot gq'_N)_e, (x\cdot gr'_N)_e, (y\cdot gr''_N)_e\} > R.$$

By our choice of R, the geodesics $[gp'_N, gq'_N]$ and $[gr'_N, gr''_N]$ intersect $B_S(e,D)$ nontrivially and hence $d([p'_N,q'_N],[r'_N,r''_N]) = d([gp'_N,gq'_N],[gr'_N,gr''_N]) < 2D$. But by our choice of K, $d([p'_N,q'_N],[r'_N,r''_N]) > 2D$, a contradiction. $\square$

Lemma 7.3 *Let G be a group acting acylindrically on S by isometries. Then every pair of distinct points of ∂S satisfies (C_3).*

Proof Let x and y be two distinct points of ∂S. By Lemma 2.8, there exists $R, K > 0$ such that $d(e,[u,v]) < R$ for every $u \in U_K(x)$ and $v \in U_K(y)$.

As the action of G on S is acylindrical, there exists $E > 0$ such that for every two points $t, w \in S$ with $d(t,w) \geqslant E$, the number of elements $g \in G$ satisfying both $d(t,gt) \leqslant 189\delta$ and $d(w,gw) \leqslant 189\delta$ is finite.

By Lemmas 2.9 and 2.14, there exists $F' > K$ such that both of $d(e,u_1)$ and $d(e,v_1)$ are strictly greater than $R+E$ and that $[u_1,v_2] \cap B_S(e,R+E)$ lies inside the 2δ–neighborhood of $[u_2,v_2]$ for every $u_1,u_2 \in U_{F'}(x)$ and every $v_1,v_2 \in U_{F'}(y)$. By

Lemma 2.10, there exists $F>0$ such that $[u_1,u_2]\subset U_{F'}(x)$ for every $u_1,u_2\in U_F(y)$ and that $[v_1,v_2]\subset U_{F'}(y)$ for every $v_1,v_2\in U_F(y)$. Using (2), we can pick open subsets U and V of ∂S containing x and y, respectively, such that $U\subset\partial U_F(x)$ and $V\subset\partial U_F(y)$. We examine (C$_3$) for U and V.

Suppose, to the contrary, that there exist four distinct points p, q, r and s such that for every four open subsets A, B, C and D of ∂S containing p, q, r and s, respectively, we have

$$\bigl|\{g\in G \mid gA\cap U,\ gB\cap U,\ gC\cap V,\ gD\cap V \text{ are all nonempty}\}\bigr|=\infty.$$

In particular, for $A=B_{\partial S}(p,1)$, $B=B_{\partial S}(q,1)$, $C=B_{\partial S}(r,1)$ and $D=B_{\partial S}(s,1)$, there exist $p_1\in A$, $q_1\in B$, $r_1\in C$, $s_1\in D$ and $g_1\in G$ such that $g_1p_1\in U$, $g_1q_1\in U$, $g_1r_1\in V$ and $g_1s_1\in V$. For $A=B_{\partial S}\bigl(p,\frac12\bigr)$, $B=B_{\partial S}\bigl(q,\frac12\bigr)$, $C=B_{\partial S}\bigl(r,\frac12\bigr)$ and $D=B_{\partial S}\bigl(s,\frac12\bigr)$, since

$$\bigl|\{g\in G \mid gA\cap U,\ gB\cap U,\ gC\cap V,\ gD\cap V \text{ are all nonempty}\}\bigr|=\infty,$$

there exist $p_2\in A$, $q_2\in B$, $r_2\in C$, $s_2\in D$ and $g_2\in G\setminus\{g_1\}$ such that $g_2p_2\in U$, $g_2q_2\in U$, $g_2r_2\in V$ and $g_2s_2\in V$. Continuing in this manner, we see that there exist four sequences $\{p_n\}_{n\geqslant1}$, $\{q_n\}_{n\geqslant1}$, $\{r_n\}_{n\geqslant1}$ and $\{s_n\}_{n\geqslant1}$ of points in ∂S and a sequence $\{g_n\}_{n\geqslant1}$ of distinct elements in G, such that

$$\max\{\rho(p,p_n),\rho(q,q_n),\rho(r,r_n),\rho(s,s_n)\}<\frac1n$$

and

$$g_np_n\in U,\quad g_nq_n\in U,\quad g_nr_n\in V,\quad g_ns_n\in V,$$

for all $n\geqslant1$.

