Algebraic & Geometric Topology 18 (2018) 1753–1798

The relative lattice path operad

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We construct a set-theoretic coloured operad $\mathcal{RL}$ that may be thought of as a combinatorial model for the Swiss cheese operad. This is the relative (or Swiss cheese) version of the lattice path operad constructed by Batanin and Berger. By adapting their condensation process we obtain a topological (resp. chain) operad that we show to be weakly equivalent to the topological (resp. chain) Swiss cheese operad.

18D50; 18G30, 55P48

1 Introduction

The Swiss cheese operad $\mathcal{SC}$ is a 2–coloured topological operad that mixes, in its m–dimensional part $\mathcal{SC}_m$, the m–dimensional and the $(m-1)$–dimensional parts of the little cubes operad $\mathcal{C}$. It was introduced by Voronov [23] as a natural way to define actions of $C_*(\mathcal{C}_m)$–algebras on $C_*(\mathcal{C}_{m-1})$–algebras and has been used by Kontsevich [15] in deformation quantization. As announced by Hoefel, Livernet and Stasheff [13], the Swiss cheese operad $\mathcal{SC}_m$ also recognizes the pair (m–fold loop space, m–fold relative loop space). The goal of this paper is to provide a convenient combinatorial model for (the operad of chains of) $\mathcal{SC}_m$, $m \geq 1$.

In [3], Batanin and Berger introduce the notion of *condensation* of a coloured operad. By applying this condensation to the *lattice path operad* $\mathcal{L}$ they obtain a model for the (operad of chains of the) little cubes operad.

We introduce the *relative lattice path operad* $\mathcal{RL}$. It is a coloured operad in the category of sets that has two types of colours (closed and open). Taking a cosimplicial object $\delta\colon \triangle \to \boldsymbol{C}$ in a cocomplete closed monoidal symmetric category $\boldsymbol{C}$, we adapt Batanin and Berger's method to obtain a functor

$$\mathcal{RL}\text{–algebra} \to \mathrm{Coend}_{\mathcal{RL}}(\delta)\text{–algebra}$$

that sends algebras over $\mathcal{RL}$ into algebras over the *condensation operad* of $\mathcal{RL}$, that is, the *SC-type* operad $\mathrm{Coend}_{\mathcal{RL}}(\delta)$.

Published: 3 April 2018 DOI: 10.2140/agt.2018.18.1753

The operad $\mathcal{RL}$ has two filtrations by suboperads $\mathcal{RL}_m$ and $\mathcal{RL}'_m$, $m \geq 1$; they differ from each other by their open/closed interacting part.

We are interested in two choices for δ:

$$\delta_{\mathrm{Top}}\colon \triangle \xrightarrow{\delta_{\mathrm{yon}}} \mathbf{Set}^{\triangle^{\mathrm{op}}} \xrightarrow{|-|} \mathbf{Top} \quad \text{and} \quad \delta_{\mathbb{Z}}\colon \triangle \xrightarrow{\delta_{\mathrm{yon}}} \mathbf{Set}^{\triangle^{\mathrm{op}}} \xrightarrow{C_*(-;\mathbb{Z})} \mathbf{Ch}(\mathbb{Z}),$$

where $\delta_{\mathrm{yon}}([n]) = \mathrm{Hom}_{\triangle}(-,[n])$ is the Yoneda functor. In this manner, the condensation of $\mathcal{RL}_m$ leads to a topological operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathrm{Top}})$ and to a chain operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$, and similarly for $\mathcal{RL}'_m$.

In the topological case, both SC-type operads $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathrm{Top}})$ and $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathrm{Top}})$ naturally act on the pair (m–fold loop space, m–fold relative loop space).

In order to relate our operads to the Swiss cheese operad, we use Berger's method of cellular decompositions. The Swiss cheese operad that we consider is denoted by $\mathcal{SC}_m$, $m \geq 1$, and is the augmented (cubical) version of Voronov's Swiss cheese operad $\mathcal{SC}_m^{\mathrm{vor}}$ defined in [23]. We construct a cellular decomposition of $\mathcal{SC}_m$ that generalizes cell decomposition of the little m–cubes operad by Berger [5]. The latter is indexed by the *extended complete graph operad* $\mathcal{K}_m$. In contrast to the nonrelative case, there are two ways to index the cells of $\mathcal{SC}_m$. This naturally leads us to consider two different indexing operads $\mathcal{RK}_m$ and $\mathcal{RK}'_m$ that may be thought of as the relative versions of $\mathcal{K}_m$. One obtains:

1.1 Theorem *Let $m \geq 1$. Any topological $\mathcal{RK}_m$–cellular operad (resp. $\mathcal{RK}'_m$–cellular operad) is weakly equivalent to the Swiss cheese operad $\mathcal{SC}_m$.*

The operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta)$ has a decomposition by "cells" that are indexed by the poset operad $\mathcal{RK}_m$. This is obtained by means of a map $c_{\mathrm{tot}}\colon \mathcal{RL}_m \to \mathcal{RK}_m$ satisfying $c_{\mathrm{tot}}(x \circ_i y) \leq c_{\mathrm{tot}}(x) \circ_i c_{\mathrm{tot}}(y)$. From this (and similar considerations for $\mathcal{RL}'_m$), one obtains:

1.2 Theorem *Let $m \geq 1$. The operads $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathrm{Top}})$ and $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathrm{Top}})$ are weakly equivalent to the topological Swiss cheese operad $\mathcal{SC}_m$. The operads $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ and $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathbb{Z}})$ are weakly equivalent to the chain Swiss cheese operad $C_*(\mathcal{SC}_m)$.*

Note that, for each $m \geq 1$, the operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ (resp. $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathbb{Z}})$) admits a weakly equivalent suboperad $\mathcal{RS}_m$ (resp. $\mathcal{RS}'_m$). These two operads $\mathcal{RS}_m$ and $\mathcal{RS}'_m$

are relative versions of the *surjection operad* $\mathcal{S}_m$ studied by McClure and Smith [19; 20] and Berger and Fresse [6].

With regards to future applications, we pay close attention to the operads $\mathcal{RL}_2$ and $\mathcal{RL}'_2$. They have a description by planar rooted trees with different types of vertices. The operad $\mathcal{RL}_2$ encodes pairs $(\mathcal{M},\mathcal{Z})$ where $\mathcal{M}$ is a multiplicative nonsymmetric operad and $\mathcal{Z}$ is bimodule over $\mathcal{M}$–$\mathcal{A}$s (left action of $\mathcal{M}$, right action of $\mathcal{A}$s) where $\mathcal{A}$s is the operad of associative algebras; such a pair is naturally endowed with a morphism of bimodules $\iota\colon \mathcal{M}\to\mathcal{Z}$. The operad $\mathcal{RL}'_2$ encodes almost similar pairs $(\mathcal{M},\mathcal{Z}')$ where, instead, $\mathcal{Z}'$ and $\iota\colon \mathcal{M}\to\mathcal{Z}'$ are in the category of bimodules over $\mathcal{A}$s–$\mathcal{M}$.

Outline of the paper Section 2 is first devoted to setting our conventions and notations on nonsymmetric, symmetric, coloured and SC-type operads. In particular, we spend time on modules and weak modules over a nonsymmetric operad; this will finally be used in Section 6. Afterwards, we define the (symmetric) *coloured SC-operads*, and we explain how we *condense* them to obtain SC-type operads.

In Section 3 we consider the (cubical) Swiss cheese operad $\mathcal{SC}$. For each integer $m\geq 1$, we construct two cellular decompositions of $\mathcal{SC}_m$, one indexed by $\mathcal{RK}_m$ and the other indexed by $\mathcal{RK}'_m$.

Section 4 concerns the relative lattice path operad $\mathcal{RL}$ which is a coloured SC-operad. By using the condensation process from Section 2 one obtains an SC-type operad $\mathrm{Coend}_{\mathcal{RL}}(\delta)$. We use results of Section 3 to prove Theorem 1.2. We end the section by exhibiting a few examples of representations of $\mathcal{RL}_m$ and $\mathcal{RL}'_m$. In particular, we show that the operads $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathrm{Top}})$ and $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathrm{Top}})$ act on the pair (m–fold loop space, m–fold relative loop space).

In Section 5 we exhibit the suboperads $\mathcal{RS}_m$ and $\mathcal{RS}'_m$ and show that the inclusions $\mathcal{RS}_m\hookrightarrow \mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ and $\mathcal{RS}'_m\hookrightarrow \mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathbb{Z}})$ are weak equivalences.

In Section 6 we focus on the operads $\mathcal{RL}_2$ and $\mathcal{RL}'_2$ and their representations. We describe $\mathcal{RL}_2$ and $\mathcal{RL}'_2$ in terms of planar rooted trees. This provides a convenient language for describing the representations of $\mathcal{RL}_2$ and $\mathcal{RL}'_2$.

Acknowledgements I would like to warmly thank both Muriel Livernet for corrections and suggestions on an earlier version of this paper and Eduardo Hoefel for his constant support. I am highly indebted to the referee for numerous enlightening comments, suggestions and corrections. I was partially supported by Bolsista da CAPES à Projeto 88881.030367/2013-01.

2 Preliminaries

All along the paper $\boldsymbol{C} = (C, \otimes, 1_{\boldsymbol{C}})$ denotes a cocomplete closed monoidal symmetric category. In particular, $\boldsymbol{C}$ is endowed with an object $0 \in \boldsymbol{C}$ such that $0 \otimes X = 0 = X \otimes 0$ for all $X \in \boldsymbol{C}$.

For two $\boldsymbol{C}$–categories $\boldsymbol{A}$ and $\boldsymbol{B}$ (ie enriched over $\boldsymbol{C}$), we denote by $\boldsymbol{A} \otimes \boldsymbol{B}$ the category with the pairs (a, b) for $a \in \boldsymbol{A}$ and $b \in \boldsymbol{B}$ as objects and

$$\mathrm{Hom}_{\boldsymbol{A}\otimes\boldsymbol{B}}((a,b),(a',b')) := \mathrm{Hom}_{\boldsymbol{A}}(a,a') \otimes \mathrm{Hom}_{\boldsymbol{B}}(b,b')$$

as hom-objects, where the tensor on the right-hand side is the tensor of $\boldsymbol{C}$.

2A Nonsymmetric operads and modules

By a *nonsymmetric* (*non-*Σ) *operad* $\mathcal{M}$ we mean a collection $\{\mathcal{M}(n)\}_{n\geq 0}$ of objects of $\boldsymbol{C}$ together with a unit $\eta\colon 1_{\boldsymbol{C}} \to \mathcal{M}(1)$ and partial composition maps

$$\circ_i\colon \mathcal{M}(m) \otimes \mathcal{M}(n) \to \mathcal{M}(m+n-1) \quad \text{for } 1 \leq i \leq m$$

that satisfy associativity and unit conditions (see [17, Definition 1.14]). One denotes by $\mathcal{A}s$ the non-Σ operad of associative algebras in $\boldsymbol{C}$ given by $\mathcal{A}s(n) = 1_{\boldsymbol{C}}$ for $n \geq 0$. A non-Σ operad $\mathcal{M}$ is called *multiplicative* if there is a morphism of operads $\alpha\colon \mathcal{A}s \to \mathcal{M}$. A morphism of multiplicative operads $\alpha\colon \mathcal{A}s \to \mathcal{M}$ and $\alpha'\colon \mathcal{A}s \to \mathcal{M}'$ is a morphism of operads $f\colon \mathcal{M} \to \mathcal{M}'$ such that $f \circ \alpha = \alpha'$.

In what follows, we recall the notions of left module, right module and bimodule over a non-Σ operad.

2.1 Definition Let $\mathcal{M}$ and $\mathcal{N}$ be two non-Σ operads and let $\mathcal{Z} = \{\mathcal{Z}(m)\}_{m\geq 0}$ be a collection of objects of $\boldsymbol{C}$. Consider the morphisms

$$\rho_i\colon \mathcal{Z}(m) \otimes \mathcal{N}(k) \to \mathcal{Z}(k+m-1) \quad \text{for } 1 \leq i \leq m \qquad \text{(right action)},$$
$$\lambda\colon \mathcal{M}(k) \otimes \mathcal{Z}(m_1) \otimes \cdots \otimes \mathcal{Z}(m_k) \to \mathcal{Z}(m_1 + \cdots + m_k) \quad \text{(left action)}.$$

Consider the following relations:

(1) **Unit condition of the left action** $\mathcal{Z}(m) = 1_{\boldsymbol{C}} \otimes \mathcal{Z}(m) \xrightarrow{\eta\otimes\mathrm{id}} \mathcal{M}(1) \otimes \mathcal{Z}(m) \xrightarrow{\lambda} \mathcal{Z}(m)$ is the identity.

(2) **Associativity of the left action** All the diagrams of the form

$$\begin{tikzcd}
\mathcal{M}(k)\otimes\mathcal{M}(l)\otimes\bigotimes_{1\le p\le k+l-1}\mathcal{Z}(m_p) \arrow[r, "(\mathrm{id}\otimes\lambda\otimes\mathrm{id})\circ\tau_{2,(3,i+1)}"] \arrow[d, "(-\circ_i-)\otimes\mathrm{id}"] & \begin{array}{c}\mathcal{M}(k)\otimes\bigotimes_{1\le p<i}\mathcal{Z}(m_p)\\ \otimes\,\mathcal{Z}(x)\otimes\bigotimes_{i+l\le p\le k+l-1}\mathcal{Z}(m_p)\end{array} \arrow[d, "\lambda"] \\
\mathcal{M}(k+l-1)\otimes\bigotimes_{1\le p\le k+l-1}\mathcal{Z}(m_p) \arrow[r, "\lambda"] & \mathcal{Z}(m_1+\cdots+m_{k+l-1})
\end{tikzcd}$$

commute, where $x := m_i+\cdots+m_{i+l-1}$, for $1\le i\le k$, and $\tau_{2,(3,i+1)}$ denotes the permutation of the second factor with the $i-1$ factors that follow.

(3) **Unit condition of the right action** $\mathcal{Z}(m)=\mathcal{Z}(m)\otimes 1_C\xrightarrow{\mathrm{id}\otimes\eta}\mathcal{Z}(m)\otimes\mathcal{N}(1)\xrightarrow{\rho_i}\mathcal{Z}(m)$ is the identity.

(4) **Associativity and commutativity of the right action on different inputs** All the diagrams of the following forms commute:

$$\begin{tikzcd}
\mathcal{Z}(m)\otimes\mathcal{N}(k)\otimes\mathcal{N}(l) \arrow[r, "\rho_t\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes(-\circ_i-)"] & \mathcal{Z}(m+k-1)\otimes\mathcal{N}(l) \arrow[d, "\rho_{t+i-1}"] \\
\mathcal{Z}(m)\otimes\mathcal{N}(k+l-1) \arrow[r, "\rho_t"] & \mathcal{Z}(m+k+l-2)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{Z}(m)\otimes\mathcal{N}(k)\otimes\mathcal{N}(l) \arrow[r, "\rho_t\otimes\mathrm{id}"] \arrow[d, "\rho_t\circ\tau_{2,3}"] & \mathcal{Z}(m+k-1)\otimes\mathcal{N}(l) \arrow[d, "\rho_{t+i-1}"] \\
\mathcal{Z}(m+l-1)\otimes\mathcal{N}(k) \arrow[r, "\rho_i"] & \mathcal{Z}(m+k+l-2)
\end{tikzcd}$$

(5) **Associativity of the right and left actions** All the diagrams of the form

$$\begin{tikzcd}
\mathcal{M}(k)\otimes\bigotimes_{1\le p\le k}\mathcal{Z}(m_p)\otimes\mathcal{N}(l) \arrow[r, "(\mathrm{id}\otimes\mathrm{id}^{\otimes s-1}\otimes\rho_i\otimes\mathrm{id}^{\otimes k-s})\circ\tau_{(s+2,k+1),k+2}"] \arrow[d, "\lambda\otimes\mathrm{id}"'] & \begin{array}{c}\mathcal{M}(k)\otimes\bigotimes_{1\le p<s}\mathcal{Z}(m_p)\\ \otimes\,\mathcal{Z}(x)\otimes\bigotimes_{s<p\le k}\mathcal{Z}(m_p)\end{array} \arrow[d, "\lambda"] \\
\mathcal{Z}(m_1+\cdots+m_k)\otimes\mathcal{N}(l) \arrow[r, "\rho_{i+m_1+\cdots+m_{s-1}-s+1}"] & \mathcal{Z}(m_1+\cdots+m_k+l-1)
\end{tikzcd}$$

commute, where $x=m_s+l-1$.

We establish the following terminology:

- $(\mathcal{Z},\lambda)$ is called a *left module* over $\mathcal{M}$ if λ satisfies (1) and (2).
- $(\mathcal{Z},\rho)$ is called a *right module* over $\mathcal{N}$ if the ρ_i satisfy (3) and (4).
- $(\mathcal{Z},\lambda,\rho)$ is called a *bimodule* over $\mathcal{M}$–$\mathcal{N}$ if λ and ρ_i satisfy (1)–(5)

2.2 Example A non-Σ operad $\mathcal{M}$ is canonically a bimodule over itself (ie over $\mathcal{M}$–$\mathcal{M}$).

Given two non-Σ operads $\mathcal{M}$ and $\mathcal{N}$, the category of bimodules over $\mathcal{M}$–$\mathcal{N}$ is denoted by $\mathbf{BiMod}_{\mathcal{M}-\mathcal{N}}$; the morphisms commute with each of the actions. A bimodule over $\mathcal{M}$–$\mathcal{M}$ is simply called a bimodule over $\mathcal{M}$.

2.3 Definition Let $\mathcal{Z}'$ be a left (resp. right, bi) module over $\mathcal{M}'$. A morphism of non-Σ operads $f\colon \mathcal{M}\to\mathcal{M}'$ endows $\mathcal{Z}'$ with a structure of left (resp. right, bi) module over $\mathcal{M}$ by pulling back the actions maps along f; we denote the resulting left (resp. right, bi) module over $\mathcal{M}$ by $f^*\mathcal{Z}'$. Explicitly, $\lambda^{f^*\mathcal{Z}'}=\lambda'\circ(f\otimes \mathrm{id}^{\otimes k})$ and $\rho_i^{f^*\mathcal{Z}'}=\rho_i'\circ(\mathrm{id}\otimes f)$.

Moreover, if $\mathcal{Z}$ is a left (resp. right, bi) module over $\mathcal{M}$, then a morphism $g\colon \mathcal{Z}\to\mathcal{Z}'$ is said *f–equivariant* if it is a morphism of left (resp. right, bi) modules over $\mathcal{M}$, $g\colon \mathcal{Z}\to f^*\mathcal{Z}'$.