By (2), $\lim_{n\to\infty}(p\cdot p_n)_e=\lim_{n\to\infty}(q\cdot q_n)_e=\lim_{n\to\infty}(r\cdot r_n)_e=\lim_{n\to\infty}(s\cdot s_n)_e=\infty$. By passing to a subsequence, we may assume that

$$\min\{(p\cdot p_n)_e,(q\cdot q_n)_e,(r\cdot r_n)_e,(s\cdot s_n)_e\}>n\quad\text{for all } n.$$

Since $(g_np_n\cdot x)_e>F$ and $(p_n\cdot p)_e>n$, there exists $p'_n\in S$ such that

$$(g_np'_n\cdot x)_e>F,\quad (p'_n\cdot p)_e>n,$$

by Lemma 2.7. Thus, there exist four sequences $\{p'_n\}_{n\geqslant1}$, $\{q'_n\}_{n\geqslant1}$, $\{r'_n\}_{n\geqslant1}$ and $\{s'_n\}_{n\geqslant1}$ of points in S such that

$$\min\{(p'_n\cdot p)_e,(q'_n\cdot q)_e,(r'_n\cdot r)_e,(s'_n\cdot s)_e\}>n$$

and

$$\min\{(g_n p'_n \cdot x)_e, (g_n q'_n \cdot x)_e, (g_n r'_n \cdot y)_e, (g_n s'_n \cdot y)_e\} > F,$$

for all $n \geqslant 1$.

For each n, use the compactness of $[p'_n, q'_n]$ and $[r'_n, s'_n]$ and choose points a'_n and b'_n in $[p'_n, q'_n]$ and $[r'_n, s'_n]$, respectively, such that $d(e, a'_n) = d(e, [p'_n, q'_n])$ and $d(e, b'_n) = d(e, [r'_n, s'_n])$. By Proposition 2.18, there exists $N > 0$ such that if $n \geqslant N$, $[a'_n, b'_n]$ will be in the 92δ–neighborhood of $[a'_N, b'_N]$.

By our choice of F and the properties of $\{p'_n\}_{n\geqslant 1}$, $\{q'_n\}_{n\geqslant 1}$, $\{r'_n\}_{n\geqslant 1}$ and $\{s'_n\}_{n\geqslant 1}$, we have

$$\min\{(g_n a'_n \cdot x), (g_n b'_n \cdot y)\} > F' \quad \text{for all } n \geqslant 1.$$

By our choice of F', we have the following properties:

(P$_1$) $d(e, [g_N a'_N, g_N b'_N]) < R$.

(P$_2$) $\min\{d(e, g_N a'_N), d(e, g_N b'_N)\} > R + E$.

(P$_3$) $[g_N a'_N, g_N b'_N] \cap B_S(R+E)$ lies inside the 2δ–neighborhood of $[g_n a'_n, g_n b'_n]$ for all $n \geqslant 1$.

Pick $c \in [g_N a'_N, g_N b'_N]$ such that $d(e, c) = d(e, [g_N a'_N, g_N b'_N]) < R$ by (P$_1$) and the compactness of $[g_N a'_N, g_N b'_N]$. By (P$_2$), there exist $t \in [g_N a'_N, c]$ and $w \in [c, g_N b'_N]$ such that $d(e, t) = d(e, w) = R + E$. As $d(e, c) < R$, we have

$$d(t, w) = d(t, c) + d(c, w) \geqslant 2E.$$

By (P$_3$), $\max\{d(t, [g_n a'_n, g_n b'_n]), d(w, [g_n a'_n, g_n b'_n])\} \leqslant 2\delta$ for all $n \geqslant N$. Since g_n is an isometry, apply g_n^{-1} and we obtain

$$\max\{d(g_n^{-1} t, [a'_n, b'_n]), d(g_n^{-1} w, [a'_n, b'_n])\} \leqslant 2\delta.$$

For each $n > N$, $[a'_n, b'_n]$ lies inside the 92δ–neighborhood of $[a'_N, b'_N]$. Thus,

$$\max\{d(g_n^{-1} t, [a'_N, b'_N]), d(g_n^{-1} w, [a'_N, b'_N])\} \leqslant 2\delta + 92\delta \leqslant 94\delta.$$

Select a point $z_{t,n}$ (resp. $z_{w,n}$) of $[a'_N, b'_N]$ such that $d(g_n^{-1} t, z_{t,n}) \leqslant 94\delta$ (resp. $d(g_n^{-1} w, z_{w,n}) \leqslant 94\delta$).

Partition $[a'_N, b'_N]$ into finitely many subpaths such that each of these subpaths has length $< \delta$. Using the pigeonhole principle, we may assume, after passing to a subsequence, that $z_{t,n}$ stays in a subpath for all $n \geqslant N + 1$. Using the pigeonhole principle once more and passing to a further subsequence, we may further assume that $z_{w,n}$ also stays in a subpath for all $n \geqslant N+1$. Thus, for all $m, n \geqslant N+1$, we have

$$d(g_m^{-1}t, g_n^{-1}t) \leqslant d(g_m^{-1}t, z_{t,m}) + d(z_{t,m}, z_{t,n}) + d(z_{t,n}, g_n^{-1}t) < 189\delta,$$
$$d(g_m^{-1}w, g_n^{-1}w) \leqslant d(g_m^{-1}w, z_{w,m}) + d(z_{w,m}, z_{w,n}) + d(z_{w,n}, g_n^{-1}w) < 189\delta.$$

As the g_n are all distinct for $n \geqslant N+1$, we have

$$d(t, g_n g_{N+1}^{-1} t) < 189\delta \quad \text{and} \quad d(w, g_n g_{N+1}^{-1} w) < 189\delta.$$

We have found infinitely many elements which move t and w by at most 189δ. As $d(t, w) > E$, this contradicts our choice of E. □

Proposition 7.4 *Let G be a group acting nonelementarily, acylindrically and isometrically on a Gromov hyperbolic space S. Then G is not virtually cyclic, has an element with north–south dynamics on ∂S and the action of G on the completely Hausdorff topological space ∂S satisfies (C).*

Recall that a topological space M is called *completely Hausdorff* if for any two distinct points $u, v \in M$, there are open sets U and V containing u and v, respectively, such that $\overline{U} \cap \overline{V} = \varnothing$.

Proof By Theorem 3.2, G is not virtually cyclic. By Osin [16, Theorem 1.1], G contains a loxodromic element g (with respect to the action of G on S). By Lemma 3.6, g has north–south dynamics on ∂S. As the action of G on S is nonelementary, it is well known that $|\Lambda(G)| = \infty$ (see [16]) and thus $|\partial S| = \infty \geqslant 3$. Let x and y be a pair of distinct points of M. Pick open sets U_1 and V_1 in M containing x and y, respectively, and satisfying (C_1) by Lemma 7.1; U_2 and V_2 in M containing x and y, respectively, and satisfying (C_2) by Lemma 7.2; and U_3 and V_3 in M containing x and y, respectively, and satisfying (C_3) by Lemma 7.3. Let $U = U_1 \cap U_2 \cap U_3$ and $V = V_1 \cap V_2 \cap V_3$. Then U and V satisfy the (C_1), (C_2) and (C_3) simultaneously. As x and y are arbitrary, Lemma 5.3 implies that the action of G on ∂S satisfies (C). □

We now turn to the other direction of Theorem 1.2.

Proposition 7.5 *Let G be a group acting on a completely Hausdorff topological space M which has at least 3 points. If there is an element $g \in G$ having north–south dynamics on M and (C_1), (C_2) and (C_3) hold for the fixed points of g, then G is either acylindrically hyperbolic or virtually cyclic.*

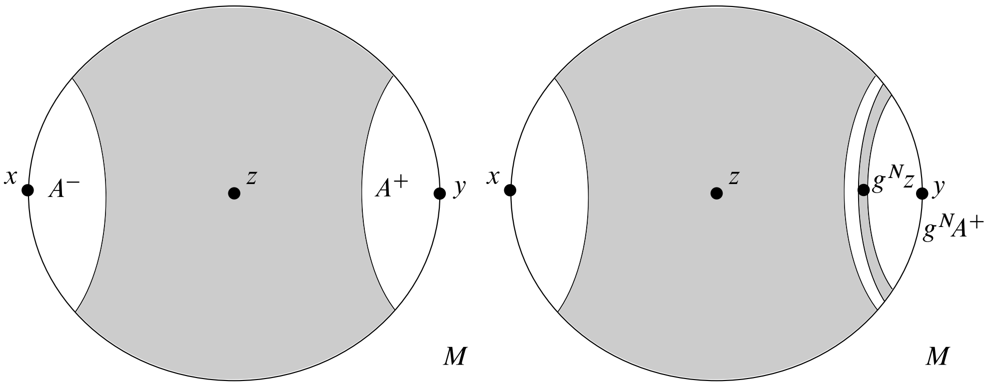

Figure 3: The annulus A (left) and the image of A under the action of g^N (right).