2.4 Example A left module over a multiplicative operad is, by pullback, a left module over $\mathcal{A}\mathrm{s}$.

In [22, Definition 4.1], the notion of *weak* bimodule over a non-Σ operad $\mathcal{M}$ (in **Top**) is introduced. In our framework, it refers to a right module over $\mathcal{M}$ together with a *weak* left action

$$\lambda_i^w\colon \mathcal{M}(k)\otimes\mathcal{Z}(m)\to\mathcal{Z}(k+m-1)\quad \text{for } 1\le i\le k$$

that satisfies natural associativity and unit conditions, ie (1) above and (a)–(c) below:

(a) **Associativity of the weak left action** Diagrams of the following form commute for all $1\le i\le k$ and $1\le j\le l$:

$$\begin{tikzcd}
\mathcal{M}(k)\otimes\mathcal{M}(l)\otimes\mathcal{Z}(m) \arrow[r, "\mathrm{id}\otimes\lambda_j^w"] \arrow[d, "(-\circ_i-)\otimes\mathrm{id}"'] & \mathcal{M}(k)\otimes\mathcal{Z}(l+m-1) \arrow[d, "\lambda_i^w"] \\
\mathcal{M}(k+l-1)\otimes\mathcal{Z}(m) \arrow[r, "\lambda_{j+i-1}^w"'] & \mathcal{Z}(k+l+m-2)
\end{tikzcd}$$

(b) **Associativity of the right and left actions** Diagrams of the following form commute for all $1 \le i \le k$ and $1 \le t \le m$:

$$\begin{tikzcd}
\mathcal{M}(k) \otimes \mathcal{Z}(m) \otimes \mathcal{M}(l) \arrow[r, "\mathrm{id}\otimes\rho_t"] \arrow[d, "\lambda_i^w\otimes\mathrm{id}"] & \mathcal{M}(k) \otimes \mathcal{Z}(m+l-1) \arrow[d, "\lambda_i^w"] \\
\mathcal{Z}(k+m-1) \otimes \mathcal{M}(l) \arrow[r, "\rho_{i+t-1}"'] & \mathcal{Z}(k+m+l-2)
\end{tikzcd}$$

(c) **Compatibility between the actions and the operad composition** Diagrams of the following form commute for all $1 \le t < i \le k$:

$$\begin{tikzcd}
\mathcal{M}(k) \otimes \mathcal{Z}(m) \otimes \mathcal{M}(l) \arrow[r, "((-\circ_t -)\otimes\mathrm{id})\circ\tau_{2,3}"] \arrow[d, "\lambda_i^w\otimes\mathrm{id}"'] & \mathcal{M}(k+l-1) \otimes \mathcal{Z}(m) \arrow[d, "\lambda_{i+l-1}^w"] \\
\mathcal{Z}(k+m-1) \otimes \mathcal{M}(l) \arrow[r, "\rho_t"] & \mathcal{Z}(k+m+l-2)
\end{tikzcd}$$

and those of the following form commute for all $1 \le i < t \le k$:

$$\begin{tikzcd}
\mathcal{M}(k) \otimes \mathcal{Z}(m) \otimes \mathcal{M}(l) \arrow[r, "((-\circ_t -)\otimes\mathrm{id})\circ\tau_{2,3}"] \arrow[d, "\lambda_i^w\otimes\mathrm{id}"'] & \mathcal{M}(k+l-1) \otimes \mathcal{Z}(m) \arrow[d, "\lambda_i^w"] \\
\mathcal{Z}(k+m-1) \otimes \mathcal{M}(l) \arrow[r, "\rho_{t+m-1}"] & \mathcal{Z}(k+m+l-2)
\end{tikzcd}$$

2.5 Example A non-Σ operad $\mathcal{M}$ is a weak bimodule over itself.

2.6 Example Consider a bimodule $\mathcal{Z}$ over $\mathcal{M}$ and suppose it comes together with a bimodule map $\iota\colon \mathcal{M} \to \mathcal{Z}$; recall that $\eta\colon 1_{\mathbf{C}} \to \mathcal{M}(1)$ denotes the unit. Precomposing the left action λ of $\mathcal{Z}$ by $\iota\eta$ at all but one input provides maps

$$\lambda_i^w := \lambda \circ (\mathrm{id} \otimes (\iota\eta)^{\otimes i-1} \otimes \mathrm{id} \otimes (\iota\eta)^{\otimes k-i})\colon \mathcal{M}(k) \otimes \mathcal{Z}(m) \to \mathcal{Z}(k+m-1)$$

that endow $\mathcal{Z}$ with a weak bimodule structure over $\mathcal{M}$. Moreover, for this structure, ι is a morphism of weak bimodules.

As observed in [22, Lemma 4.2], one of the interesting features of the *weak* bimodules is the following.

2.7 Lemma *The structure of a cosimplicial object is equivalent to the structure of a weak bimodule over $\mathcal{A}s$.*

2B Coloured operads

Coloured (symmetric) operads are also known as *small symmetric multicategories*; see [16, Definition 2.2.21]. For our purposes one uses left actions for the symmetric groups Σ_n, so that a coloured operad with set of colours Col (or a Col–coloured operad) consists of the following:

- for each $k \geq 0$ and each $(k+1)$–tuple $(e_1, \dots, e_k; e)$ of colours $e_i, e \in \mathrm{Col}$, an object $\mathcal{O}(e_1, \dots, e_k; e)$ in $\boldsymbol{C}$;
- for each colour e, a unit $1_{\boldsymbol{C}} \to \mathcal{O}(e; e)$;
- for each $1 \leq i \leq k$, $(e_1, \dots, e_k; e)$ and $(f_1, \dots, f_l; e_i)$, substitution maps
$$\circ_i\colon \mathcal{O}(e_1, \dots, e_k; e) \otimes \mathcal{O}(f_1, \dots, f_l; e_i) \\ \to \mathcal{O}(e_1, \dots, e_{i-1}, f_1, \dots, f_l, e_{i+1}, \dots, e_k; e);$$
- for each $\sigma \in \Sigma_k$, a map $\sigma_*\colon \mathcal{O}(e_1, \dots, e_k; e) \to \mathcal{O}(e_{\sigma^{-1}(1)}, \dots, e_{\sigma^{-1}(k)}; e)$.

Substitution maps are required to satisfy the natural unit, associativity and equivariance axioms.

2.8 Example A *symmetric* operad $\mathcal{P}$ is a 1–coloured operad with
$$\mathcal{P}(k) = \mathcal{P}(\underbrace{*, \dots, *}_{k}; *).$$

For a coloured operad $\mathcal{O}$, the category of its unary operations is called its *underlying category* and is denoted by $\mathcal{O}_u$: the objects of $\mathcal{O}_u$ are the colours; the morphisms are $\mathcal{O}_u(e, f) := \mathcal{O}(e; f)$ for $e, f \in \mathrm{Col}$. The operadic structure of $\mathcal{O}$ is encoded as functors
$$\mathcal{O}(\underbrace{-, \dots, -}_{k}; -)\colon (\mathcal{O}_u^{\mathrm{op}})^{\otimes k} \otimes \mathcal{O}_u \to \boldsymbol{C}, \quad k \geq 0.$$

Recall that an $\mathcal{O}$–algebra X is a family $\{X(e)\}_{e \in \mathrm{Col}}$ of objects $X(e) \in \boldsymbol{C}$ equipped with morphisms

$$\mathcal{O}(e_1, \dots, e_k; e) \otimes X(e_1) \otimes \dots \otimes X(e_k) \to X(e), \quad e_1, \dots, e_k, e \in \mathrm{Col}, \tag{2-1}$$

subject to the natural unit, associative and equivariance axioms.

2.9 Definition Let $\boldsymbol{C}$ be a *monoidal model category*; see [14]. A morphism of Col–coloured operads $\mathcal{O} \to \mathcal{O}'$ is a *weak equivalence* if and only if each of its components $\mathcal{O}(n_1, \dots, n_k; n) \to \mathcal{O}'(n_1, \dots, n_k; n)$ is a weak equivalence. Two Col–coloured operads $\mathcal{O}$ and $\mathcal{O}'$ are said *weakly equivalent* if there is a zig-zag of weak equivalences of Col–coloured operads $\mathcal{O} \leftarrow \cdot \rightarrow \mathcal{O}'$.

2.10 Remark The weak equivalence as described above is part of the model category structure on the category of Col–coloured operads as established in [7, Section 3] whenever $\boldsymbol{C}$ satisfies properties of [7, Theorem 2.1].

2C SC-operads and coloured SC-operads

The SC-type operads (see [1]) are a special type of 2–coloured operads whose structure mimics that of the Swiss cheese operad introduced in [23]. Explicitly, by an *SC-type operad* (or SC-operad), one means a $\{\mathfrak{cl}, \mathfrak{op}\}$–coloured operad $\mathcal{O}$ such that $\mathcal{O}(c_1, \dots, c_n; \mathfrak{cl}) = 0$ if there exists a $1 \leq i \leq n$ such that $c_i = \mathfrak{op}$. The colour $\mathfrak{cl}$ is called the *closed* colour; the colour $\mathfrak{op}$ is called the *open* colour.

Let us define the coloured version of the SC-type operads.

2.11 Definition Let Col be a set (of colours). A *coloured SC-operad* is a Col–coloured operad $\mathcal{O}$ that satisfies the following hypotheses:

(H1) $\mathrm{Col} = \mathrm{Col}_{\mathfrak{cl}} \sqcup \mathrm{Col}_{\mathfrak{op}}$.

(H2) The collection of the $\mathcal{O}(e_1, \dots, e_k; e)$ for $e_i, e \in \mathrm{Col}_{\mathfrak{cl}}$ and $k \geq 0$ forms a suboperad of $\mathcal{O}$.

(H3) The collection of the $\mathcal{O}(e_1, \dots, e_j; e)$ for $e_i, e \in \mathrm{Col}_{\mathfrak{op}}$ and $j \geq 0$ forms a suboperad of $\mathcal{O}$.

(H4) $\mathcal{O}(e_1, \dots, e_j; e) = 0 \in \boldsymbol{C}$ for any $e \in \mathrm{Col}_{\mathfrak{cl}}$ if there exists $1 \leq i \leq j$ such that $e_i \in \mathrm{Col}_{\mathfrak{op}}$, where $j \geq 1$.

The suboperad in (H2) is called the *closed part of* $\mathcal{O}$; the suboperad in (H3) is called the *open part of* $\mathcal{O}$. For $c \in \{\mathfrak{cl}; \mathfrak{op}\}$, a colour of Col_c is called *colour of type* c or c *colour*.

2.12 Example An SC-type operad is a Col–coloured SC-operad with $\mathrm{Col}_{\mathfrak{cl}} = \{\mathfrak{cl}\}$ and $\mathrm{Col}_{\mathfrak{op}} = \{\mathfrak{op}\}$.

The underlying category $\mathcal{O}_u$ of a coloured SC-operad contains two particular categories:

- $\mathcal{O}_u^{\mathfrak{cl}}$ is the subcategory of $\mathcal{O}_u$ with objects the colours in $\mathrm{Col}_{\mathfrak{cl}}$ and morphisms the $\mathcal{O}_u(e, f)$ for $e, f \in \mathrm{Col}_{\mathfrak{cl}}$;
- $\mathcal{O}_u^{\mathfrak{op}}$ is the subcategory of $\mathcal{O}_u$ with objects the colours in $\mathrm{Col}_{\mathfrak{op}}$ and morphisms the $\mathcal{O}_u(e, f)$ for $e, f \in \mathrm{Col}_{\mathfrak{op}}$.

Also, from the hypothesis on $\mathcal{O}$, an $\mathcal{O}$–algebra X can be seen as a pair $(X_{\mathfrak{cl}}, X_{\mathfrak{op}})$ where $X_{\mathfrak{cl}}$ is the subfamily $\{X_{\mathfrak{cl}}(e)\}_{e\in \mathrm{Col}_{\mathfrak{cl}}}$ and $X_{\mathfrak{op}}$ is the subfamily $\{X_{\mathfrak{op}}(e)\}_{e\in \mathrm{Col}_{\mathfrak{op}}}$.

2D SC functor-operads

Here one defines the SC analogues to the functor-operads and their algebras. The notion of functor-operad is introduced in [21] and generalizes the notion of operad; see also [3].

Let us fix two $\boldsymbol{C}$–categories $\boldsymbol{A}$ and $\boldsymbol{B}$, and let $k \geq 0$.

For a collection of $\boldsymbol{C}$–functors $\{\xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\colon \boldsymbol{A}_1\otimes\cdots\otimes \boldsymbol{A}_k \to \boldsymbol{A}_{k+1}\}_{\boldsymbol{A}_i\in\{\boldsymbol{A},\boldsymbol{B}\}}$ and for $\sigma\in\Sigma_k$, we denote by $\xi^{\sigma}_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\colon \boldsymbol{A}_1\otimes\cdots\otimes \boldsymbol{A}_k \to \boldsymbol{A}_{k+1}$ the functor

$$\xi^{\sigma}_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}(X_1,\dots,X_k) = \xi_{\boldsymbol{A}_{\sigma^{-1}(1)},\dots,\boldsymbol{A}_{\sigma^{-1}(k)};\boldsymbol{A}_{k+1}}(X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(k)}).$$

A collection of $\boldsymbol{C}$–functors $\{\xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\colon \boldsymbol{A}_1\otimes\cdots\otimes \boldsymbol{A}_k \to \boldsymbol{A}_{k+1}\}_{\boldsymbol{A}_i\in\{\boldsymbol{A},\boldsymbol{B}\}}$ is called *twisted symmetric* if there exist $\boldsymbol{C}$–natural transformations

$$\phi_{\sigma,\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\colon \xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}} \to \xi^{\sigma}_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}$$

for $\sigma\in\Sigma_k$ such that $\phi_{\sigma_1\sigma_2,\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}} = (\phi_{\sigma_1,\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}})^{\sigma_2}\phi_{\sigma_2,\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}$ and such that $\phi_{\mathrm{id},\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}$ is the identity transformation where id denotes the neutral element of Σ_k.

2.13 Definition An *SC functor-operad* $\xi = \{\xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\}_{k,\boldsymbol{A}_i}$ over $(\boldsymbol{A},\boldsymbol{B})$ is the data, for each $k\geq 0$, of a twisted symmetric collection

$$\xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\colon \boldsymbol{A}_1\otimes\cdots\otimes \boldsymbol{A}_k \to \boldsymbol{A}_{k+1}$$

indexed by the $(k+1)$–tuples $(\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1})$ of categories in $\{\boldsymbol{A},\boldsymbol{B}\}$ such that $\boldsymbol{A}_{k+1}=\boldsymbol{B}$ whenever there exists $1\leq i\leq k$ such that $\boldsymbol{A}_i=\boldsymbol{B}$. Such a collection is required to be endowed with natural transformations

$$\begin{aligned}\mu_{[\boldsymbol{A}]_{1,i_1},\dots,[\boldsymbol{A}]_{k,i_k};\boldsymbol{A}_{k+1}}\colon \xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\circ(\xi_{\boldsymbol{A}_{1,1},\dots,\boldsymbol{A}_{1,i_1};\boldsymbol{A}_1}\otimes\cdots\otimes\xi_{\boldsymbol{A}_{k,1},\dots,\boldsymbol{A}_{k,i_k};\boldsymbol{A}_k})&\\ \to \xi_{\boldsymbol{A}_{1,1},\dots,\boldsymbol{A}_{k,i_k};\boldsymbol{A}_{k+1}}&\end{aligned}$$

for $i_1,\dots,i_k\geq 0$, where $[\boldsymbol{A}]_{a,b} = (\boldsymbol{A}_{a,1},\dots,\boldsymbol{A}_{a,b};\boldsymbol{A}_a)$. These natural transformations have to satisfy the following three conditions:

(1) For $\boldsymbol{A}_0\in\{\boldsymbol{A},\boldsymbol{B}\}$, the functor $\xi_{\boldsymbol{A}_0;\boldsymbol{A}_0}$ is the identity, and

$$\xi_{\boldsymbol{A}_{k+1};\boldsymbol{A}_{k+1}}\circ\overline{\xi}_k = \overline{\xi}_k = \overline{\xi}_k\circ(\xi_{\boldsymbol{A}_1;\boldsymbol{A}_1}\otimes\cdots\otimes\xi_{\boldsymbol{A}_k;\boldsymbol{A}_k}),$$

where we let $\overline{\xi}_k$ denote $\xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}$, and where the equalities are obtained via $\mu_{(\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1});\boldsymbol{A}_{k+1}}$ and $\mu_{(\boldsymbol{A}_1;\boldsymbol{A}_1),\dots,(\boldsymbol{A}_k;\boldsymbol{A}_k);\boldsymbol{A}_{k+1}}$, respectively.

(2) The natural transformations $\mu_{[\boldsymbol{A}]_{1,i_1},\dots,[\boldsymbol{A}]_{k,i_k};\boldsymbol{A}_{k+1}}$ are associative.

(3) All diagrams of the form

$$\begin{tikzcd}
\xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\circ(\xi_{\boldsymbol{A}_{1,1},\dots,\boldsymbol{A}_{1,i_1};\boldsymbol{A}_1}\otimes\cdots\otimes\xi_{\boldsymbol{A}_{k,1},\dots,\boldsymbol{A}_{k,i_k};\boldsymbol{A}_k}) \arrow[r] \arrow[d, "\phi_\sigma\circ(\phi_{\sigma_1}\otimes\cdots\otimes\phi_{\sigma_{k+j}})"'] & \xi_{\boldsymbol{A}_{1,1},\dots,\boldsymbol{A}_{k,i_k};\boldsymbol{A}_{k+1}} \arrow[d, "\phi_{\sigma(\sigma_1,\dots,\sigma_{k+j})}"'] \\
\xi^{\sigma}_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\circ(\xi^{\sigma_1}_{\boldsymbol{A}_{1,1},\dots,\boldsymbol{A}_{1,i_1};\boldsymbol{A}_1}\otimes\cdots\otimes\xi^{\sigma_k}_{\boldsymbol{A}_{k,1},\dots,\boldsymbol{A}_{k,i_k};\boldsymbol{A}_k}) \arrow[r] & \xi^{\sigma(\sigma_1,\dots,\sigma_{k+j})}_{\boldsymbol{A}_{1,1},\dots,\boldsymbol{A}_{k,i_k};\boldsymbol{A}_{k+1}}
\end{tikzcd}$$

commute, where the horizontal maps are $\mu_{[\boldsymbol{A}]_{1,i_1},\dots,[\boldsymbol{A}]_{k,i_k};\boldsymbol{A}_{k+1}}$.

2.14 Definition Let $\xi=\{\xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\}_{k\geq 0}$ be an SC functor-operad over $(\boldsymbol{A},\boldsymbol{B})$. A *$\xi$–algebra* X is a pair $(X_{\boldsymbol{A}},X_{\boldsymbol{B}})\in\boldsymbol{A}\otimes\boldsymbol{B}$ equipped with morphisms in $\boldsymbol{A}_{k+1}$,

$$\alpha_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\colon \xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}(X_{\boldsymbol{A}_1},\dots,X_{\boldsymbol{A}_k})\to X_{\boldsymbol{A}_{k+1}},\quad k\geq 0,$$

subject to the following conditions:

(1) $\alpha_{\boldsymbol{A}_1}=1_{X_{\boldsymbol{A}_1}}$.

(2) $\alpha_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}\circ\phi_\sigma=\alpha_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}$, for all $\sigma\in\Sigma_k$.