Remark 7.6 The existence of a loxodromic element does not follow from the assumption that the action of G satisfies (C). For example, let $G=\mathbb{Z}\times\mathbb{Z}$, let $M=\mathbb{R}^2$ and let G act on M by integral translations, ie $(m,n)\cdot(x,y)=(x+m,y+n)$ for all $(m,n)\in G$ and $(x,y)\in M$. As G acts on M properly discontinuously and M is locally compact, it is easy to see that the action of G on M satisfies (C). Nevertheless, no element of G can fix exactly two points of M.

Proof Let x and y be the fixed points of g. The idea is to construct a specific annulus system on M, obtain a Gromov hyperbolic space and then verify that there is a loxodromic WPD element. The construction is illustrated by Figure 3. Since M has at least three points, there is some $z\in M\setminus\{x,y\}$. Pick open sets U and V containing x and y, respectively, and satisfying (C_1), (C_2) and (C_3). By shrinking U and V if necessary, we may assume that $\overline{U}\cap\overline{V}=\varnothing$ and that $z\notin\overline{U}\cup\overline{V}$, as M is a completely Hausdorff space. Let

$$A^-=\overline{U},\quad A^+=\overline{V}.$$

Then A^- and A^+ are two closed sets such that $x\in\operatorname{int}A^-$, $y\in\operatorname{int}A^+$, $A^-\cap A^+=\varnothing$ and $A^-\cup A^+\neq M$. In Figure 3, left, the white closed half-disc containing x (resp. y) is A^- (resp. A^+). The gray shaded region is $M\setminus(A^-\cup A^+)$. Let

$$\mathcal{A}=\{h(\pm A)\mid h\in G\},$$

where $A=(A^-,A^+)$. Then $\mathcal{A}$ is a symmetric G–invariant annulus system. Define the crossratio $(\cdot,\cdot\,|\,\cdot,\cdot)$ and the quasimetric ρ in the same manner as in Section 6.

We proceed to verify that $\mathcal{A}$ satisfies (A1) and (A2). Suppose (A1) does not hold, then there exist four points p, q, r and s such that $p \neq q$, $r \neq s$ and $(p,q \mid r,s) = \infty$. By the definition of $(\cdot,\cdot \mid \cdot,\cdot)$, we see that p, q, r and s are pairwise distinct and, by switching p with q and r with s if necessary, we may assume that there exist infinitely many elements $h \in G$ such that $hp, hq \in U$ and $hr, hs \in V$. Thus, for all open sets P, Q, R and W in M containing p, q, r and s, respectively, we have infinitely many elements $h \in G$ such that $hP \cap U$, $hQ \cap U$, $hR \cap V$ and $hW \cap V$ are all nonempty and (C_3) is violated.

The verification for (A2) is similar. Suppose (A2) does not hold; then there exist four sequences of points $\{p_n\}_{n\geqslant 1}, \{q_n\}_{n\geqslant 1}, \{r_n\}_{n\geqslant 1}, \{s_n\}_{n\geqslant 1} \subset M$ such that for each n, $(p_n, q_n \mid r_n, s_n) > n$ and $(p_n, r_n \mid q_n, s_n) > n$. We will choose a sequence $\{h_n\}_{n\geqslant 1}$ of distinct elements of G such that $h_n U \cap U$, $h_n U \cap V$, $h_n V \cap U$ and $h_n V \cap V$ are nonempty for all $n \geqslant 1$. The verification of (A2) will then be complete since, by applying (C_1) with $p = x$ and $q = y$, we see that there are only finitely many elements $h \in G$ with $hU \cap U$, $hU \cap V$, $hV \cap U$ and $hV \cap V$ all nonempty, a contradiction.