(3) All the diagrams of the following form commute:

$$\begin{tikzcd}
\overline{\xi}_k\circ(\xi_{[\boldsymbol{A}]_{1,i_1}}(X_{1,i_1})\otimes\cdots\otimes\xi_{[\boldsymbol{A}]_{k,i_k}}(X_{k,i_k})) \arrow[r] \arrow[d, "\overline{\xi}_k(\alpha_{[\boldsymbol{A}]_{1,i_1}}\otimes\cdots\otimes\alpha_{[\boldsymbol{A}]_{k,i_k}})"] & \xi_{\boldsymbol{A}_{1,1},\dots,\boldsymbol{A}_{k,i_k};\boldsymbol{A}_{k+1}}(X_{\boldsymbol{A}_{1,1}},\dots,X_{\boldsymbol{A}_{k,i_k}}) \arrow[d, "\alpha_{\boldsymbol{A}_{1,1},\dots,\boldsymbol{A}_{k,i_k};\boldsymbol{A}_{k+1}}"] \\
\overline{\xi}_k(X_{\boldsymbol{A}_1},\dots,X_{\boldsymbol{A}_{k+1}}) \arrow[r, "\alpha_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}"'] & X_{\boldsymbol{A}_{k+1}}
\end{tikzcd}$$

where $X_{a,b}=X_{\boldsymbol{A}_{a,1}},\dots,X_{\boldsymbol{A}_{a,b}}$, $\overline{\xi}_k=\xi_{\boldsymbol{A}_1,\dots,\boldsymbol{A}_k;\boldsymbol{A}_{k+1}}$, $[A]_{a,b}=(\boldsymbol{A}_{a,1},\dots,\boldsymbol{A}_{a,b};\boldsymbol{A}_a)$, and the top horizontal map is $\mu_{[\boldsymbol{A}]_{1,i_1},\dots,[\boldsymbol{A}]_{k,i_k};\boldsymbol{A}_{k+1}}$.

2.15 Remark An SC functor-operad over $(\boldsymbol{A},\boldsymbol{B})$ is a particular example of an *internal symmetric operad* in $\mathbf{End}_{\boldsymbol{A},\boldsymbol{B}}$ (the endomorphism SC-type operad of $(\boldsymbol{A},\boldsymbol{B})$ in **Cat**). The notion of internal symmetric operad was introduced in [2, Definition 9.3].

2E Condensation

In this section, we explain how we extend the condensation process described in [3, Section 1] to the case of coloured SC-operads.

Let $\mathcal{O}$ be a coloured SC-operad and let $\delta=(\delta_{\mathfrak{cl}},\delta_{\mathfrak{op}})$ be a pair of functors $\delta_{\mathfrak{cl}}\colon \mathcal{O}_u^{\mathfrak{cl}}\to \boldsymbol{C}$ and $\delta_{\mathfrak{op}}\colon \mathcal{O}_u^{\mathfrak{op}}\to \boldsymbol{C}$. We will define the *condensation operad* of $\mathcal{O}$, denoted by $\mathrm{Coend}_{\mathcal{O}}(\delta)$, and an associated functor

$$\mathcal{O}\text{–algebra}\to \mathrm{Coend}_{\mathcal{O}}(\delta)\text{–algebra}.$$

The operad $\mathrm{Coend}_{\mathcal{O}}(\delta)$ is obtained by condensing each type of colours into one colour, so that it is an SC-type operad. It is obtained in two steps.

From $\mathcal{O}$ to the SC functor-operad $\xi(\mathcal{O})$ Recall that, by hypotheses (H2) and (H3) from Definition 2.11, both $\mathcal{O}_u^{\mathfrak{cl}}$ and $\mathcal{O}_u^{\mathfrak{op}}$ are $\boldsymbol{C}$–categories. Moreover, the category $\boldsymbol{C}^{\mathcal{O}_u^{\mathfrak{cl}}}$ (resp. $\boldsymbol{C}^{\mathcal{O}_u^{\mathfrak{op}}}$) of $\boldsymbol{C}$–functors from $\mathcal{O}_u^{\mathfrak{cl}}$ (resp. from $\mathcal{O}_u^{\mathfrak{cl}}$) to $\boldsymbol{C}$ is a $\boldsymbol{C}$–category. For $k\geq 0$ and $(c_1,\dots,c_k;c_{k+1})$ a tuple of elements in $\{\mathfrak{cl};\mathfrak{op}\}$ satisfying

$$c_{k+1}=\mathfrak{op}\quad\text{if there exists } 1\leq i\leq k \text{ such that } c_i=\mathfrak{op}, \tag{2-2}$$

one lets $\boldsymbol{A}_i:=\boldsymbol{C}^{\mathcal{O}_u^{c_i}}$ and one defines the $\boldsymbol{C}$–functor

$$\xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}\colon \boldsymbol{A}_1\otimes\cdots\otimes\boldsymbol{A}_k\to\boldsymbol{A}_{k+1}$$

as the coend

$$\begin{aligned}\xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}&(X_{c_1},\dots,X_{c_k})(n)\\ &=\mathcal{O}(\underbrace{-,\dots,-}_{k};n)\otimes_{\mathcal{O}_u^{c_1}\otimes\cdots\otimes\mathcal{O}_u^{c_k}}X_{c_1}(-)\otimes\cdots\otimes X_{c_k}(-).\end{aligned}$$

We have an SC analogue to [3, Proposition 1.8] or [9]:

2.16 Proposition *The functors $\xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}$ extend to an SC functor-operad $\xi(\mathcal{O})$ such that the category of $\mathcal{O}$–algebras and the category of $\xi(\mathcal{O})$–algebras are isomorphic.*

Proof A straightforward verification, along the lines of [9], shows that the family of the $\xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}$ forms an SC functor-operad.

Via the hypotheses (H2) and (H3) from Definition 2.11, an $\mathcal{O}$–algebra X can be seen as a pair $(X_{\mathfrak{cl}},X_{\mathfrak{op}})$, where $X_{\mathfrak{cl}}$ and $X_{\mathfrak{op}}$ are functors $X_{\mathfrak{cl}}\colon\mathcal{O}_u^{\mathfrak{cl}}\to\boldsymbol{C}$ and $X_{\mathfrak{op}}\colon\mathcal{O}_u^{\mathfrak{op}}\to\boldsymbol{C}$, respectively. Therefore, the maps (2-1) induce the following maps for $e_{k+1} \in \mathrm{Col}$:

$$\xi_{c_1,\dots,c_k;c_{k+1}}(\mathcal{O})(X_{c_1},\dots,X_{c_k})(e_{k+1}) = \\ \int^{e_1,\dots,e_k} \mathcal{O}(e_1,\dots,e_k;e_{k+1}) \otimes X_{c_1}(e_1) \otimes \cdots \otimes X_{c_k}(e_k) \to X_{c_{k+1}}(e_{k+1}).$$

They form the maps $\alpha_{c_1,\dots,c_k;c_{k+1}} : \xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}(X_{c_1},\dots,X_{c_k}) \to X_{c_{k+1}}$ for $k \geq 0$. We conclude that X is a $\xi(\mathcal{O})$–algebra because of the unit, associativity and equivariance properties of maps (2-1). The isomorphism follows from the universal property of the coend. $\square$

From $\xi(\mathcal{O})$ to the coendomorphism operad $\mathrm{Coend}_{\mathcal{O}}(\delta)$ The operad $\mathrm{Coend}_{\mathcal{O}}(\delta)$ is the coendomorphism operad of the SC functor-operad $\xi(\mathcal{O})$. Explicitly,

$$\mathrm{Coend}_{\mathcal{O}}(\delta)(c_1,\dots,c_k;c_{k+1}) = \mathrm{Hom}_{\boldsymbol{C}^{\mathcal{O}_u^{c_{k+1}}}}(\delta_{c_{k+1}}, \xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}(\delta_{c_1},\dots,\delta_{c_k}))$$

for $(c_1,\dots,c_k;c_{k+1})$ satisfying (2-2). The composition maps

$$\mathrm{Coend}_{\mathcal{O}}(\delta)(c_1,\dots,c_k;c_{k+1}) \otimes \mathrm{Coend}_{\mathcal{O}}(\delta)(c_{1,1},\dots,c_{1,i_1};c_1) \otimes \cdots \\ \otimes \mathrm{Coend}_{\mathcal{O}}(\delta)(c_{k,1},\dots,c_{k,i_k};c_k) \\ \longrightarrow \mathrm{Coend}_{\mathcal{O}}(\delta)(c_{1,1},\dots,c_{k,i_k};c_{k+1})$$

are given by sending maps $f \otimes g_1 \otimes \cdots \otimes g_k$ to the composite

$$\delta_{c_{k+1}} \xrightarrow{f} \xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}(\delta_{c_1},\dots,\delta_{c_k}) \\ \xrightarrow{\xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}(g_1,\dots,g_k)} \xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}\big(\xi(\mathcal{O})_{[c]_{1,i_1}}(\delta),\dots,\xi(\mathcal{O})_{[c]_{k,i_k}}(\delta)\big) \\ \xrightarrow{\alpha_{c_1,\dots,c_k;c_{k+1}}} \xi(\mathcal{O})_{c_{1,1},\dots,c_{k,i_k};c_{k+1}}(\delta_{c_{1,1}},\dots,\delta_{c_{k,i_k}}).$$

The action of Σ_k on $\mathrm{Coend}_{\mathcal{O}}(\delta)$ is given by postcomposing with the natural transformations

$$\phi_{\sigma,c_1,\dots,c_k}(\delta_{c_1},\dots,\delta_{c_k}) \colon \xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}(\delta_{c_1},\dots,\delta_{c_k}) \\ \to \xi(\mathcal{O})_{c_{\sigma^{-1}(1)},\dots,c_{\sigma^{-1}(k)};c_{k+1}}(\delta_{c_{\sigma^{-1}(1)}},\dots,\delta_{c_{\sigma^{-1}(k)}}).$$

Given an $\mathcal{O}$–algebra $X = (X_{\mathfrak{cl}}, X_{\mathfrak{op}})$, we set

$$\mathrm{Tot}_{\delta_{\mathfrak{cl}}} X_{\mathfrak{cl}} := \mathrm{Hom}_{\boldsymbol{C}^{\mathcal{O}_u^{\mathfrak{cl}}}}(\delta_{\mathfrak{cl}}, X_{\mathfrak{cl}}) \quad \text{and} \quad \mathrm{Tot}_{\delta_{\mathfrak{op}}} X_{\mathfrak{op}} := \mathrm{Hom}_{\boldsymbol{C}^{\mathcal{O}_u^{\mathfrak{op}}}}(\delta_{\mathfrak{op}}, X_{\mathfrak{op}}).$$

Since X is a $\xi(\mathcal{O})$–algebra by Proposition 2.16 and since $\mathrm{Coend}_{\mathcal{O}}(\delta)$ is the coendomorphism operad of $\xi(\mathcal{O})$, it follows that the pair $(\mathrm{Tot}_{\delta_{\mathfrak{cl}}} X_{\mathfrak{cl}}, \mathrm{Tot}_{\delta_{\mathfrak{op}}} X_{\mathfrak{op}})$ is a $\mathrm{Coend}_{\mathcal{O}}(\delta)$–algebra: the action maps

$$\begin{aligned}(2\text{-}3)\quad \mathrm{Coend}_{\xi(\mathcal{O})}(\delta)(c_1,\dots,c_k;c_{k+1})\otimes \mathrm{Tot}_{\delta_{c_1}} X_{c_1}\otimes\cdots\otimes \mathrm{Tot}_{\delta_{c_k}} X_{c_k}&\\ \to \mathrm{Tot}_{\delta_{c_{k+1}}} X_{c_{k+1}}&\end{aligned}$$

are given by sending maps $f\otimes g_1\otimes\cdots\otimes g_k$ to the composite

$$\begin{aligned}\delta_{c_{k+1}} \xrightarrow{f} \xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}(\delta_{c_1},\dots,\delta_{c_k})&\\ \xrightarrow{\xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}(g_1,\dots,g_k)} \xi(\mathcal{O})_{c_1,\dots,c_k;c_{k+1}}(X_{c_1},\dots,X_{c_k})&\\ \xrightarrow{\alpha_{c_1,\dots,c_k;c_{k+1}}} X_{c_{k+1}}.&\end{aligned}$$

Unit, associative and equivariance properties of the maps (2-3) are deduced from the SC functor-operad properties of $\xi(\mathcal{O})$.

3 Cellular decompositions of the Swiss cheese operad

The little cubes operad $\mathcal{C}$ has a cellular decomposition indexed by the extended complete graph operad $\mathcal{K}$; see [5] and [8, Section 4.1]. We extend this result to the Swiss cheese operads $\mathcal{SC}_m$, $m\geq 1$, which provides a recognition principle for SC-type operads. In particular, we construct a poset operad $\mathcal{RK}_m$ that indexes the cells $(\mathcal{SC}_m)^{(\alpha)}$ of $\mathcal{SC}_m$. This leads to a zig-zag of weak equivalences of operads

$$\mathcal{SC}_m \xleftarrow{\sim} \mathrm{hocolim}_{\alpha\in\mathcal{RK}_m}(\mathcal{SC}_m)^{(\alpha)} \xrightarrow{\sim} \mathcal{BRK}_m$$

between the Swiss cheese operad $\mathcal{SC}_m$ and the operad of the geometric realization of the nerve of $\mathcal{RK}_m$. There is a second way to index the cells $(\mathcal{SC}_m)^{(\alpha)}$; this is done by another poset operad $\mathcal{RK}'_m$, providing a similar zig-zag.

3A The Swiss cheese operad

The Swiss cheese operad that we use is the cubical version of the one defined in [15].

Let $m\geq 1$. Let Sym: $\mathbb{R}^m\to\mathbb{R}^m$ be the reflection $\mathrm{Sym}(x_1,\dots,x_m)=(x_1,\dots,-x_m)$, and let Half_+ be the upper half space

$$\mathrm{Half}_+=\{(x_1,\dots,x_m)\in\mathbb{R}^m \mid x_m>0\}.$$

The standard cube C_0 in $\mathbb{R}^m$ is $C_0=[-1,1]^{\times m}$. A cube C in the standard cube is of the form $C=[x_1,y_1]\times[x_2,y_2]\times\cdots\times[x_m,y_m]$ with $-1<x_j<y_j<1$ for $1\leq j\leq m$.

3.1 Definition For $n\geq 0$ and $c_i,c\in\{\mathfrak{cl},\mathfrak{op}\}$ we define a topological Σ_n–space $\mathcal{SC}_m(c_1,\dots,c_n;c)$ as the empty set if $c=\mathfrak{cl}$ and there exists $1\leq i\leq n$ such that $c_i=\mathfrak{op}$; for the other cases, it is defined as follows:

- The space of the little m–cubes operad $\mathcal{C}^{(m)}(n)$ defined in [18] for $c=\mathfrak{cl}$.
- The empty set if $n=0$.
- The one-point space if $n=1$.
- In the case $s+t=n\geq 2$ with $s,t\geq 0$ such that s colours c_i are $\mathfrak{cl}$ and t colours c_j are $\mathfrak{op}$, the space of configuration of $2s+t$ disjoint cubes $(C_1,\dots,C_{2s+t})$ in the standard cube $C_0\in\mathbb{R}^m$ such that $\mathrm{Sym}(C_i)=C_{i+s}$ for $1\leq i\leq s$ and $\mathrm{Sym}(C_i)=C_i$ for $2s+1\leq i\leq 2s+t$ and such that all the cubes $(C_1,\dots,C_s)$ are in the upper half space.

3.2 Remark Because of the symmetry conditions imposed by Sym, we may think of $\mathcal{SC}_m(c_1,\dots,c_n;\mathfrak{op})$ as the configuration space of cubes $(C_1,\dots,C_s)$ and semicubes $(C_{s+1},\dots,C_{s+t})$ lying in the standard semicube $\mathrm{Half}_+\cap C_0$.

Similarly to the little m–cubes operad $\mathcal{C}^{(m)}$, the composition maps

$$\circ_i\colon \mathcal{SC}_m(c_1,\dots,c_n;c)\times\mathcal{SC}_m(d_1,\dots,d_r;c_i)\\ \to\mathcal{SC}_m(c_1,\dots,c_{i-1},d_1,\dots,d_r,c_{i+1},\dots,c_n;c)$$

are defined as substitutions of cubes. We denote the resulting SC-type operad by $\mathcal{SC}_m$.

3B The SC extended complete graph operad

We define the *SC* (*or relative*) *extended complete graph* operad $\mathcal{RK}$. It is an SC-type operad in the category of posets. We provide two filtrations by suboperads $\{\mathcal{RK}_m\}_{m\geq 1}$ and $\{\mathcal{RK}'_m\}_{m\geq 1}$. Their closed parts are isomorphic to $\mathcal{K}_m$ and their open parts are isomorphic to $\mathcal{K}_{m-1}$, where $\{\mathcal{K}_m\}_{m\geq 1}$ denotes the extended complete graph operad defined in [8, Section 4.1].

3B1 Definition of $\mathcal{RK}$ Given n colours $c_i \in \{\mathfrak{cl}, \mathfrak{op}\}$, we denote by $\{\widetilde{c}_1, \dots, \widetilde{c}_n\}$ the set with

$$\widetilde{c}_i = \begin{cases} i & \text{if } c_i = \mathfrak{cl}, \\ \underline{i} & \text{if } c_i = \mathfrak{op}. \end{cases} \tag{3-1}$$

A colouring and an orientation (μ, σ) on the complete graph on $\{\widetilde{c}_1, \dots, \widetilde{c}_n\}$ is, for each $1 \le i < j \le n$, a strict positive natural number $\mu_{i,j} \in \mathbb{N}^{>0}$ and an orientation $\sigma_{i,j}$ (that is, $\widetilde{c}_i \to \widetilde{c}_j$ or $\widetilde{c}_i \leftarrow \widetilde{c}_j$). A *monochromatic acyclic orientation* of a complete graph is a colouring and an orientation such that there exist no oriented cycles with the same colour, ie there are no configurations of the form $\widetilde{c}_{i_1} \to \widetilde{c}_{i_2} \to \dots \to \widetilde{c}_{i_k} \to \widetilde{c}_{i_1}$ with $\mu_{i_1,i_2} = \mu_{i_2,i_3} = \dots = \mu_{i_{k-1},i_k} = \mu_{i_k,i_1}$. A *marked* monochromatic acyclic orientation $(\mu, \sigma)^c$ is a monochromatic acyclic orientation (μ, σ) together with a colour $c \in \{\mathfrak{cl}, \mathfrak{op}\}$.

If there exists an i such that $c_i = \mathfrak{op}$, then we define $\mathcal{RK}(c_1, \dots, c_n; \mathfrak{cl})$ as the empty set. Otherwise, $\mathcal{RK}(c_1, \dots, c_n; c)$ is the set of the marked monochromatic acyclic orientations $(\mu, \sigma)^c$ of the complete graph on $\{\widetilde{c}_1, \dots, \widetilde{c}_n\}$.

The poset structure is given by

$$(\mu, \sigma)^c \le (\mu', \sigma')^c \quad \Longleftrightarrow \quad (\mu_{i,j}, \sigma_{i,j}) = (\mu'_{i,j}, \sigma'_{i,j}) \text{ or } \mu_{i,j} < \mu'_{i,j} \text{ for all } i < j.$$

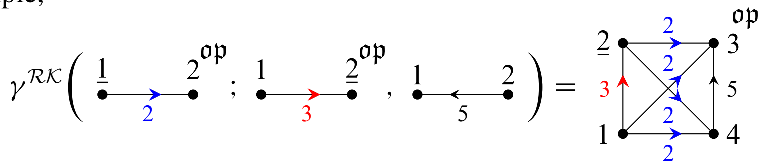
Given a permutation $\sigma \in \Sigma_n$ and an element $(\mu, \tau)^c \in \mathcal{RK}(c_1, \dots, c_n; c)$, the resulting element $\sigma \cdot (\mu, \tau)^c \in \mathcal{RK}(c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(n)}; c)$ is given by permuting the numbers i by σ leaving the underline, the orientation, and the colouring unchanged. For example, the edges $\underline{i} \to j$ of $(\mu, \tau)^c$ with colours $\mu_{i,j}$ become the edges $\underline{\sigma(i)} \to \sigma(j)$ with the same colours $\mu_{i,j}$.

The compositions

$$\gamma^{\mathcal{RK}} \colon \mathcal{RK}(c_1, \dots, c_n; c) \times \mathcal{RK}(c_{1,1}, \dots, c_{1,k_1}; c_1) \times \dots \times \mathcal{RK}(c_{n,1}, \dots, c_{n,k_n}; c_n) \\ \to \mathcal{RK}(c_{1,1}, \dots, c_{n,k_n}; c)$$

send a tuple $(\alpha; \alpha_1, \dots, \alpha_n)$ of $\mathcal{RK}(c_1, \dots, c_n; c) \times \dots \times \mathcal{RK}(c_{n,1}, \dots, c_{n,k_n}; c_n)$ to an element in $\mathcal{RK}(c_{1,1}, \dots, c_{n,k_n}; c)$ obtained as follows. The subcomplete graph with the vertices in the same block $\{c_{i,1}, \dots, c_{i,k_i}\}$ is oriented and coloured as $\alpha_i \in \mathcal{RK}(c_{i,1}, \dots, c_{i,k_i}; c_i)$; the edges with vertices in two different blocks are oriented and coloured as the edges between the corresponding vertices in $\alpha \in \mathcal{RK}(c_1, \dots, c_n; c)$.

For example,



3B2 Filtrations of $\mathcal{RK}$ We define two different filtrations of $\mathcal{RK}$ by suboperads $(\mathcal{RK}_m)_{m\geq 1}$ and $(\mathcal{RK}'_m)_{m\geq 1}$.

For $m \geq 1$, the suboperad $\mathcal{RK}_m \subset \mathcal{RK}$ is defined as follows. The closed part is

$$\mathcal{RK}_m(\mathfrak{cl},\dots,\mathfrak{cl};\mathfrak{cl}) = \{(\mu,\sigma)^{\mathfrak{cl}} \in \mathcal{RK}(\mathfrak{cl},\dots,\mathfrak{cl};\mathfrak{cl}) \mid \mu_{i,j} \leq m \text{ for all } i<j\}.$$

The nonclosed part is defined, for an $(n+1)$–tuple of colours $(c_1,\dots,c_n;\mathfrak{op})$, by

$$\mathcal{RK}_m(c_1,\dots,c_n;\mathfrak{op}) = \{(\mu,\sigma)^{\mathfrak{op}} \in \mathcal{RK}(c_1,\dots,c_n;\mathfrak{op}) \mid \mu_{i,j} \leq \widehat{m}\},$$

where

$$\widehat{m} = \begin{cases} m & \text{if } c_i = c_j = \mathfrak{cl}, \\ m-1 & \text{if } c_i = c_j = \mathfrak{op}, \\ m-1 & \text{if } \underline{i} \to j \text{ or } i \leftarrow \underline{j}, \\ m & \text{if } i \to \underline{j} \text{ or } \underline{i} \leftarrow j. \end{cases}$$

The second suboperad $\mathcal{RK}'_m \subset \mathcal{RK}$ is obtained by exchanging the last two conditions on $\mu_{i,j}$ above. Explicitly,

$$\mathcal{RK}'_m(\mathfrak{cl},\dots,\mathfrak{cl};\mathfrak{cl}) = \mathcal{RK}_m(\mathfrak{cl},\dots,\mathfrak{cl};\mathfrak{cl})$$

and

$$\mathcal{RK}'_m(c_1,\dots,c_n;\mathfrak{op}) = \{(\mu,\sigma)^{\mathfrak{op}} \in \mathcal{RK}(c_1,\dots,c_n;\mathfrak{op}) \mid \mu_{i,j} \leq \widehat{m}'\},$$

where

$$\widehat{m}' = \begin{cases} m & \text{if } c_i = c_j = \mathfrak{cl}, \\ m-1 & \text{if } c_i = c_j = \mathfrak{op}, \\ m & \text{if } \underline{i} \to j \text{ or } i \leftarrow \underline{j}, \\ m-1 & \text{if } i \to \underline{j} \text{ or } \underline{i} \leftarrow j. \end{cases}$$

3.3 Remark For $m=1$ the conditions where $\mu_{i,j} \leq m-1$ cannot be satisfied. It follows that $\mathcal{RK}_1(c_1,\dots,c_n;\mathfrak{op})$ and $\mathcal{RK}'_1(c_1,\dots,c_n;\mathfrak{op})$ are empty whenever the tuple $(c_1,\dots,c_n)$ has more than one open colour.

3C Cellular decompositions of the Swiss cheese operad

The idea of cellular decomposition of operads comes from [5]. It consists of cellular decompositions of each space that are compatible with the operad structure.

Recall from [3, Section 3.1] that, given a topological space X and a poset $\mathcal{A}$, an *$\mathcal{A}$–cellulation* of X is a functor $\Theta\colon \mathcal{A} \to \mathbf{Top}$ such that

(1) $\operatorname{colim}_{\alpha\in\mathcal{A}} \Theta(\alpha) \cong X$;

(2) for each $\alpha \in \mathcal{A}$, the canonical map $\operatorname{colim}_{\beta<\alpha} \Theta(\beta) \to \Theta(\alpha)$ is a closed cofibration;

(3) for each $\alpha \in \mathcal{A}$, the "cell" $\Theta(\alpha)$ is contractible.

Such an $\mathcal{A}$–cellulation provides a zig-zag of weak equivalences

$$X \cong \operatorname{colim}_{\alpha\in\mathcal{A}} \Theta(\alpha) \leftarrow \operatorname{hocolim}_{\alpha\in\mathcal{A}} \Theta(\alpha) \to \operatorname{hocolim}_{\alpha\in\mathcal{A}}(*) \cong \mathcal{B}\mathcal{A}, \tag{3-2}$$

where $\mathcal{B}\mathcal{A}$ denotes the realization of the nerve of the category $\mathcal{A}$. Moreover, if X and $\mathcal{A}$ are operads and if the cellular decomposition of X is compatible with its operadic structure, then all the objects in (3-2) are operads and the weak equivalences are morphisms of operads. It is straightforward to check that this holds for SC-type operads for which the notion of a compatible cellular decomposition is as follows.

3.4 Definition Let $\mathcal{A}$ be a poset SC-type operad. A topological SC-type operad $\mathcal{O}$ is called an *$\mathcal{A}$–cellular operad* if, for each $(n+1)$–tuple of colours $(c_1, \dots, c_n; c)$, there is an $\mathcal{A}(c_1, \dots, c_n; c)$–cellulation of $\mathcal{O}(c_1, \dots, c_n; c)$,

$$\Theta_{c_1,\dots,c_n;c}\colon \mathcal{A}(c_1, \dots, c_n; c) \to \mathbf{Top},$$

subject to the following two compatibilities:

(1) **Compatibility with the Σ_n–action**

$$\Theta_{c_{\sigma^{-1}(1)},\dots,c_{\sigma^{-1}(n)};c}(\sigma\cdot\alpha) = \sigma\cdot\Theta_{c_1,\dots,c_n;c}(\alpha)$$

for all $\sigma \in \Sigma_n$ and $\alpha \in \mathcal{A}(c_1, \dots, c_n; c)$.

(2) **Compatibility with the operadic composition**

$$\gamma^{\mathcal{O}}\big(\Theta_{c_1,\dots,c_n;c}(\alpha) \times \Theta_{c_{1,1},\dots,c_{1,k_1};c_1}(\alpha_1) \times \cdots \times \Theta_{c_{n,1},\dots,c_{n,k_n};c_n}(\alpha_n)\big) \subseteq \Theta_{c_{1,1},\dots,c_{n,k_n};c}(\gamma^{\mathcal{A}}(\alpha;\alpha_1,\dots,\alpha_n))$$

for all variables c, c_i, $c_{i,j}$, α and α_i, where $\gamma^{\mathcal{O}}$ and $\gamma^{\mathcal{A}}$ denote the composition maps of $\mathcal{O}$ and $\mathcal{A}$, respectively.

In what follows, one shows that Berger's cellular decomposition of the little m–cubes operad (see [5, Theorem 1.16] and [8]) extends to $\mathcal{SC}_m$.

3.5 Theorem *Let $m \geq 1$. The Swiss cheese operad $\mathcal{SC}_m$ has the structure of an $\mathcal{RK}_m$–cellular operad as well as the structure of an $\mathcal{RK}'_m$–cellular operad.*

Proof We start by the $\mathcal{RK}'_m$–cellular decomposition; we explain at the end of the proof how to proceed for the other decomposition.

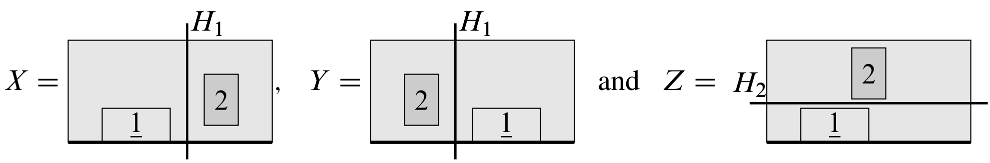
We use the description of $\mathcal{SC}_m$ via cubes and semicubes given in Remark 3.2. The integer $m \geq 1$ is fixed.

3.6 Notation For C_1 either a cube or a semicube and C_2 either a cube or a semicube, we write $C_1 \square_\mu C_2$ if they are separated by a hyperplane H_i orthogonal to the i^{th} coordinate axis for some $i \leq \mu$ such that whenever there is no separating hyperplane H_i for $i < \mu$, the left element C_1 lies in the *negative* side of H_μ and C_2 lies in the *positive* side of H_μ.

For $\alpha = (\mu, \sigma)^c \in \mathcal{RK}'_m(c_1, \dots, c_k; c)$, we define $\mathcal{SC}_m(c_1, \dots, c_k; c)^{(\alpha)}$ to be the cell

$$\{(C_1, \dots, C_k) \in \mathcal{SC}_m(c_1, \dots, c_k; c) \mid C_i \square_{\mu_{i,j}} C_j \text{ if } \tilde{c}_i \to \tilde{c}_j \text{ and } C_j \square_{\mu_{i,j}} C_i \text{ if } \tilde{c}_i \leftarrow \tilde{c}_j\}.$$

For example, consider the configurations



in $\mathcal{SC}_2(\mathfrak{op}, \mathfrak{cl}; \mathfrak{op})$, and the elements

$$\alpha = \overset{1}{\bullet} \xrightarrow[1]{} \overset{2^{\mathfrak{op}}}{\bullet}, \quad \beta = \overset{1}{\bullet} \xleftarrow[1]{} \overset{2^{\mathfrak{op}}}{\bullet} \quad \text{and} \quad \gamma = \overset{1}{\bullet} \xrightarrow[2]{} \overset{2^{\mathfrak{op}}}{\bullet}$$

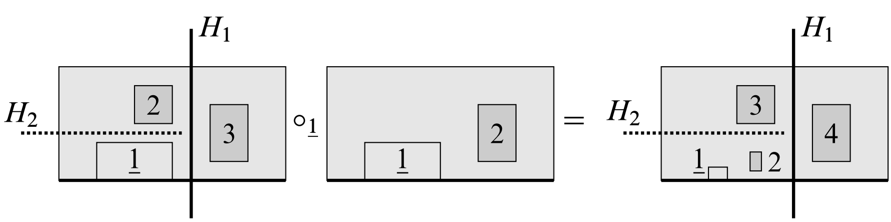
that form $\mathcal{RK}'_2(\mathfrak{op}, \mathfrak{cl}; \mathfrak{op})$. The cell $\mathcal{SC}_2(\mathfrak{op}, \mathfrak{cl}; \mathfrak{op})^{(\alpha)}$ is made of configurations of type X; the cell $\mathcal{SC}_2(\mathfrak{op}, \mathfrak{cl}; \mathfrak{op})^{(\beta)}$ is made of configurations of type Y; and the cell $\mathcal{SC}_2(\mathfrak{op}, \mathfrak{cl}; \mathfrak{op})^{(\gamma)}$ is made of configurations of types X, Y and Z, where those of type Z intersect with configurations of type X or Y whenever C_1 and C_2 are separated by 2 hyperplanes H_1 and H_2.

As a remark, note that in $\mathcal{SC}_m$, whenever C_1 is a semicube and C_2 is a cube, if H_m exists, then C_1 lies in the negative side of H_m (eg Z). From this one can see that, with this definition of the cells, the condition $\mu_{i,j} \leq m-1$ if $i \to \underline{j}$ given in the definition of $\mathcal{RK}'_m$ is necessary for ensuring the contractibility of the cells. Indeed, if one removes the above condition, then there exists

$$\delta_m = \underline{1} \xleftarrow[m]{} 2^{\mathfrak{op}}$$

that indexes the space $\mathcal{SC}_m(\mathfrak{op},\mathfrak{cl};\mathfrak{op})^{(\delta_m)}$, which is homotopic to the $(m-2)$–sphere (for $m=2$, one has $\mathcal{SC}_2(\mathfrak{op},\mathfrak{cl};\mathfrak{op})^{(\delta_2)} = \mathcal{SC}_2(\mathfrak{op},\mathfrak{cl};\mathfrak{op})^{(\alpha)} \sqcup \mathcal{SC}_2(\mathfrak{op},\mathfrak{cl};\mathfrak{op})^{(\beta)}$).

Our definition recovers that of Berger when considering the open part and the closed part of $\mathcal{SC}_m$ separately. The main input of our construction, then, resides in the interaction between cubes and semicubes. At first sight it could appear useless to consider separating hyperplanes between cubes and semicubes since contractibility is not hindered by their relative positions (for example, $\mathcal{SC}_m(\mathfrak{op},\mathfrak{cl};\mathfrak{op})$ is contractible). However it is important to do so for the operadic composition of the cells. This is because, in a configuration of cubes and semicubes, if one substitutes a semicube, then a cube may appear; the position of such a cube has to be compared with that of the other cubes. For instance, consider the following substitution:



Then in the resulting configuration (the right-hand side standard semicube), knowing the position of 2 relative to 3 and 4 requires the knowledge of the position of $\underline{1}$ relative to 2 and 3 in the first term (the left-hand side standard semicube).

In what concerns the contractibility of the cells, both closed and open cells are known as being contractible; the same argument as [5, Theorem 1.16] shows that open/closed cells also are.

The fact that $\operatorname{colim}_{\alpha \in \mathcal{RK}'_m(c_1,\dots,c_k;c)} \mathcal{SC}_m(c_1,\dots,c_k;c)^{(\alpha)} \cong \mathcal{SC}_m(c_1,\dots,c_k;c)$ essentially follows from the following two facts:

(1) If $x \in \mathcal{SC}_m(c_1,\dots,c_k;c)$ then there exists an $\alpha \in \mathcal{RK}'_m(c_1,\dots,c_k;c)$ such that $x \in \mathcal{SC}_m(c_1,\dots,c_k;c)^{(\alpha)}$. This is immediate since any two cubes/semicubes are separated by a hyperplane. For example, if C_i is a semicube and C_j a cube, then $\alpha = (\mu,\sigma)^c$ may be chosen such that $\mu_{i,j} = m$ and $\sigma_{i,j} = \underline{i} \to j$.

(2) If $\alpha = (\mu,\sigma)^c$ and $\beta = (\mu',\sigma')^c$ are not comparable (neither $\alpha \le \beta$ nor $\alpha \ge \beta$) and are such that there exists

$$x = (C_1,\dots,C_k) \in \mathcal{SC}_m(c_1,\dots,c_k;c)^{(\alpha)} \cap \mathcal{SC}_m(c_1,\dots,c_k;c)^{(\beta)},$$

then $x \in \mathcal{SC}_m(c_1,\dots,c_k;c)^{(\gamma)}$ for a γ such that $\gamma < \alpha$ and $\gamma < \beta$. This follows from the observation that, for each $i < j$ such that $(\mu_{i,j},\sigma_{i,j})$ and $(\mu'_{i,j},\sigma'_{i,j})$ are not comparable, there exists $\mu''_{i,j} < \mu_{i,j} = \mu'_{i,j}$ such that $C_i \square_{\mu''_{i,j}} C_j$ or $C_j \square_{\mu''_{i,j}} C_i$; let us denote by $\sigma''_{i,j}$ the corresponding orientation. The element $\gamma = (\mu'',\sigma'')^c$ is given by $(\mu''_{i,j},\sigma''_{i,j})$ as above whenever $i < j$ indexes incomparable components and by $(\mu''_{i,j},\sigma''_{i,j}) = \min\{(\mu_{i,j},\sigma_{i,j}),(\mu'_{i,j},\sigma'_{i,j})\}$ otherwise.

The other cellular decomposition (indexed by $\mathcal{RK}_m$) is obtained by exchanging, in Notation 3.6 above, the terms *negative* and *positive*. In what concerns the closed and the open parts, such an exchange is not relevant. However, as we have remarked before, whenever C_1 is a semicube and C_2 is a cube, if H_m exists, then C_1 lies in the negative side of H_m, hence the condition $\mu_{i,j} \le m-1$ if $i \leftarrow \underline{j}$ given in the definition of $\mathcal{RK}_m$. □

4 The operad $\mathcal{RL}$

4A Definition of the operad $\mathcal{RL}$

We describe a coloured SC-operad $\mathcal{RL}$ in the category of sets, **Set**.

The operad $\mathcal{RL}$ has two natural filtrations by suboperads $\mathcal{RL}_m$ and $\mathcal{RL}'_m$ for $m \ge 1$. For each $m \ge 1$, we can think of $\mathcal{RL}_m$ and $\mathcal{RL}'_m$ as mixes between the suboperads $\mathcal{L}_m$ and $\mathcal{L}_{m-1}$ of the lattice paths operad $\mathcal{L}$ introduced in [3]. The operad $\mathcal{RL}$ has two types of colours, the closed colours $\mathbb{N}$ and the open colours $\underline{\mathbb{N}}$, while $\mathcal{L}$ has one type of colours $\mathbb{N}$.

We generalize [3, Section 2] which serves as a basis for this section. In particular we refer to [loc. cit.] for the definition of the category of bipointed small categories $\mathbf{Cat}_{*,*}$, ordinals $[n]$ and for the tensor product of ordinals $[i] \otimes [j]$. Decorating each object of the ordinal $[n]$ with an underline gives the *underlined ordinal* $[\underline{n}]$. One denotes by $\underline{\mathbb{N}}$ the set of the natural numbers decorated by an underline; for $\underline{n} \in \underline{\mathbb{N}}$ and $k \in \mathbb{Z}_{\geq -n}$ one lets $\underline{n} + k := \underline{n+k}$. Let us denote by $\mathrm{ev}\colon \mathbb{N} \sqcup \underline{\mathbb{N}} \to \mathbb{N}$ the map defined by $\mathrm{ev}(e) = n$ if $e = n$ or $e = \underline{n}$.

Definition of $\mathcal{RL}$ The set of colours of $\mathcal{RL}$ is $\mathrm{Col} = \mathrm{Col}_{\mathfrak{cl}} \sqcup \mathrm{Col}_{\mathfrak{op}}$, where $\mathrm{Col}_{\mathfrak{cl}} := \mathbb{N}$ and $\mathrm{Col}_{\mathfrak{op}} := \underline{\mathbb{N}}$. Hence, $n \in \mathrm{Col}_{\mathfrak{cl}}$ whereas $\underline{n} \in \mathrm{Col}_{\mathfrak{op}}$.