First we choose h_1. Since $(p_1, q_1 \mid r_1, s_1) > 1$ and $(p_1, r_1 \mid q_1, s_1) > 1$, by renaming p_1, q_1, r_1 and s_1 if necessary, we may assume that there exist h_1' and h_1'' such that

$$\{p_1, q_1\} < h_1' A < \{r_1, s_1\}, \quad \{p_1, r_1\} < h_1'' A < \{q_1, s_1\}.$$

In other words,

$$p_1 \in h_1' U \cap h_1'' U, \quad q_1 \in h_1' U \cap h_1'' V, \quad r_1 \in h_1' V \cap h_1'' U, \quad s_1 \in h_1' V \cap h_1'' V.$$

Let $h_1 = h_1'^{-1} h_1''$ and we see that $h_1 U \cap U$, $h_1 U \cap V$, $h_1 V \cap U$ and $h_1 V \cap V$ are all nonempty.

Suppose that we have chosen $h_1, \dots, h_{n-1}$. Since we have $(p_n, q_n \mid r_n, s_n) > n$ and $(p_n, r_n \mid q_n, s_n) > n$, there are two elements $h_n', h_n'' \in G$ such that $h_n'^{-1} h_n''$ is not one of $h_1, \dots, h_{n-1}$ and that (by renaming p_n, q_n, r_n and s_n if necessary)

$$\{p_n, q_n\} < h_n' A < \{r_n, s_n\}, \quad \{p_n, r_n\} < h_n'' A < \{q_n, s_n\}.$$

In other words,

$$p_n \in h_n' U \cap h_n'' U, \quad q_n \in h_n' U \cap h_n'' V, \quad r_n \in h_n' V \cap h_n'' U, \quad s_n \in h_n' V \cap h_n'' V.$$

Let $h_n = h_n'^{-1} h_n''$ and we see that $h_n U \cap U$, $h_n U \cap V$, $h_n V \cap U$ and $h_n V \cap V$ are all nonempty and that $h_1, \dots, h_n$ are all distinct. This finishes the verification of (A2).

Below, we are going to show that g is loxodromic and satisfies ($\star$) with respect to the action of G on $\Theta_3(M)$. Once this is done, Propositions 6.4 and 6.6 and Lemma 6.7 will imply that G admits an isometric action on some Gromov hyperbolic space with g being a loxodromic WPD element, and then Theorem 3.2 will imply that G is either virtually cyclic or acylindrically hyperbolic, which finishes the proof.

Since g has north–south dynamics on M with fixed points x and y, there exists a positive integer N such that $g^N(M \setminus \operatorname{int} A^-) \subset \operatorname{int} A^+$. Figure 3, right, illustrates the dynamics of g^N on M: g^N maps the large gray shaded area onto the small gray shaded band inside of A^+ and compresses A^+ into the small white half-disc around b labeled by $g^N A^+$. From the figure, it is easy to see inequalities (6) and (9) below. Let $a=(x,y,z)$. To prove that g is loxodromic, it suffices to show that $\rho(a, g^{nN}a) \geqslant n-1$ for all positive integers n. Fix a positive integer n. Observe that x and y are fixed by g, hence $x \in g^N(\operatorname{int} A^-)$ and $y \in g^{(n-1)N}(\operatorname{int} A^+)$. Consequently,

$$\{x\} < g^N A, \quad \{y\} > g^{(n-1)N} A. \tag{6}$$

Note that g is a bijection on M, thus $g^N(M \setminus \operatorname{int} A^-) \subset \operatorname{int} A^+$ is equivalent to

$$g^N(\operatorname{int} A^-) \cup \operatorname{int} A^+ = M. \tag{7}$$

As a consequence, $A < g^N A$. Multiplying both sides of this inequality by g^N, g^{2N}, etc, we have the chain of inequalities

$$g^N A < g^{2N} A < \cdots < g^{(n-1)N} A. \tag{8}$$

Since $z \notin \operatorname{int} A^- \cup \operatorname{int} A^+$, equality (7) also implies

$$\{z\} < g^N A, \quad A < \{g^N z\}. \tag{9}$$

The second inequality of (9) is equivalent to

$$g^{(n-1)N} A < \{g^{nN} z\}. \tag{10}$$

Combining inequalities (6), (8), (9) and (10), we obtain

$$\{x,z\} < g^N A < g^{2N} A < \cdots < g^{(n-1)N} A < \{g^{nN} z, y\}. \tag{11}$$

Thus, $\rho(a, g^{nN}a) \geqslant (x,z \mid g^{nN}z, y) \geqslant n-1$ and loxodromicity is proved.