For a $(k+1)$–tuple of colours $(e_1, \dots, e_k; e)$ in Col, the set $\mathcal{RL}(e_1, \dots, e_k; e)$ is defined as

- the empty set $\varnothing$ if $e \in \mathrm{Col}_{\mathfrak{cl}}$ and if there is an i such that $e_i \in \mathrm{Col}_{\mathfrak{op}}$,
- $\mathbf{Cat}_{*,*}([e+1], [e_1+1] \otimes [e_2+1] \otimes \cdots \otimes [e_k+1])$ otherwise.

The substitutions maps are the natural extension to that of $\mathcal{L}$; that is, they are given by tensor and composition in $\mathbf{Cat}_{*,*}$.

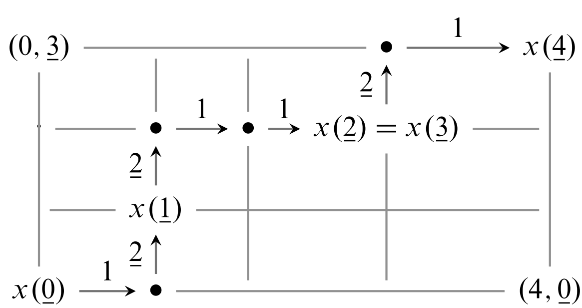
4.1 Remark Accordingly, one recovers the lattice path operad

$$\begin{aligned}\mathcal{RL}(n_1, \dots, n_k; n) &= \mathcal{L}(n_1, \dots, n_k; n) \\ &= \mathbf{Cat}_{*,*}([n+1], [n_1+1] \otimes [n_2+1] \otimes \cdots \otimes [n_k+1])\end{aligned}$$

for all $(k+1)$–tuples $(n_1, \dots, n_k; n)$ of colours in $\mathrm{Col}_{\mathfrak{cl}} = \mathbb{N}$.

For instance, an element $x \in \mathcal{RL}(n_1, \underline{n_2}; \underline{n})$ is a functor $x\colon [\underline{n+1}] \to [n_1+1] \otimes [\underline{n_2+1}]$ that sends $(0, \underline{n+1})$ to $((0, \underline{0}), (n_1+1, \underline{n_2+1}))$ and is determined by the image of the n remaining objects of $[\underline{n+1}]$ and the morphisms into the lattice $[n_1+1] \otimes [\underline{n_2+1}]$.

4.2 Example The following lattice path x belongs to $\mathcal{RL}(3, \underline{2}; \underline{3})$:

(4-1) 

Integer strings representation In [3, Section 2.2] a description of $\mathcal{L}$ in terms of integer strings is given. One has the obvious similar bijective correspondence for $\mathcal{RL}$, where one considers natural numbers and underlined natural numbers; the latter correspond to the open colours. Additionally, we put an extra label according to the nature of the output colour. Explicitly, given a lattice path $x \in \mathcal{RL}(e_1, \dots, e_k; e)$, one runs through it starting from $x(0)$ (or $x(\underline{0})$) and ending at $x(e+1)$. Along the way, each time one meets an edge that is parallel to the i–axis one writes down i if $e_i \in \mathrm{Col}_{\mathfrak{cl}}$ and $\underline{i}$ if $e_i \in \mathrm{Col}_{\mathfrak{op}}$; one writes down a vertical bar each time one meets an $x(a)$ for $1 \le \mathrm{ev}(a) \le \mathrm{ev}(e)$. One adds an $\mathfrak{op}$ if the output colour belongs to $\mathrm{Col}_{\mathfrak{op}}$.

The $\mathrm{ev}(e)$ vertical bars of the integer string x subdivide it into $\mathrm{ev}(e)+1$ (possibly empty) *substrings*. The substrings are indexed by $[\mathrm{ev}(e)]$.

4.3 Example The lattice path x given in (4-1) corresponds to the integer string $(1\underline{2}|\underline{2}11||\underline{2}1)^{\mathfrak{op}}$.

4.4 Example $(121)^{\mathfrak{op}} \in \mathcal{RL}(1,0;\underline{0})$ whereas $(121) \in \mathcal{RL}(1,0;0)$.

Let us exhibit the corresponding composition on integer string representations via an example.

4.5 Example $(1\underline{2}|1\underline{4}\underline{2}31||\underline{2}\underline{4})^{\mathfrak{op}} \circ_{\underline{2}} (1\underline{3}|21\underline{3}|\underline{3}1)^{\mathfrak{op}} = (12\underline{4}|1\underline{6}32\underline{4}51||\underline{4}2\underline{6})^{\mathfrak{op}}$.

We renumber the integer string $(1\underline{2}|1\underline{4}\underline{2}31||\underline{2}\underline{4})^{\mathfrak{op}}$ by increasing the integers greater than $\underline{2}$ by 2 (which is one less than the number of integers in the second integer string) to obtain $(1\underline{2}|1\underline{6}\underline{2}51||\underline{2}\underline{6})^{\mathfrak{op}}$. We increase the integers of the second integer string by 1 (one less than the value of $\underline{2}$) to obtain $(2\underline{4}|32\underline{4}|\underline{4}2)^{\mathfrak{op}}$. Finally, we replace the three occurrences of $\underline{2}$ by the three substrings $2\underline{4}$, $32\underline{4}$ and $\underline{4}2$.

The action of the symmetric group $\sigma \cdot x \in \mathcal{RL}(e_{\sigma^{-1}(1)}, \dots, e_{\sigma^{-1}(k)}; e)$ is obtained by permuting the number i (resp. $\underline{i}$) of the integer string representation of x by the number $\sigma(i)$ (resp. $\underline{\sigma(i)}$).

4.6 Example For $x = (1\underline{2}|3\underline{2}11||\underline{2}1)^{\mathfrak{op}}$ and $\sigma(1)=2$, $\sigma(2)=3$, $\sigma(3)=1$, one has $\sigma \cdot x = (2\underline{3}|1\underline{3}22||\underline{3}2)^{\mathfrak{op}}$.

The underlying category of $\mathcal{RL}$ It follows directly from [3, Section 2.4] and the definition of $\mathcal{RL}$ that the two underlying subcategories $(\mathcal{RL})_u^{\mathfrak{cl}}$ and $(\mathcal{RL})_u^{\mathfrak{op}}$ are (canonically isomorphic to) the simplicial category $\triangle$. Thus, for each $k \geq 0$, one has a functor

$$\mathcal{RL}(-,\dots,-;-)\colon (\triangle^{\mathrm{op}})^k \times \triangle \to \mathbf{Set},$$

that is, a multisimplicial/cosimplicial set.

Dual interpretation of $\mathcal{RL}$ For later use, let us mention that $\mathcal{RL}$ has a dual interpretation as given in [3, Lemma 2.7]. In particular, an element $x \in \mathcal{RL}(e_1,\dots,e_k;e)$ determines a k–tuple $(x_1,\dots,x_k) \in \triangle([e_1],[e]) \times \cdots \times \triangle([e_k],[e])$. The value of $x_i(r)$ corresponds to the substring of the integer string x in which the $(r+1)^{\mathrm{st}}$ occurrence of i (or $\underline{i}$) appears. We refer to [loc. cit.] for more details.

4.7 Example The integer string x from Example 4.3 determines the pair $(x_1,x_2) \in \triangle([3],[\underline{3}]) \times \triangle([\underline{2}],[\underline{3}])$ given by

$$x_1\colon (0,1,2,3) \mapsto (\underline{0},\underline{1},\underline{1},\underline{3}) \quad \text{and} \quad x_2\colon (\underline{0},\underline{1},\underline{2}) \mapsto (\underline{0},\underline{1},\underline{3}).$$

Filtrations of $\mathcal{RL}$ by suboperads Let us define two maps

$$c_{i,j}, \widehat{c}_{i,j}\colon \mathcal{RL}(e_1,\dots,e_k;e) \to \mathbb{N}.$$

The map $c_{i,j}$ is defined as in [3, Section 2.9] and will be used to defined the complexity index for the closed and the open parts of $\mathcal{RL}$. The map $\widehat{c}_{i,j}$ will be used to control the interaction between the closed and the open part.

For $1 \leq i < j \leq k$, we denote by

$$\phi_{ij}\colon \mathcal{RL}(e_1,\dots,e_k;e) \to \mathcal{RL}(e_i,e_j;e)$$

the projection induced by the canonical projection

$$p_{ij}\colon [e_1+1] \otimes \cdots \otimes [e_k+1] \to [e_i+1] \otimes [e_j+1].$$

For $x \in \mathcal{RL}(e_1,\dots,e_k;e)$ and $1 \leq i < j \leq k$, we define $c_{ij}(x)$ as the number of changes of directions in the lattice paths $\phi_{ij}(x)$.

The second number $\widehat{c}_{i,j}(x)$ is defined as follows. Recall that if $x \in \mathcal{RL}(e_1,\dots,e_k;e)$, then its integer string representation is, in particular, a sequence of numbers (underlined or not) between 1 and k. For $1 \leq i \leq k$, we set i^- (resp. $\underline{i}^-$) to be the first occurrence of i (resp. $\underline{i}$) in the integer string representation. Equivalently, i^- (resp. $\underline{i}^-$) is the first edge of the lattice x which is in the i^{th} direction. We write $r^- < s^-$ if the element r^- precedes s^-.

For $1 \le i < j \le k$, we set

$$\widehat{c}_{i,j}(x) = \begin{cases} c_{i,j}(x) & \text{if } i^- < \underline{j}^-, \\ c_{i,j}(x)+1 & \text{if } i^- > \underline{j}^-, \\ c_{i,j}(x)+1 & \text{if } \underline{i}^- < j^-, \\ c_{i,j}(x) & \text{if } \underline{i}^- > j^-. \end{cases} \tag{4-2}$$

For $m \ge 1$, we define $\mathcal{RL}_m(e_1, \dots, e_k; e)$ as the set of elements $x \in \mathcal{RL}(e_1, \dots, e_k; e)$ satisfying the three conditions

$$\max_{(i,j)} c_{i,j}(x) \le m, \quad \max_{(\underline{i},\underline{j})} c_{i,j}(x) \le m-1 \quad \text{and} \quad \max_{(\underline{i},j) \text{ or } (i,\underline{j})} \widehat{c}_{i,j}(x) \le m.$$

Changing the number defined in (4-2) to

$$\widehat{c}'_{i,j}(x) = \begin{cases} c_{i,j}(x)+1 & \text{if } i^- < \underline{j}^-, \\ c_{i,j}(x) & \text{if } i^- > \underline{j}^-, \\ c_{i,j}(x) & \text{if } \underline{i}^- < j^-, \\ c_{i,j}(x)+1 & \text{if } \underline{i}^- > j^-, \end{cases} \tag{4-3}$$

provides another filtration of $\mathcal{RL}$ by suboperads $\mathcal{RL}'_m$. Explicitly, $\mathcal{RL}'_m(e_1, \dots, e_k; e)$ is the set of elements $x \in \mathcal{RL}(e_1, \dots, e_k; e)$ satisfying the three conditions

$$\max_{(i,j)} c_{i,j}(x) \le m, \quad \max_{(\underline{i},\underline{j})} c_{i,j}(x) \le m-1 \quad \text{and} \quad \max_{(\underline{i},j) \text{ or } (i,\underline{j})} \widehat{c}'_{i,j}(x) \le m.$$

A relation between these two filtrations and the two cellular decompositions of $\mathcal{SC}_m$ from Theorem 3.5 is given in the next section.

4B The operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta)$ as an SC-type operad

Given a functor $\delta\colon \triangle \to \boldsymbol{C}$, where $\boldsymbol{C}$ is a *monoidal model category*, by following the method developed in [3, Sections 3.5–3.6], we construct a zig-zag of weak equivalences of operads

$$\mathrm{Coend}_{\mathcal{RL}_m}(\delta) \leftarrow \mathrm{Coend}_{\widehat{\mathcal{RL}}_m}(\delta) \rightarrow \mathcal{B}_\delta \mathcal{RK}_m \tag{4-4}$$

whenever δ satisfies some conditions. Here, $\mathcal{B}_\delta \mathcal{A}$ denotes the δ–realization of the nerve of the category $\mathcal{A}$. The intermediate operad $\widehat{\mathcal{RL}}_m$ is defined using homotopy colimits in $\boldsymbol{C}$ applied on a decomposition of $\mathcal{RL}_m$ indexed by $\mathcal{RK}_m$.

We use most of the material from and the same conventions as in [3, Sections 3.5–3.6]. In particular, we require δ to be a *standard system of simplices*. This confers on homotopy colimits good properties, such as compatibility with the symmetric monoidal structure of $\boldsymbol{C}$. The functors δ_{yon}, δ_{Top} and $\delta_{\mathbb{Z}}$ defined hereafter are standard systems of simplices.

Let

$$\delta_{\text{Top}}\colon \triangle \xrightarrow{\delta_{\text{yon}}} \mathbf{Set}^{\triangle^{\text{op}}} \xrightarrow{|-|} \mathbf{Top} \quad \text{and} \quad \delta_{\mathbb{Z}}\colon \triangle \xrightarrow{\delta_{\text{yon}}} \mathbf{Set}^{\triangle^{\text{op}}} \xrightarrow{C_*(-;\mathbb{Z})} \mathbf{Ch}(\mathbb{Z})$$

be the two functors, where

- $\delta_{\text{yon}}([n]) = \text{Hom}_{\triangle}(-,[n])$ is the Yoneda functor,
- $|-|\colon \mathbf{Set}^{\triangle^{\text{op}}} \to \mathbf{Top}$ is the geometric realization, and
- $C_*(-;\mathbb{Z})\colon \mathbf{Set}^{\triangle^{\text{op}}} \to \mathbf{Ch}(\mathbb{Z})$ is the normalized chain complex.

Let us recall that for $\delta = (\delta_{\mathfrak{cl}}, \delta_{\mathfrak{op}})$ with $\delta_{\mathfrak{cl}}, \delta_{\mathfrak{op}}\colon \triangle \to \boldsymbol{C}$, the functor $\xi(\mathcal{RL}_m)_{c_1,\dots,c_k;c}(\delta)$ denotes the δ–realization

$$\xi(\mathcal{RL}_m)_{c_1,\dots,c_k;c}(\delta)(n) = \mathcal{RL}_m(\underbrace{-,\dots,-}_{k};n) \otimes_{\triangle^k} \delta_{c_1}(-) \otimes \cdots \otimes \delta_{c_k}(-),$$

where we use implicitly the strong monoidal functor $\mathbf{Set} \to \boldsymbol{C}$, $E \mapsto \coprod_{e\in E} 1_{\boldsymbol{C}}$.

We use the same functor $\delta_{\mathfrak{cl}} = \delta_{\mathfrak{op}}$ and we denote it by δ. This way, by using condensation from Section 2E, we define the SC-type operad $\text{Coend}_{\mathcal{RL}_m}(\delta)$ and similarly for $\text{Coend}_{\mathcal{RL}'_m}(\delta)$.

The idea for providing the zig-zag (4-4) follows that of (3-2) and relies on a decomposition of $\xi(\mathcal{RL}_m)_{c_1,\dots,c_k;c}(\delta)$ by "cells" $\xi(\mathcal{RL}_m)_\alpha(\delta)$ indexed by the $\alpha \in \mathcal{RK}_m(c_1,\dots,c_k;c)$. Under some properties, the right-sided weak equivalence results from "the contraction of the cells", that is, from weak equivalences $\xi(\mathcal{RL}_m)_\alpha(\delta) \to I$, where I denotes the constant cosimplicial object $I^n = 1_{\boldsymbol{C}}$; the left-sided weak equivalence is induced by the natural map $\text{hocolim}\,\xi(\mathcal{RL}_m)_\alpha(\delta) \to \text{colim}\,\xi(\mathcal{RL}_m)_\alpha(\delta)$.

In fact, Batanin and Berger [3, Theorem 3.8] show that, in the closed case, (4-4) holds provided that $\mathcal{L}_m$ is *strongly δ–reductive*. Here we extend their result to $\mathcal{RL}_m$. For consistency we recall the notion of *strong δ–reductivity* in our context. Let us also recall that a weak equivalence in $\boldsymbol{C}$ is called *universal* if any pullback of it is again a weak equivalence.

4.8 Definition Let δ be a standard system of simplices in $\boldsymbol{C}$. The operad $\mathcal{RL}_m$ is called *δ–reductive* if for any $n \geq 0$ and $k \geq 0$ and any colours $c_i, c \in \{\mathfrak{cl}; \mathfrak{op}\}$

satisfying (2-2), the map $\xi(\mathcal{RL}_m)_{c_1,\dots,c_k;c}(\delta)^n \to \xi(\mathcal{RL}_m)_{c_1,\dots,c_k;c}(\delta)^0$ is a universal weak equivalence.

The operad $\mathcal{RL}_m$ is called *strongly δ–reductive* if in addition the induced maps $\mathrm{Coend}_{\mathcal{RL}_m}(\delta)(c_1,\dots,c_k;c) \to \xi(\mathcal{RL}_m)_{c_1,\dots,c_k;c}(\delta)^0$ are universal weak equivalences in $\boldsymbol{C}$.

4.9 Theorem *Let δ be a standard system of simplices in $\boldsymbol{C}$. If the operad $\mathcal{RL}_m$ (resp. $\mathcal{RL}'_m$) is strongly δ–reductive, the operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta)$ (resp. $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta)$) is weakly equivalent to $\mathcal{B}_\delta\mathcal{RK}_m$ (resp. $\mathcal{B}_\delta\mathcal{RK}'_m$).*

Proof The cells $\xi(\mathcal{RL}_m)_\alpha(\delta)$ are obtained via a map

$$c_{\mathrm{tot}}\colon \mathcal{RL}_m(e_1,\dots,e_k;e) \to \mathcal{RK}_m(c_1,\dots,c_k;c)$$

for each $(k+1)$–tuple of colours $(c_1,\dots,c_k;c)$ in $\{\mathfrak{cl},\mathfrak{op}\}$ and $e_i \in \mathrm{Col}_{c_i}$, $e \in \mathrm{Col}_c$. Such a map is defined in (4-7) below and recovers the maps from [3, Proposition 3.4] in the closed case (ie $c=\mathfrak{cl}$). Note that there is a slight inaccuracy in [3] since, as we will show in Example 4.12 and Lemma 4.11, including in the closed case, the map c_{tot} is not a morphism of coloured operads but instead satisfies

$$c_{\mathrm{tot}}(x \circ_i y) \le c_{\mathrm{tot}}(x) \circ_i c_{\mathrm{tot}}(y). \tag{4-5}$$

Such an inequality is sufficient to apply the method developed in [3, Sections 3.5–3.6], however. Indeed, for $\alpha \in \mathcal{RK}_m(c_1,\dots,c_k;c)$ and $e_i \in \mathrm{Col}_{c_i}$, $e \in \mathrm{Col}_c$, let us define

$$(\mathcal{RL}_m)_\alpha(e_1,\dots,e_k;e) := \{x \in \mathcal{RL}_m(e_1,\dots,e_k;e) \mid c_{\mathrm{tot}}(x) \le \alpha\}. \tag{4-6}$$

It follows that $\mathcal{RL}_m(e_1,\dots,e_k;e) = \mathrm{colim}_{\mathcal{RK}_m(c_1,\dots,c_k;c)}(\mathcal{RL}_m)_\alpha(e_1,\dots,e_k;e)$; the inequality (4-5) ensures the compatibility of the decomposition with the operadic structures, so that it implies that this is an equality of coloured operads. Moreover, the operad $\widehat{\mathcal{RL}}_m$, given by

$$\widehat{\mathcal{RL}}_m(e_1,\dots,e_k;e) = \mathrm{hocolim}_{\mathcal{RK}_m(c_1,\dots,c_k;c)}(\mathcal{RL}_m)_\alpha(e_1,\dots,e_k;e),$$

is an operad (again, because of (4-5) and because of the compatibility of hocolim with symmetric monoidal structure). It turns out that (4-6) forms a multisimplicial subcomplex of $\mathcal{RL}_m(-,\dots,-;e)$, so it makes sense to take its δ–realization $\xi(\mathcal{RL}_m)_\alpha(\delta)$. In fact, as cosimplicial objects, $\xi(\mathcal{RL}_m)_{c_1,\dots,c_k;c}(\delta) = \mathrm{colim}_{\mathcal{RK}_m(c_1,\dots,c_k;c)}\,\xi(\mathcal{RL}_m)_\alpha(\delta)$, and $\xi(\widehat{\mathcal{RL}}_m)_{c_1,\dots,c_k;c}(\delta)$ identifies with $\mathrm{hocolim}_{\mathcal{RK}_m(c_1,\dots,c_k;c)}\,\xi(\mathcal{RL}_m)_\alpha(\delta)$.

From these considerations, it is straightforward to verify that the proof of Theorem 3.8 in [3] generalizes to our case: the δ–reductivity implies that $\xi(\mathcal{RL}_m)_\alpha(\delta) \to I$ is a weak equivalence, and via the strongly δ–reductivity, the left-sided weak equivalence of the zig-zag (4-4) results from the weak equivalence $\operatorname{hocolim} \xi(\mathcal{RL}_m)_\alpha(\delta)^0 \to \operatorname{colim} \xi(\mathcal{RL}_m)_\alpha(\delta)^0$. □

4.10 Proposition *If δ is δ_{Top} or $\delta_{\mathbb{Z}}$, the operads $\mathcal{RL}_m$ and $\mathcal{RL}'_m$ are strongly δ–reductive. Consequently, the operads $\mathrm{Coend}_{\mathcal{RL}_m}(\delta)$ and $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta)$ are weakly equivalent to the topological* (*resp. chain*) *Swiss cheese operad $\mathcal{SC}_m$* (*resp. $C_*\mathcal{SC}_m$*) *if δ is δ_{Top}* (*resp. $\delta_{\mathbb{Z}}$*)*.*

Proof Again, this is a straightforward generalization of [3, Examples 3.10], where it is shown that $\mathcal{L}_m$ is strongly δ–reductive if δ is δ_{Top} or $\delta_{\mathbb{Z}}$. The very same method applies in our context by considering $\xi(\mathcal{RL}_m)_{c_1,\dots,c_k;c}(\delta)$ instead of $\xi(\mathcal{L}_m)_k(\delta)$. □

For a $(k+1)$–tuple $(c_1,\dots,c_k;c)$ of colours in $\{\mathfrak{cl},\mathfrak{op}\}$ and $e_i \in \mathrm{Col}_{c_i}$, $e \in \mathrm{Col}_c$, let

$$c_{\mathrm{tot}}\colon \mathcal{RL}_m(e_1,\dots,e_k;e) \to \mathcal{RK}_m(c_1,\dots,c_k;c)$$

be as follows. Recall the notation $\widetilde{c}_i$ from (3-1). The element $c_{\mathrm{tot}}(x) = (\mu,\sigma) \in \mathcal{RK}_m$ is defined, for $1 \le i < j \le k$, by

$$\mu_{i,j} = c_{i,j}(x) \qquad \text{and} \qquad \sigma_{i,j} = \begin{cases} \widetilde{c}_i \to \widetilde{c}_j & \text{if } \widetilde{c}_i^{\,-} < \widetilde{c}_j^{\,-}, \\ \widetilde{c}_i \leftarrow \widetilde{c}_j & \text{if } \widetilde{c}_i^{\,-} > \widetilde{c}_j^{\,-}. \end{cases} \tag{4-7}$$

Similarly, let $c'_{\mathrm{tot}}\colon \mathcal{RL}'_m \to \mathcal{RK}'_m$ be the map defined by the formula (4-7) for $x \in \mathcal{RL}'_m$.

4.11 Lemma *For all $x, y \in \mathcal{RL}_m$ and i such that $x \circ_i y$ makes sense, one has the inequality $c_{\mathrm{tot}}(x \circ_i y) \le c_{\mathrm{tot}}(x) \circ_i c_{\mathrm{tot}}(y)$. For all $x, y \in \mathcal{RL}'_m$ and i such that $x \circ_i y$ makes sense, one has the inequality $c'_{\mathrm{tot}}(x \circ_i y) \le c'_{\mathrm{tot}}(x) \circ_i c'_{\mathrm{tot}}(y)$.*

Proof We show the first assertion; the second one is similar. In the following arguments the type of colours does not matter, so we abusively forget about the underline. Let $x \in \mathcal{RL}_m(e_1,\dots,e_p;e)$ and $y \in \mathcal{RL}_m(f_1,\dots,f_q;e_i)$ for some $1 \le i \le p$.

For an integer $a \ne i$, one defines $a' := a$ if $a < i$ and $a' := a+q-1$ if $a > i$.

Recall that, by definition, the complete graph $(\mu,\sigma)=c_{\mathrm{tot}}(x)\circ_i c_{\mathrm{tot}}(y)$ is given by

$$\begin{aligned}
(\mu_{k+i-1,l+i-1},\sigma_{k+i-1,l+i-1}) &= (c_{k,l}(y),\sigma_{k,l}(c_{\mathrm{tot}}(y))) && \text{for } 1\le k<l\le q;\\
(\mu_{r',s'},\sigma_{r',s'}) &= (c_{r,s}(x),\sigma_{r,s}(c_{\mathrm{tot}}(x))) && \text{for } 1\le r<s\le p \\
& && \text{such that } i\notin\{r,s\};\\
(\mu_{r,k+i-1},\sigma_{r,k+i-1}) &= (c_{r,i}(x),\sigma_{r,i}(c_{\mathrm{tot}}(x))) && \text{for } 1\le r<i,\ \ 1\le k\le q;\\
(\mu_{k+i-1,s'},\sigma_{k+i-1,s'}) &= (c_{i,s}(x),\sigma_{i,s}(c_{\mathrm{tot}}(x))) && \text{for } i<s\le p,\ \ 1\le k\le q.
\end{aligned}$$

On the other hand, recall that the integer string y is subdivided in n_i+1 substrings that are delimited by the n_i vertical bars ($n_i:=\mathrm{ev}(e_i)$). Recall that the integer string $x\circ_i y$ is obtained by substituting the b^{th} occurrence of i by the b^{th} substring (indexed by $b-1$) of y together with a reindexation of the values. It follows that $c_{k+i-1,l+i-1}(x\circ_i y)=c_{k,l}(y)$ and that the order in which the pair $((k+i-1)^-,(l+i-1)^-)$ appears in $x\circ_i y$ is the same as the order in which the pair (k^-,l^-) appears in y. Similarly, one has $c_{r',s'}(x\circ_i y)=c_{r,s}(x)$, and the order in which (r'^-,s'^-) appears in $x\circ_i y$ is the same as the order in which the pair (r^-,s^-) appears in x.

Moreover, it is straightforward to see that $c_{r,k+i-1}(x\circ_i y)\le c_{r,i}(x)$ and that the equality holds if and only if k is present in at least one substring per switch between r and i. Such an equality is illustrated below; there is at least one k in each of the $c_{r,i}(x)$ blocks of substrings:

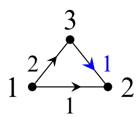
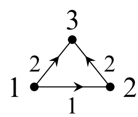
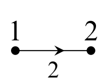
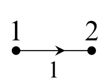
$$x=(i^-\ i\ \cdots\ i\ r^-\ i\ r\ \cdots\ r\ i\ \cdots\ i)$$
$$y=(\underbrace{\overbrace{\qquad}^{0}\mid\overbrace{\ k^-\ }^{1}\mid\cdots\mid\overbrace{\qquad}}_{1}\mid\underbrace{\overbrace{\ k\ }}_{2}\mid\cdots\mid\underbrace{\overbrace{\ k\ }\mid\cdots\mid\overbrace{\ k\ }^{n_i}}_{c_{r,i}(x)})$$

Here, only the integers r, i and k are written; the arrows indicate the substitutions.

In the equality case, the order in which $(r^-,(k+i-1)^-)$ appears in $x\circ_i y$ is the same as the order in which (r^-,i^-) appears in x. Similar considerations hold for the inequality $c_{k+i-1,s'}(x\circ_i y)\le c_{i,s}(x)$.

It follows from the above paragraphs that $c_{\mathrm{tot}}(x\circ_i y)\le c_{\mathrm{tot}}(x)\circ_i c_{\mathrm{tot}}(y)$. □

4.12 Example Here is an example of x and y such that $c_{\mathrm{tot}}(x\circ_i y)<c_{\mathrm{tot}}(x)\circ_i c_{\mathrm{tot}}(y)$. For $x=(121|1)$ and $y=(1|12|2)$, one has $(121|1)\circ_1(1|12|2)=(1312|2)$, and

$c_{\mathrm{tot}}(x\circ_1 y)=$  $\le$  $=$  $\circ_1$  $=c_{\mathrm{tot}}(x)\circ_1 c_{\mathrm{tot}}(y)$.

4C Action on cochains

One has the obvious relative version of [3, Proposition 2.20]:

4.13 Proposition *Let X and Y be two simplicial sets equipped with a simplicial map $f\colon Y \to X$. The pair $(\mathbb{Z}[X], \mathbb{Z}[Y])$ (resp. $(\mathrm{Hom}(\mathbb{Z}[X], \mathbb{Z}), \mathrm{Hom}(\mathbb{Z}[Y], \mathbb{Z}))$) is a coalgebra (resp. an algebra) over $\mathcal{RL}$. In particular, $\mathrm{Coend}_{\mathcal{RL}}(\delta_{\mathbb{Z}})$ acts on $(C^*(X;\mathbb{Z}), C^*(Y;\mathbb{Z}))$.*

Proof The $\mathcal{RL}$–coaction in the closed case is as in [3, Proposition 2.20]. Otherwise,

$$\mathbb{Z}[\mathcal{RL}(e_1, \dots, e_k; \underline{n})] \otimes \mathbb{Z}[Y_n] \to \text{tensor products of } \mathbb{Z}[X_{n_i}] \text{ and } \mathbb{Z}[Y_{n_j}]$$

is given by $x \otimes y \mapsto$ tensor products of $x_i^*(y)$ and $f(x_j^*(y))$, where the x_i and x_j are the components of the dual interpretation of x from Section 4A; ie x corresponds to the k–tuple $(x_1, \dots, x_k) \in \triangle([e_1], [e]) \times \cdots \times \triangle([e_k], [e])$. Since f is a map of simplicial sets, the result directly follows from [loc. cit.]. □

4D Action on iterated relative loop spaces

In this section we show that, for any two topological spaces $Y \subset X$ pointed at the same point $*$, the operads $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathrm{Top}})$ and $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathrm{Top}})$ act on the pair $(\Omega^m X, \Omega^m(X, Y))$. Here, $\Omega^m X$ denotes the m–fold loop space of X, and $\Omega^m(X, Y)$ denotes the m–fold relative loop spaces of (X, Y).

For $m \geq 1$ let Δ^m be the simplicial m–simplex and $\partial\Delta^m$ its boundary. For $0 \leq p \leq m$, let Λ_p^m be the m–horn at p. Explicitly, $(\Lambda_p^m)_k \subset \Delta_k^m$ is given by elements of the form $yz\colon [k] \to [m]$ for $z\colon [k] \to [n]$ and $y\colon [n] \to [m]$ with $n < m$, and if $n = m-1$, then $p \in \mathrm{Im}(y)$. For more details on these simplicial sets we refer to [11, pages 6–7]. It immediately follows from the definitions that:

4.14 Lemma *Let $x \in \Delta^m$. If $x \notin \Lambda_p^m$, then there is a $k \in \{m-1, m\}$ such that $x = \iota\rho\colon [n] \twoheadrightarrow [k] \hookrightarrow [m]$. Moreover, if $k = m-1$, then $\iota = \partial_p\colon [m-1] \hookrightarrow [m]$ is the p–face map ($p \notin \mathrm{Im}(\partial_p)$); if $k = m$, then $\iota = \mathrm{id}\colon [m] \to [m]$ is the identity map. In particular, if $x \notin \partial\Delta^m$, then $k = m$; hence x is surjective.*

Let $\mathbb{S}^m = \Delta^m/\partial\Delta^m$ be the simplicial m–sphere. Let $K_{\mathfrak{cl}}[m] := (\mathbb{S}^m, *)$ and $K_{\mathfrak{op}}^p[m] := (\Delta^m/\Lambda_p^m, \partial\Delta^m/\Lambda_p^m)$. Both $K_{\mathfrak{cl}}[m]$ and $K_{\mathfrak{op}}^p[m]$ are simplicial objects in the category **CFin** whose objects are pairs (A, B) of finite sets pointed at the same point with $* \subset B \subset A$ and whose morphisms $f\colon (A, B) \to (A', B')$ are morphisms of pointed sets $f\colon A \to A'$ such that $f(B) \subset B'$. The *wedge sum* is given by $(A,B) \vee (A',B') = (A \vee A', B \vee B')$.

The inclusion $\Lambda_p^m \subset \partial\Delta^m$ induces the projection $\pi\colon K_{\mathfrak{op}}^p[m] \to K_{\mathfrak{cl}}[m]$, which is a map of simplicial objects in **CFin**.

4.15 Proposition *Let $m \geq 1$ and $0 \leq p \leq m$. For p odd, the pair $(K_{\mathfrak{cl}}[m], K_{\mathfrak{op}}^p[m])$ is a coalgebra over $\mathcal{RL}_m$ in* **CFin**. *For p even, the pair $(K_{\mathfrak{cl}}[m], K_{\mathfrak{op}}^p[m])$ is a coalgebra over $\mathcal{RL}'_m$ in* **CFin**.

Proof Let us fix $m \geq 1$. In order to simplify notation, we denote $K_{\mathfrak{cl}}[m]$ (resp. $K_{\mathfrak{op}}^p[m]$) by $K^{\mathfrak{cl}}$ (resp. $K^{\mathfrak{op}}$). An element $x \in \mathcal{RL}(e_1,\dots,e_k;e)$ induces maps

$$x^*\colon K_n^{\mathfrak{cl}} \to K_{e_1}^{\mathfrak{cl}} \times \cdots \times K_{e_k}^{\mathfrak{cl}} \quad \text{if } e = n \qquad \text{and} \qquad \hat{x}^*\colon K_{\underline{n}}^{\mathfrak{op}} \to K_{e_1}^{c_1} \times \cdots \times K_{e_k}^{c_k} \quad \text{if } e = \underline{n}.$$

In what concerns the closed part ($e = n$), this is [3, Proposition 2.16]. For $e = \underline{n}$, let $(x_1,\dots,x_k) \in \Delta([e_1],[\underline{n}]) \times \cdots \times \Delta([e_k],[\underline{n}])$ be the k–tuple corresponding to x in the dual interpretation of $\mathcal{RL}$; see Section 4A. For $1 \leq i \leq k$, let $n_i = \mathrm{ev}(e_i)$. For $y \in \Delta^m/\Lambda_p^m$, the i^{th} component of $\hat{x}^*(y)$ is given by

$$\hat{x}_i^*(y) = \begin{cases} x_i^*(y) & \text{if } x_i\colon [\underline{n_i}] \to [\underline{n}], \\ \pi(x_i^*(y)) & \text{if } x_i\colon [n_i] \to [\underline{n}]. \end{cases}$$

We prove that if $x \in \mathcal{RL}_m$ for p odd (resp. $x \in \mathcal{RL}'_m$ for p even) then $\hat{x}^*(y)$ belongs to $K_{e_1}^{c_1} \vee \cdots \vee K_{e_k}^{c_k}$. From now on, let us suppose that there are $i \neq j$ such that $x_i^*(y)$ and $x_j^*(y)$ are not at the base point.

Suppose i corresponds to an open colour (ie $e_i = \underline{n_i}$). In this case, $x_i^*(y)\colon [n_i] \to [n] \to [m]$ is not at the base point, ie $x_i^*(y) \notin \Lambda_p^m$. From Lemma 4.14, this means that $\mathrm{Im}(x_i) \cap y^{-1}(r) \neq \varnothing$ for each $r \in [m]$ such that $r \neq p$.

(1) Suppose j corresponds to an open colour. This means that $x_j^*(y_s)$ is not at the base point and then $\mathrm{Im}(x_j) \cap y^{-1}(r) \neq \varnothing$ for each $r \in [m]$ such that $r \neq p$. Therefore, $\underline{i}$ and $\underline{j}$ appear in m common fibres of y. Consequently, $c_{i,j}(x) \geq m > m-1$.

(2) Suppose j corresponds to a closed colour. This means that $\pi(x_j^*(y)) \notin \partial\Delta^m$, that is, $x_j^*(y)\colon [n_j] \to [n] \to [m]$ does not belong to $\partial\Delta^m$ and thus is surjective (Lemma 4.14). It follows that $\mathrm{Im}(x_j)$ intersects all the fibres of y; that is, $\mathrm{Im}(x_j) \cap y^{-1}(r) \neq \varnothing$ for each $r \in [m]$. In particular, $y\colon [n] \to [m]$ is surjective, and then there are two cases for x_i:

(a) $\mathrm{Im}(x_i) \cap y^{-1}(p) \neq \varnothing$ (ie $\mathrm{Im}(x_i) \cap y^{-1}(r) \neq \varnothing$ for each $r \in [m]$), and so $c_{i,j}(x) \geq m+1 > m$ (because $\underline{i}$ and j appear in $m+1$ common fibres of y).

(b) $\mathrm{Im}(x_i) \cap y^{-1}(p) = \varnothing$. When p is odd, this implies that $c_{i,j}(x) > m$ if $j^- < \underline{i}^-$ and $c_{i,j}(x) > m-1$ if $j^- > \underline{i}^-$. When p is even, this implies that $c_{i,j}(x) > m$ if $j^- > \underline{i}^-$ and $c_{i,j}(x) > m-1$ if $j^- < \underline{i}^-$.