In order to prove ($\star$), we proceed as follows. Given $\epsilon > 0$, let

$$L > \epsilon + 2, \quad K = (2L+1)N \tag{12}$$

be integers. By (11) and (12), we have $\{x,z\} < A_1 < A_2 < \cdots < A_{2L} < \{g^K z, y\}$, where

$$A_i = g^{iN} A \tag{13}$$

for all $1 \leqslant i \leqslant 2L$. Let us make the following observation:

Lemma 7.7 *Let $a = (x,y,z) \in \Theta_3(M)$. If $w = (w_1,w_2,w_3) \in \Theta_3(M)$ and $\rho(a,w) < \epsilon$, then at least two of w_1, w_2 and w_3 lie in A_L^-. Similarly, if $\rho(g^K a, w) < \epsilon$, then at least two of w_1, w_2 and w_3 lie in A_L^+.*

Proof Suppose that $w_i, w_j \notin A_L^-$ for some $1 \leqslant i < j \leqslant 3$. Since $\operatorname{int} A_{L-1}^+ \cup \operatorname{int} A_L^- = M$ by (7) and (13), we have $\{w_i, w_j\} \in \operatorname{int} A_{L-1}^+$ and consequently $\{x,z\} < A_1 < A_2 < \cdots < A_{L-1} < \{w_i, w_j\}$. By (12) and the definition of the quasimetric ρ, we have $\rho(a,w) > \epsilon + 1$. This proves the first part.

Similarly, suppose $w_i, w_j \notin A_L^+$ for some $1 \leqslant i < j \leqslant 3$. Again, using (7) and (13), we obtain $\operatorname{int} A_{L+1}^- \cup \operatorname{int} A_L^+ = M$. Thus $\{w_i, w_j\} \in \operatorname{int} A_{L+1}^-$ and consequently $\{w_i, w_j\} < A_{L+1} < A_{L+2} < \cdots < A_{2L} < \{g^K z, y\}$. As above, this implies $\rho(g^K a, w) > \epsilon + 2$ and proves the second part. □

Now suppose that there is an infinite sequence of distinct elements $\{h_n\}_{n \geq 1} \subset G$ such that $\rho(a, h_n a) < \epsilon$ and $\rho(g^K a, h_n g^K a) < \epsilon$ for all n. Since $\rho(a, h_n a) < \epsilon$ and $\rho(g^K a, h_n g^K a) < \epsilon$, by Lemma 7.7, for every n, at least two of $h_n x$, $h_n y$ and $h_n z$ lie in A_L^- and at least two of $h_n g^K x$, $h_n g^K y$ and $h_n g^K z$ lie in A_L^+. There is a subsequence $\{h_{n_k}\}$ and four points $u_1 \neq u_2$ and $v_1 \neq v_2$ such that $u_1, u_2 \in \{x,y,z\}$ and $v_1, v_2 \in \{g^K x, g^K y, g^K z\}$ and that $h_{n_k} u_1, h_{n_k} u_2 \in A_L^-$ and $h_{n_k} v_1, h_{n_k} v_2 \in A_L^+$. In particular, we see that u_1, u_2, v_1 and v_2 are four distinct points and that $(u_1, u_2 \mid v_1, v_2) = \infty$, which already contradicts the previously proved axiom (A1). This proves that g satisfies $(\star)$ with respect to the action $G \curvearrowright \Theta_3(M)$ and we are done. □

Corollary 7.8 *Let G be a group which admits an action on a completely Hausdorff space satisfying (C) and contains an element with north–south dynamics. Then G is either acylindrically hyperbolic or virtually cyclic.*

Theorem 1.2 is now an obvious consequence of Proposition 7.4 and Corollary 7.8.