Let us illustrate what happens in (b) when p is odd and $j^- < \underline{i}^-$. In the integer string representation of x,

$$x = (\underbrace{\overbrace{j\,\underline{i}}^{0} \mid \cdots \mid \overbrace{\underline{i}\,j}^{p-2} \mid \overbrace{j\,\underline{i}}^{p-1}}_{c_{i,j}(x)_{|0,p-1} \geq p} \mid \overbrace{j}^{p} \mid \underbrace{\overbrace{j\,\underline{i}}^{p+1} \mid \cdots \mid \overbrace{\underline{i}\,j}^{m-1} \mid \overbrace{j\,\underline{i}}^{m}}_{c_{i,j}(x)_{|p+1,m} \geq m-p}) \quad \text{with } j^- < \underline{i}^-,$$

we only represent (relevant occurrences of) the integers $\underline{i}$ and j; the $m+1$ substrings represent the fibres $y^{-1}(r)$ for $r \in [m]$, so that if $n > m$, then these substrings would be subdivided (to end up with n vertical bars). Condition (2) says that the j are present in each of the $m+1$ substrings. Case (b) says that the $\underline{i}$ are present in all the $m+1$ substrings except the substring p. The number of switches $c_{i,j}(x)_{|0,p-1}$ (resp. $c_{i,j}(x)_{|p+1,m}$) between $\underline{i}$ and j in the first p substrings (resp. in the last $m-p$ substrings) is then $\geq p$ (resp. $\geq m-p$). If $c_{i,j}(x)_{|0,p-1} = p$, the conditions $j^- < \underline{i}^-$ and p odd imply that, in the substring $p-1$, the occurrences of j all appear before that of $\underline{i}$. Therefore, because of the presence of j in the substring p, there is an additional switch, ie $c_{i,j}(x) \geq c_{i,j}(x)_{|0,p-1} + 1 + c_{i,j}(x)_{|p+1,m} \geq p+1+m-p > m$. □

4.16 Corollary *Let $* \subset Y \subset X$ be topological spaces. For $m \geq 1$, the pair $(\Omega^m X, \Omega^m(X,Y))$ is an algebra over $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathrm{Top}})$ and over $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathrm{Top}})$.*

Proof The m–fold relative loop space $\Omega^m(X,Y) = \mathrm{Hom}_{\mathrm{Top}_*}(|K^p_{\mathfrak{op}}[m]|_{\delta_{\mathrm{Top}}}, (X,Y))$ is, by adjunction, homeomorphic to the δ_{Top}–totalization of $(X,Y,*)^{(K^p_{\mathfrak{op}}[m],*)}$. Similarly, the m–fold loop space $\Omega^m X = \mathrm{Hom}_{\mathrm{Top}_*}(|K_{\mathfrak{cl}}[m]|_{\delta_{\mathrm{Top}}}, X)$ is homeomorphic to the δ_{Top}–totalization of $(X,*)^{(K_{\mathfrak{cl}}[m],*)}$. Proposition 4.15 implies that the pair $((X,*)^{(K_{\mathfrak{cl}}[m],*)}, (X,Y,*)^{(K^p_{\mathfrak{op}}[m],*)})$ is an $\mathcal{RL}_m$–algebra for p odd and an $\mathcal{RL}'_m$–algebra for p even. The result follows from condensation (Section 2E). □

5 The relative surjection operad

We define two SC-type operads $\mathcal{RS}_m$ and $\mathcal{RS}'_m$ that are suboperads of $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ and $\mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathbb{Z}})$, respectively. We show that these inclusions are weak equivalences.

Since we are using $\delta_{\mathbb{Z}}$–realization, the Dold–Kan correspondence provides a convenient way to present the cosimplicial chain complex $\xi_{c_1,\dots,c_k;c}(\mathcal{RL}_m)(\delta_{\mathbb{Z}})$ as well as the operad $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$. We closely follow [4, Section 3] in which this point of view is adopted. In particular, we refer to [4, Section 3.3] for normalized totalizations of (co)simplicial abelian groups; we adopt the same notation except that since we have defined $\mathcal{RL}_m$ in the category of sets, we apply $\mathbb{Z}[-]$ to make it a multisimplicial/cosimplicial abelian group. This way, one identifies $\mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ with $\overline{\mathrm{Nor}}\,\underline{\mathrm{Nor}}(\mathbb{Z}[\mathcal{RL}_m])$, where $\underline{\mathrm{Nor}}(-)$ stands for the normalized realization of multisimplicial abelian groups and $\overline{\mathrm{Nor}}(-)$ stands for the normalized totalization of cosimplicial dg-abelian groups.

Let $(c_1,\dots,c_k;c)$ be a $(k+1)$–tuple of colours in $\{\mathfrak{cl},\mathfrak{op}\}$. As complexes, we set

$$\mathcal{RS}_m(c_1,\dots,c_k;c)^* := \underline{\mathrm{Nor}}(\mathbb{Z}[\mathcal{RL}_m(-,\dots,-;e)]),$$

where e is of type c such that $\mathrm{ev}(e)=0$ and the colours in the i^{th} input are of type c_i for $1\le i\le k$. More explicitly, $\mathcal{RS}_m(c_1,\dots,c_k;c)^*$ is obtained from $\bigoplus_{*=-(n_1+\cdots+n_k)}\mathbb{Z}[\mathcal{RL}_m(e_1,\dots,e_k;e)]$, where e is of type c such that $\mathrm{ev}(e)=0$ and $n_i:=\mathrm{ev}(e_i)$ with e_i of type c_i for $1\le i\le k$, by modding out the images of the simplicial degeneracies.

We closely follow [4, Equations (4) and (5)], in which the following *whiskering* map and partial compositions are defined for the closed case.

On an integer string $x\in\mathcal{RL}_m(e_1,\dots,e_k;e)$ with $\mathrm{ev}(e)=0$, the n–whiskering $w_n(x)$ is a signed sum of integer strings obtained from x by copying integers and adding a vertical bar between each copy (eg $i\mapsto i|i|i$) with the requirement that the total number of vertical bars is n. Note that there is no bar between two adjacent integers with different values (eg $i|j$ is not allowed).

5.1 Example $(\underline{1}|\underline{1}2|2|2)^{\mathfrak{op}}$ is a term of $w_3((\underline{1}2)^{\mathfrak{op}})$.

The whiskering

$$w\colon \mathcal{RS}_m\to\overline{\mathrm{Nor}}\,\underline{\mathrm{Nor}}(\mathbb{Z}[\mathcal{RL}_m])$$

is defined by $x\mapsto\Pi\, w_n(x)$ for $x\in\mathcal{RL}_m(e_1,\dots,e_k;e)$ with $\mathrm{ev}(e)=0$, and is extended by linearity.

Let $f\in\mathcal{RS}_m(c_1,\dots,c_k;c)$ and $g\in\mathcal{RS}_m(d_1,\dots,d_j;c_i)$ be two integer strings. We define their partial composition by

$$f\circ_i^{\mathcal{RS}_m} g = f\circ_i w_{\mathrm{val}_i(f)}g, \tag{5-1}$$

where $\mathrm{val}_i(f)$ is one less than the number of occurrences of $\widetilde{c}_i$ in the integer string f. We extend the partial compositions $\circ_i^{\mathcal{RS}_m}$ by linearity.

We define an operad $\mathcal{RS}'_m$ in the same way as $\mathcal{RS}_m$ by replacing $\mathcal{RL}_m$ by $\mathcal{RL}'_m$ in the above paragraphs.

5.2 Proposition *The partial compositions $\circ_i^{\mathcal{RS}_m}$ and $\circ_i^{\mathcal{RS}'_m}$ respectively endow $\mathcal{RS}_m$ and $\mathcal{RS}'_m$ with operad structures. The inclusions $w\colon \mathcal{RS}_m \hookrightarrow \mathrm{Coend}_{\mathcal{RL}_m}(\delta_{\mathbb{Z}})$ and $w'\colon \mathcal{RS}'_m \hookrightarrow \mathrm{Coend}_{\mathcal{RL}'_m}(\delta_{\mathbb{Z}}))$ are weak equivalences of operads.*

Proof Except for signs, the fact that w is compatible with the operadic structures is straightforward from the definition. To get signs for the whiskering, we can proceed by a laborious induction, by requiring the whiskering to be a morphism of operads. One can also remark that, concerning signs, the type of the colours (closed or open) does not matter. This is because closed and open colours obey the same (co)simplicial and composition rules (with different constraints). Thus, signs can be chosen using the same method as in the nonrelative case (disregarding *for signs only* the colour): One could proceed as proposed in [4, Proposition 3.2], that is, embed $\mathcal{RS}_m$ into the operad of coendomorphisms of the chains of a high-dimensional simplex. Then choose signs in such a way that they match with those of [6, Section 2.2] in the closed and in the open cases and, in fact, in the open/closed case, disregarding type of colours when choosing signs. This can be compared with Proposition 4.13, by asking the map f to be id: $\Delta^n \to \Delta^n$.

Let us denote by

$$\pi\colon \overline{\mathrm{Nor}}\,\underline{\mathrm{Nor}}(\mathbb{Z}[\mathcal{RL}_m]) \to \underline{\mathrm{Nor}}(\mathbb{Z}[\mathcal{RL}_m])^0 = \mathcal{RS}_m$$

the map induced by the projection $p\colon \underline{\mathrm{Nor}}(\mathbb{Z}[\mathcal{RL}_m])^{\cdot} \to \underline{\mathrm{Nor}}(\mathbb{Z}[\mathcal{RL}_m])^0$. Such a map is a weak equivalence by Proposition 4.10. Moreover it satisfies $\pi \circ w = \mathrm{id}$. Thus, w is a weak equivalence. □

6 The operads $\mathcal{RL}_2$ and $\mathcal{RL}'_2$

6A The operad $\mathcal{RL}_2$ in term of trees

6A1 The sets of $\mathcal{RT}$ In what follows we will consider *planar rooted trees*; we refer to [17, Part II, Section 1.5] for the terminology. Our trees have only one external edge, called the *root*; all the other edges have 2 adjacent vertices. The external vertices (vertices that are adjacent to only one edge) are called the *leaves*. Given a vertex v of a rooted tree T, the minimal subtree of T containing both the root and v has only one edge originating from v; such an edge is called the *output of* v. The edges originating from a vertex that are not the output are called the *inputs*. In our planar trees, the set of the edges originating from a vertex v is cyclically ordered in the clockwise direction. This induces a linear order on the set of the inputs of v. This also canonically endows the set of the leaves with a linear order.

One considers planar rooted trees with 4 types of vertices: white round-shaped $\circ$, called *closed vertices*; white square-shaped $\square$, called *open vertices*; black round-shaped $\bullet$, called *neutral vertices*; and black arrow-shaped $\blacktriangle$, called the *arrows*. A *white* vertex is either a closed vertex or an open vertex.

6.1 Definition Let T be a planar rooted tree. Let ν be a vertex of T. We denote by T_ν the maximal subtree of T such that the output of ν is the root of T_ν.

Let $(e_1,\dots,e_k,e)$ be a $(k+1)$–tuple of colours in Col. The set $\mathcal{RT}(e_1,\dots,e_k;e)$ is the empty set if $e\in\mathrm{Col}_{\mathfrak{cl}}$ and there exists a i such that $e_i\in\mathrm{Col}_{\mathfrak{op}}$; otherwise, $\mathcal{RT}(e_1,\dots,e_k;e)$ is the set of equivalence classes of planar rooted trees T satisfying the following:

- The set of the arrows is a subset of the leaves of T and is of cardinal $\mathrm{ev}(e)$.
- The set of closed vertices is labelled by the set $\{i\in\{1,\dots,k\}\mid e_i=n_i\}$ and open vertices by $\{i\in\{1,\dots,k\}\mid e_i=\underline{n}_i\}$ in such a way that
  - (F1) the vertex labelled by $i\in\{1,\dots,k\}$ has $\mathrm{ev}(e_i)$ inputs,
  - (F2) there is no white vertex above an open vertex; ie if ν is an open vertex, then in the tree T_ν the vertex ν is the unique white vertex.
- If e is an open colour (ie $e=\underline{n}$ for some n), then the root of T is decorated by an $\mathfrak{op}$.

The equivalence class is the same as in [10, Section 3.2.1]. Explicitly, it is the finest one in which two planar rooted trees are equivalent if one of them can be obtained from the other by either

- the contraction of an edge with neutral adjacent vertices, or
- removing an neutral vertex with only one edge originating from it and joining the two edges adjacent to this vertex into one edge.

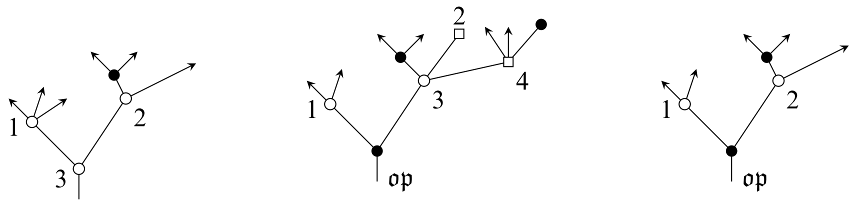


Figure 1: Examples of trees in $\mathcal{RT}$: a tree of $\mathcal{RT}(3,2,2;6)$ (left), a tree of $\mathcal{RT}(2,\underline{0},3,\underline{3};\underline{5})$ (middle), and a tree of $\mathcal{RT}(2,2;\underline{5})$ (right).

6.2 Remark In the closed case, the trees of $\mathcal{RT}(n_1,\dots,n_k;n)$ have no open vertices.

The operadic structure of $\mathcal{RT}$ is explicitly given in the next section and corresponds to the substitution of trees into white vertices.

6A2 Correspondence between $\mathcal{RT}$ and $\mathcal{RL}_2$ Let us start by constructing a bijection of sets $\Phi\colon \mathcal{RT}(e_1,\dots,e_k;e)\to\mathcal{RL}_2(e_1,\dots,e_k;e)$ for each $(k+1)$–tuple $(e_1,\dots,e_k;e)$ of colours.

The map Φ For a $T\in\mathcal{RT}(e_1,\dots,e_k;e)$, let us construct an integer string in $\mathcal{RL}_2(e_1,\dots,e_k;e)$ as follows. One runs through the tree T in clockwise direction starting from the root in such a way that one passes exactly two times on each edge (once per direction). On our way, each time one meets a closed (resp. an open) vertex labelled by an $i\in\{1,\dots,k\}$ one writes down the corresponding label i (resp. the corresponding label with an underline $\underline{i}$), and each time one meets an arrow one writes down a vertical bar. One adds an extra label $\mathfrak{op}$ if the root is decorated by $\mathfrak{op}$.

The map Φ^{-1} To an integer string representation one assigns a tree with one closed (resp. open) vertex for each different integer (resp. underlined integer) and one arrow for each vertical bar. The white vertices have one input less than the number of occurrences for the corresponding integer; the corresponding tree is constructed such that its order fits with the reading (from the left to the right) of the integer string. One adds an extra label $\mathfrak{op}$ on the root if the integer string is decorated by $\mathfrak{op}$. Note that when two equal integers (or two vertical bars) are adjacent in the integer string this forces the creation of a neutral vertex.

6.3 Example Via Φ, the tree from Figure 1 (middle) corresponds to the integer string $(1|1|13||3\underline{2}3\underline{4}|\underline{4}|\underline{4}\underline{4}3)^{\mathfrak{op}}$.

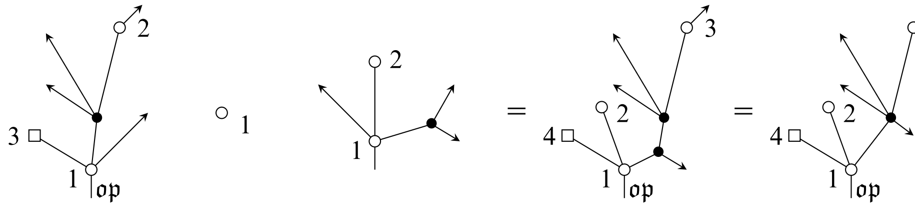


Figure 2: An example of composition in $\mathcal{RT}$

We endow $\mathcal{RT}$ with an operadic structure by transferring the composition maps of $\mathcal{RL}_2$ along the bijections above. Explicitly, $T \circ_v T' := \Phi^{-1}(\Phi(T) \circ_v \Phi(T'))$ for all composable $T, T' \in \mathcal{RT}$. One can check that the composition maps in $\mathcal{RT}$ are given by substitution of planar rooted trees into white vertices: Let $T \in \mathcal{RT}(e_1, \dots, e_k; e)$ and let v be a vertex of T labelled by e_i. For a tree $T' \in \mathcal{RT}(f_1, \dots, f_j; e_i)$, the tree $T \circ_v T' \in \mathcal{RT}(e_1, \dots, e_{i-1}, f_1, \dots, f_j, e_{i+1}, \dots, e_k; e)$ is given as follows. The vertex v of T is substituted by the tree T' in such a way that

- the root of T' is identified with the root of v,
- the ordered set of the $n_i = \mathrm{ev}(e_i)$ arrows of T' is identified with the ordered set of the n_i inputs of v.

The set of the white vertices of $T \circ_v T'$ is labelled by the set $\{1, \dots, k+j-1\}$ associated to $(e_1, \dots, e_{i-1}, f_1, \dots, f_j, e_{i+1}, \dots, e_k)$. See Figure 2.

By construction one has:

6.4 Proposition *The coloured operads $\mathcal{RL}_2$ and $\mathcal{RT}$ are isomorphic.*

6B The operad $\mathcal{RL}'_2$ in term of trees

This section is the analogue of the previous section for $\mathcal{RL}'_2$. We define an operad $\mathcal{RT}'$ as follows. For a $(k+1)$–tuple $(e_1, \dots, e_k; e)$ of colours in Col, the set $\mathcal{RT}'(e_1, \dots, e_k; e)$ is defined as $\mathcal{RT}(e_1, \dots, e_k; e)$ is, except that the condition (F2) from Section 6A1 is replaced by the following condition:

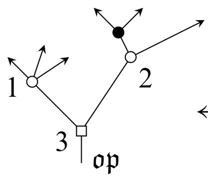
(F2′) There is no white vertex below an open vertex; ie if ν is a white vertex of T, then in the tree T_ν either the vertex ν is the unique open vertex or there is no open vertex.

In particular, for the closed part, $\mathcal{RT}'(n_1,\dots,n_k;n)=\mathcal{RT}(n_1,\dots,n_k;n)$.

The operadic composition for $\mathcal{RT}'$ is the same as for $\mathcal{RT}$, and we have the analogue of Proposition 6.4:

6.5 Proposition *The coloured operads $\mathcal{RL}'_2$ and $\mathcal{RT}'$ are isomorphic.*

Here is an example of an element of $\mathcal{RT}'(3,2,\underline{2};\underline{6})$ and its corresponding element in $\mathcal{RL}'_2(3,2,\underline{2};\underline{6})$:



$$\longleftrightarrow (\underline{3}1|1|1|1\underline{3}2||2|2\underline{3})^{\mathfrak{op}}.$$

6C A few remarks on the operads $\mathcal{RS}_2$ and $\mathcal{RS}'_2$

6.6 Proposition *As an operad, $\mathcal{RS}_2$ is generated by the elements*

$$\begin{aligned}
&\mu_{\mathfrak{cl}}=(12),\quad T_k=(1213\cdots 1k1)\quad \textit{for } k\geq 2 \ (\textit{closed part}),\\
&\mu_{\mathfrak{op}}=(\underline{1}\underline{2})^{\mathfrak{op}}\ (\textit{open part}),\\
&T_{\underline{j}}=(1\underline{2}1\underline{3}\cdots 1\underline{j}1)^{\mathfrak{op}}\quad \textit{for } j\geq 2,\\
&\mathrm{inc}=(1)^{\mathfrak{op}},
\end{aligned}$$

and the two unit elements $\mathrm{id}_{\mathfrak{cl}}=(1)$ and $\mathrm{id}_{\mathfrak{op}}=(\underline{1})^{\mathfrak{op}}$. As an operad, $\mathcal{RS}'_2$ is generated by

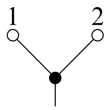
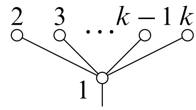
$$\mu_{\mathfrak{cl}},\quad T_k,\quad \mu_{\mathfrak{op}},\quad T'_{\underline{j}},\quad \mathrm{inc},\quad \mathrm{id}_{\mathfrak{cl}},\quad \mathrm{id}_{\mathfrak{op}}$$

for $k\geq 2$ and $j\geq 2$, where $T'_{\underline{j}}=(\underline{1}2\underline{1}3\cdots\underline{1}j\underline{1})^{\mathfrak{op}}$.