By a result of Balasubramanya [1, Theorem 1.2], an acylindrically hyperbolic group G admits a nonelementary acylindrical and isometric action on one of its Cayley graphs Γ

which is quasi-isometric to a tree T. Note that the boundaries $\partial\Gamma$ and ∂T of Γ and T, respectively, can be naturally identified by a homeomorphism. If, in addition, G is countable, then the construction in [1] actually implies that the boundary ∂T of T can be naturally identified, by a homeomorphism, with the *Baire space*, which can be defined as $\mathbb{N}^{\mathbb{N}}$ with the product topology or the set of irrational numbers with the usual topology (see Engelking [7, Theorem 1.3.13] for details).

By Proposition 7.4, G acts on the Baire space by an action satisfying (C) and has an element with north–south dynamics.

Conversely, if G is a countable group that is not virtually cyclic with an action on the Baire space satisfying (C) and contains an element with north–south dynamics, Corollary 7.8 implies that G is acylindrically hyperbolic. Theorem 1.5 is proved.

Theorem 4.3 and Lemma 5.1 imply that if G is a nonelementary convergence group acting on a compact metrizable topological space M, then G is not virtually cyclic, has an element with north–south dynamics on M and the action of G on M satisfies (C), thus Corollary 1.3 follows from Theorem 1.2 directly. As mentioned in the introduction, the converse of Corollary 1.3 is not true. In fact, we have the following general statement:

Proposition 7.9 *Let $G = \langle X \mid \mathcal{R} \rangle$ be a group generated by X with relations $\mathcal{R}$. If X consists of elements of infinite order and the commutativity graph of X is connected, then any convergence action of G on a compact metrizable topological space is elementary.*

Here the commutativity graph of X is the undirected graph with vertex set X and edge set consisting of pairs $(x, y) \in X^2$ for every $x, y \in X$ with their commutator $xyx^{-1}y^{-1}$ equal to the identity. The proof of Proposition 7.9 is similar to Karlsson and Noskov [14], which proves that groups such as $\mathrm{SL}_n(\mathbb{Z})$ and Artin braid groups can only have elementary actions on hyperbolic-type bordifications.

Proof Suppose that G acts on a compact metrizable topological space M by a convergence action. Let x be an element of X. As x has infinite order, it is either parabolic or loxodromic by Remark 4.2. We split our argument into two cases.

Case 1 (x is a parabolic element) Let y be any element of X. As the commutativity graph of X is connected, there exists a path in this graph from x to y labeled by $x = x_1, x_2, \dots, x_n = y$. Since elements of X have infinite order, each of $x_2, \dots, x_n$

is either parabolic or loxodromic. Let $a \in M$ be the fixed point of x. As x_1 commutes with x_2, we have

$$x_1 x_2 a = x_2 x_1 a = x_2 a.$$

In other words, $x_2 a$ is a fixed point of x_1. Since x_1 fixes a unique point, x_2 fixes a. Then x_2 cannot be a loxodromic element since, otherwise, the fact that x_2 shares the fixed point a with x_1 will contradict Remark 4.2. Thus, x_2 is a parabolic element fixing a. The above argument with x_2 and x_3 in place of x_1 and x_2 shows that x_3 is also a parabolic element fixing a, and then we can apply the argument with x_3 and x_4 in place of x_1 and x_2. Continue in this manner and we see that $y = x_n$ is a parabolic element fixing a. As y is arbitrary, we conclude that G fixes a and thus is elementary.

Case 2 (x is a loxodromic element) Let y be any element of X. As the commutativity graph of X is connected, there exists a path in this graph from x to y labeled by $x = x_1, x_2, \dots, x_n = y$. Since elements of X have infinite order, each of $x_2, \dots, x_n$ is either parabolic or loxodromic. Let $a, b \in M$ be the fixed points of x. As x_1 commutes with x_2, we have

$$x_1 x_2 a = x_2 x_1 a = x_2 a, \quad x_1 x_2 b = x_2 x_1 b = x_2 b.$$

In other words, $x_2 a$ and $x_2 b$ are two fixed points of x_1. Since x_1 fixes exactly two points, x_2 either permutes a and b or fixes a and b pointwise. If x_2 permutes a and b, then since x_2 is either parabolic or loxodromic, it fixes at least a point $c \in M$ and, obviously, $c \neq a, b$. Note that x_2^2 has infinite order and fixes the three points a, b and c, contradicting Remark 4.2.

Thus, x_2 fixes a and b pointwise and is a loxodromic element. The above argument with x_2 and x_3 in place of x_1 and x_2 shows that x_3 is also a loxodromic element fixing a and b, and then we can apply the argument with x_3 and x_4 in place of x_1 and x_2. Continue in this manner and we see that $y = x_n$ is a loxodromic element fixing a and b. As y is arbitrary, we conclude that G fixes a and b and thus is elementary. □

Proposition 7.9 implies that various mapping class groups and right-angled Artin groups provide counterexamples for the converse of Corollary 1.3.

Corollary 7.10 *Mapping class groups of closed orientable surfaces with genus $\geqslant 2$ and noncyclic directly indecomposable right-angled Artin groups corresponding to connected graphs are acylindrically hyperbolic groups failing to be nonelementary convergence groups.*

Proof By Osin [16], these groups are acylindrically hyperbolic. For mapping class groups of a closed surface with genus $\geqslant 2$, the commutativity graph corresponding to a generating set due to Wajnryb [20, Theorem 2] is connected. The fact that a right-angled Artin group corresponding to a connected graph has some generating set with connected commutativity graph just follows from the definition. Thus, none of these groups can be a nonelementary convergence group, by Proposition 7.9. □

References

[1] **S H Balasubramanya**, *Acylindrical group actions on quasi-trees*, Algebr. Geom. Topol. 17 (2017) 2145–2176 MR

[2] **M Bestvina**, **K Fujiwara**, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. 6 (2002) 69–89 MR

[3] **B H Bowditch**, *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc. 11 (1998) 643–667 MR

[4] **B H Bowditch**, *Convergence groups and configuration spaces*, from "Geometric group theory down under" (J Cossey, W D Neumann, M Shapiro, editors), de Gruyter, Berlin (1999) 23–54 MR

[5] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. 319, Springer (1999) MR

[6] **F Dahmani**, **V Guirardel**, **D Osin**, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Mem. Amer. Math. Soc. 1156, Amer. Math. Soc., Providence, RI (2017) MR

[7] **R Engelking**, *Dimension theory*, North-Holland Mathematical Library 19, North-Holland, Amsterdam (1978) MR

[8] **E M Freden**, *Negatively curved groups have the convergence property, I*, Ann. Acad. Sci. Fenn. Ser. A I Math. 20 (1995) 333–348 MR

[9] **F W Gehring**, **G J Martin**, *Discrete quasiconformal groups, I*, Proc. London Math. Soc. 55 (1987) 331–358 MR

[10] **M Gromov**, *Hyperbolic groups*, from "Essays in group theory" (S M Gersten, editor), Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR

[11] **M Hamann**, *Group actions on metric spaces: fixed points and free subgroups*, Abh. Math. Semin. Univ. Hambg. 87 (2017) 245–263 MR

[12] **M Hull**, *Small cancellation in acylindrically hyperbolic groups*, Groups Geom. Dyn. 10 (2016) 1077–1119 MR

[13] **A Karlsson**, *Free subgroups of groups with nontrivial Floyd boundary*, Comm. Algebra 31 (2003) 5361–5376 MR

[14] **A Karlsson**, **G A Noskov**, *Some groups having only elementary actions on metric spaces with hyperbolic boundaries*, Geom. Dedicata 104 (2004) 119–137 MR

[15] **N Monod**, **Y Shalom**, *Orbit equivalence rigidity and bounded cohomology*, Ann. of Math. 164 (2006) 825–878 MR

[16] **D Osin**, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016) 851–888 MR

[17] **D Osin**, *Groups acting acylindrically on hyperbolic spaces*, from "Proceedings of the ICM" (B Sirakov, P N de Souza, M Viana, editors), volume 1, World Sci. (2018) 915–936

[18] **P Tukia**, *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand J. Math. 23 (1994) 157–187 MR

[19] **J Väisälä**, *Gromov hyperbolic spaces*, Expo. Math. 23 (2005) 187–231 MR

[20] **B Wajnryb**, *A simple presentation for the mapping class group of an orientable surface*, Israel J. Math. 45 (1983) 157–174 MR

[21] **A Yaman**, *A topological characterisation of relatively hyperbolic groups*, J. Reine Angew. Math. 566 (2004) 41–89 MR

[22] **W-y Yang**, *Growth tightness for groups with contracting elements*, Math. Proc. Cambridge Philos. Soc. 157 (2014) 297–319 MR

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Received: 30 October 2017 Revised: 10 January 2019