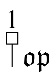
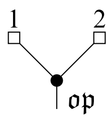
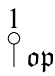
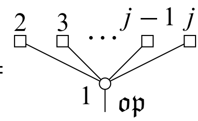
Proof We prove the statement for $\mathcal{RS}_2$; the case $\mathcal{RS}'_2$ is similar. We suppose by induction on N that any integer string of $\mathcal{RS}_2$ with N different integers is obtained by operadic compositions of elements cited in the statement. The cases $N=1$ and $N=2$ are trivially verified.

In what follows we abusively do not mark the distinction between underlined and nonunderlined integers. Let x be an integer string of $\mathcal{RS}_2$ with $N+1$ different integers. Because of the filtration condition (4-2), x can be written as a sequence $(A_1\cdots A_n)$, where the A_i are nonempty sequences of integers such that if j belongs to A_i, then $j\notin A_s$ for $s\neq i$. Moreover, because of the symmetric group action, one can suppose that the integers of A_i are smaller than the integers of A_j whenever $i < j$. In this case, if $n > 1$, then $x = (\alpha \circ_1 (A_1 \cdots A_{n-1})) \circ_{\max A_{n-1}+1} (\widetilde{A}_n)$, where $\widetilde{A}_n$ is obtained from A_n by decreasing each number by $\max A_{n-1}$, and α is $(12), (\underline{1}\underline{2})^{\mathfrak{op}}, (\underline{1}2)^{\mathfrak{op}}, (1\underline{2})^{\mathfrak{op}}$ or $(12)^{\mathfrak{op}}$. Since the A_i are not empty, $(A_1 \cdots A_{n-1})$ as well as A_n have at most N different integers and thus satisfy the induction hypotheses. If $n = 1$ then either x is T_k (or $T_{\underline{k}}$) for some k, or x is such that $A_1 = jB_1 jB_2 j \cdots jB_p j$ with $1 \le p < N$ and for some integer j. Thus there exists at least one B_{i_0} that contains $2 \le q \le N-(p-1)$ elements, and $x = (jB_1 j \cdots B_{i_0-1} jajB_{i_0+1} j \cdots jB_p j) \circ_a \widetilde{B}_{i_0}$ for some a, which concludes the proof. $\square$

Via Proposition 6.4, the generators of $\mathcal{RS}_2$ described in Proposition 6.6 correspond to the trees

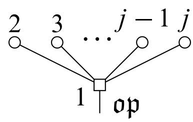
$\mathrm{id}_{\mathfrak{cl}} =$ , $\mu_{\mathfrak{cl}} =$ , $T_k =$ 

and

$\mathrm{id}_{\mathfrak{op}} =$ , $\mu_{\mathfrak{op}} =$ , $\mathrm{inc} =$ , $T_{\underline{j}} =$ 

(open part)

for the closed and nonclosed parts, respectively. For $\mathcal{RS}'_2$ the element $T'_{\underline{j}}$ corresponds to



The operadic structure of $\mathcal{RS}_2$ is described in terms of trees by means of Proposition 6.4; the whiskering w defined in Section 5 has a corresponding map on $\mathcal{RT}$ (roughly, it consists in adding arrows linked to the white vertices) and the composition of two trees is given by transferring the formula (5-1) to $\mathcal{RT}$. One has a similar description for $\mathcal{RS}'_2$. For the closed part, details are given in [4] with a slightly different convention. A full description of the operad $\mathcal{RS}'_2$ in terms of similar trees is given in [12].

6D The algebras over $\mathcal{RL}_2$

We describe the algebras over $\mathcal{RL}_2$, where $\mathcal{RL}_2$ is implicitly seen in $\boldsymbol{C}$ by means of the strong symmetric monoidal functor $\mathbf{Set} \to \boldsymbol{C}$, $E \mapsto \coprod_{e \in E} 1_{\boldsymbol{C}}$. Precisely, we show that $\mathcal{RL}_2$ encodes the pairs $(\mathcal{M}, \mathcal{Z})$ subject to the following conditions:

(I) $\mathcal{M}$ is a multiplicative non-Σ operad.

(II) $\mathcal{Z}$ is in $\mathbf{BiMod}_{\mathcal{M}-\mathcal{A}s}$, and there is a morphism $\iota\colon \mathcal{M} \to \mathcal{Z}$ in $\mathbf{BiMod}_{\mathcal{M}-\mathcal{A}s}$.

Let $\boldsymbol{E}$ be the category with objects the pairs $(\mathcal{M}, \mathcal{Z})$ satisfying the two conditions (I) and (II) above; morphisms are the pairs $(f, g)\colon (\mathcal{M}, \mathcal{Z}) \to (\mathcal{M}', \mathcal{Z}')$ subject to the following conditions:

- $f\colon \mathcal{M} \to \mathcal{M}'$ is a morphism of multiplicative non-Σ operads,
- $g\colon \mathcal{Z} \to \mathcal{Z}'$ is an f–equivariant morphism of left modules over $\mathcal{M}$,
- g is a morphism of bimodules over $\mathcal{A}s$, and
- $\iota' \circ f = g \circ \iota$.

6.7 Remark Note that $\boldsymbol{E}$ is well defined since, by the last condition $\iota' \circ f = g \circ \iota$, the morphism ι of (II) associated to $(\mathcal{M}, \mathcal{Z})$ is unique. The pair $(\mathcal{A}s, \mathcal{A}s)$ is the initial object of $\boldsymbol{E}$.

6.8 Remark As observed in Example 2.6, since here we have a morphism $\mathcal{A}s \to \mathcal{Z}$, it follows from (II) that $\mathcal{Z}$ is, in particular, a weak bimodule over $\mathcal{A}s$ (see also the proof of Proposition 6.9). By Lemma 2.7 this is equivalent to saying that $\mathcal{Z}$ is endowed with a cosimplicial structure. Likewise, the multiplicative structure of $\mathcal{M}$ endows it with a cosimplicial structure. These are the cosimplicial structures involved in the δ–totalization that gives rise to the $\mathrm{Coend}_{\mathcal{RL}_2}(\delta)$–algebra $(\mathrm{Tot}_\delta \mathcal{M}, \mathrm{Tot}_\delta \mathcal{Z})$.

6.9 Proposition *The category of $\mathcal{RL}_2$–algebras in $\boldsymbol{C}$ is isomorphic to the category $\boldsymbol{E}$.*

Proof We use the interpretation of $\mathcal{RL}_2$ in terms of planar trees; see Proposition 6.4.

Let $(\mathcal{M}, \mathcal{Z})$ be an $\mathcal{RL}_2$–algebra. This means that for each $T \in \mathcal{RL}_2(e_1, \dots, e_p; e)$ one has a corresponding operation

$$T \in \mathrm{Hom}(\textit{products of } \mathcal{M}(n_i) \textit{ and } \mathcal{Z}(n_j), \mathcal{Z}(n))$$

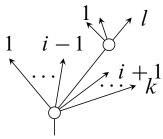
or

$$T \in \mathrm{Hom}(\textit{products of } \mathcal{M}(n_i), \mathcal{M}(n))$$

according to the type of the output e, where $n_s = \mathrm{ev}(e_s)$ is the value of e_s, $1 \le s \le p$ and $n = \mathrm{ev}(e)$.

Let us write explicitly the induced structures on $\mathcal{M}$ and $\mathcal{Z}$.

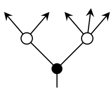
(1) **Operadic structure on $\mathcal{M}$** The partial compositions on elementary trees are given by



$$\longleftrightarrow \quad \circ_i : \mathcal{M}(k) \otimes \mathcal{M}(l) \to \mathcal{M}(k+l-1).$$

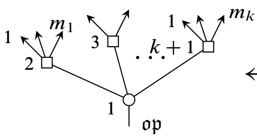
The operadic structure of $\mathcal{RL}_2$ provides the associativity condition for partial compositions of $\mathcal{M}$. This is similar to (3) below, which we describe in detail.

(2) **Multiplicative structure on $\mathcal{M}$** The morphism $\alpha\colon \mathcal{A}s(k) \to \mathcal{M}(k)$ is given by the corolla in a neutral vertex with k inputs. For $k=1$, one has that $\alpha = \eta\colon 1_{\boldsymbol{C}} \to \mathcal{M}(1)$ is the unit. Note that the isomorphism $\mathcal{A}s(k) \otimes \mathcal{A}s(l) \to \mathcal{A}s(k+l-1)$ corresponds to the equivalence relation made on the neutral vertices (see Definition 6.1). The multiplication in $\mathcal{M}$ is given by operations as



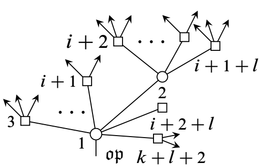
$$: \mathcal{M}(2) \otimes \mathcal{M}(3) = \mathcal{A}s(2) \otimes \mathcal{M}(2) \otimes \mathcal{M}(3) \to \mathcal{M}(5).$$

(3) **Left action of $\mathcal{M}$ on $\mathcal{Z}$** The k–corollas in a closed vertex with k open vertices at the inputs give the left action λ:

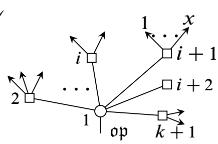


$$\longleftrightarrow \quad \lambda\colon \mathcal{M}(k) \otimes \mathcal{Z}(m_1) \otimes \cdots \otimes \mathcal{Z}(m_k) \to \mathcal{Z}(m_1 + \cdots + m_k).$$

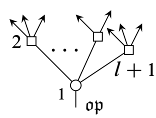
The operadic structure of $\mathcal{RL}_2$ gives the associativity of the left action:



$$= \sigma_{(2,i),i+1} \left(\vphantom{\Big|} \right.$$

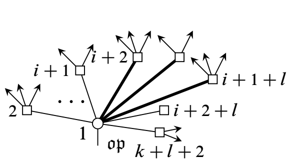


$$\circ_{i+1}$$

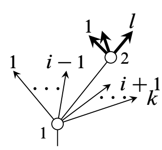


$$\left. \vphantom{\Big|} \right)$$

$$=$$



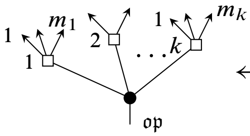
$$\circ_1$$



,

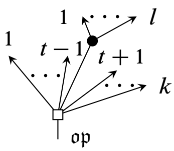
where i, k, l, x are as in Definition 2.12.1 and $\sigma_{(2,i),i+1}$ is the block permutation ($\sigma_{(2,i),i+1}(s) = s+1$ for $2 \le s \le i$, $\sigma_{(2,i),i+1}(i+1) = 2$ and $\sigma_{(2,i),i+1}(s) = s$ otherwise). The first decomposition corresponds to the top-right path in the diagram of Definition 2.12.1; the second decomposition corresponds to the left-bottom path.

Note that, by precomposing $\lambda^{\mathcal{M}}$ with $\alpha\colon \mathcal{A}\mathrm{s} \to \mathcal{M}$, one has a left action of $\mathcal{A}\mathrm{s}$:



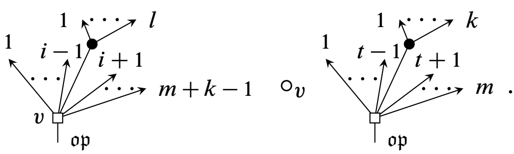
$$\longleftrightarrow\quad \lambda^{\mathcal{A}\mathrm{s}}\colon \mathcal{A}\mathrm{s}(k)\otimes\mathcal{Z}(m_1)\otimes\cdots\otimes\mathcal{Z}(m_k)\to\mathcal{Z}(m_1+\cdots+m_k).$$

(4) **Right action of $\mathcal{A}\mathrm{s}$ on $\mathcal{Z}$** This is given by



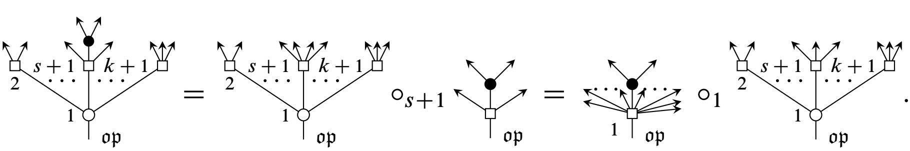
$$\longleftrightarrow\quad \rho_t\colon \mathcal{Z}(k)=\mathcal{Z}(k)\otimes\mathcal{A}\mathrm{s}(l)\to\mathcal{Z}(k+l-1).$$

The associativity for the right action is given by operadic composition of $\mathcal{RL}_2$. Precisely, for the left-sided square of Definition 2.12.1, $\mathcal{A}\mathrm{s}(k)\otimes\mathcal{A}\mathrm{s}(l)\cong\mathcal{A}\mathrm{s}(k+l-1)$ leads to the left-bottom path, while the top-right path corresponds to the decomposition of the above tree as

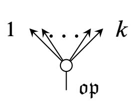


The right-sided square of Definition 2.12.1 is obtained similarly.

(5) **Associativity of the left $\mathcal{M}$–action and right $\mathcal{A}\mathrm{s}$–action on $\mathcal{Z}$** This is the square of Definition 2.12.1 and is obtained by considering trees as

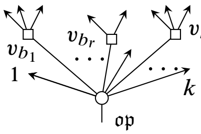


(6) **The morphism $\iota\colon \mathcal{M}\to\mathcal{Z}$** This is given by corollas in a closed vertex with an open output:



$$\longleftrightarrow\quad \iota\colon \mathcal{M}(k)\to\mathcal{Z}(k).$$

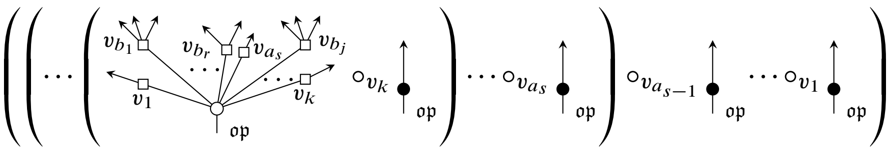
Note that the operations such as those given by k–corollas in a closed vertex with $1 \le j \le k$ open vertices at inputs $1 \le b_1 < \cdots < b_j \le k$,



$$\longleftrightarrow \lambda_{b_1,\dots,b_j} : \mathcal{M}(k) \otimes \mathcal{Z}(m_1) \otimes \cdots \otimes \mathcal{Z}(m_j) \to \mathcal{Z}\Big(k - j + \sum_{1 \le p \le j} m_p\Big),$$

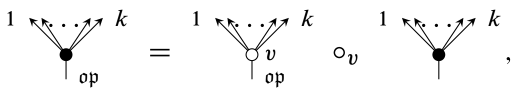
can be obtained by precomposing the left action λ by $\mathcal{A}\mathrm{s}(1) \to \mathcal{Z}(1)$ at the all inputs other than the b_i. This results from the fact that such a k–corolla is obtained as

(6-1)

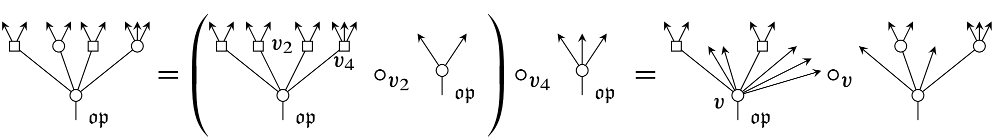


where the a_s are the inputs other than the b_j ($a_1 = 1$ in the picture above). In particular, the weak left action is obtained whenever $j = 1$ and the left action corresponds to $j = k$. Doing this to the corollas of $\lambda^{\mathcal{A}\mathrm{s}}$ in (3) above endows $\mathcal{Z}$ with a structure of weak bimodule over $\mathcal{A}\mathrm{s}$.

Let us remark that, because of the decomposition



the morphism $\mathcal{A}\mathrm{s} \to \mathcal{Z}$ given by the left-sided tree corresponds to $\iota\alpha$: $\mathcal{A}\mathrm{s} \to \mathcal{M} \to \mathcal{Z}$. More generally, note that any tree can be obtained as a composition of *elementary trees*: trees as in (1)–(4), (6) above and corollas in a neutral vertex. Therefore, for $(\mathcal{M}, \mathcal{Z}) \in \boldsymbol{E}$, the operation corresponding to a given a tree T is defined as given by any of the decompositions of T in elementary trees; the independence of this operation regarding the different decompositions is ensured by the properties of $(\mathcal{M}, \mathcal{Z})$. For instance, the two decompositions



correspond to (2 iterations of) the diagram of Definition 2.1(2), plus the fact that ι is a left module map and η: $1_{\boldsymbol{C}} \to \mathcal{M}$ is the unit. Here, the penultimate tree is not elementary but it admits a decomposition into elementary trees as in (6-1). $\square$

6E The algebras over $\mathcal{RL}_2'$

We show that the operad $\mathcal{RL}_2'$ encodes the pairs $(\mathcal{M}, \mathcal{Z})$ subject to the following conditions:

(I′) $\mathcal{M}$ is a multiplicative non-Σ operad.

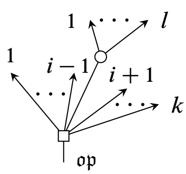
(II′) $\mathcal{Z}$ is in $\mathbf{BiMod}_{\mathcal{A}s-\mathcal{M}}$ with a morphism $\iota\colon \mathcal{M} \to \mathcal{Z}$ in $\mathbf{BiMod}_{\mathcal{A}s-\mathcal{M}}$.

Let $\boldsymbol{E}'$ be the category with objects the pairs $(\mathcal{M}, \mathcal{Z})$ satisfying the two conditions (I′) and (II′) above; morphisms are the pairs $(f, g)\colon (\mathcal{M}, \mathcal{Z}) \to (\mathcal{M}', \mathcal{Z}')$ subject to the following conditions:

- $f\colon \mathcal{M} \to \mathcal{M}'$ is a morphism of multiplicative non-Σ operads,
- $g\colon \mathcal{Z} \to \mathcal{Z}'$ is an f–equivariant morphism of right modules over $\mathcal{M}$,
- g is a morphism of bimodules over $\mathcal{A}s$, and
- $\iota' \circ f = g \circ \iota$.

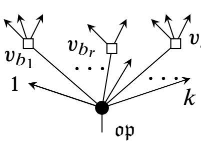
6.10 Proposition *The category of $\mathcal{RL}_2'$–algebras in $\boldsymbol{C}$ is isomorphic to the category $\boldsymbol{E}'$.*

Proof The only significant differences with Proposition 6.9 are the following. The right action of $\mathcal{M}$ on $\mathcal{Z}$ is given by the k–corollas in an open vertex with a closed vertex at an input:



$$\longleftrightarrow \rho_i\colon \mathcal{Z}(k) \otimes \mathcal{M}(l) \to \mathcal{Z}(k+l-1).$$

The operations such as those given by k–corollas in a neutral vertex with $1 \le j \le k$ open vertices at inputs $1 \le b_1 < \cdots < b_j \le k$,



$$\longleftrightarrow \lambda_{b_1,\dots,b_j}\colon \mathcal{A}s(k) \otimes \mathcal{Z}(m_1) \otimes \cdots \otimes \mathcal{Z}(m_j) \to \mathcal{Z}\Big(k - j + \sum_{1 \le p \le j} m_p\Big),$$

are obtained via the left action of $\mathcal{A}s$ on $\mathcal{Z}$, using the fact that $\iota\alpha\colon \mathcal{A}s \to \mathcal{M} \to \mathcal{Z}$ is a morphism of left modules. □

References

[1] **M A Batanin**, *Symmetrisation of n–operads and compactification of real configuration spaces*, Adv. Math. 211 (2007) 684–725 MR

[2] **M A Batanin**, *The Eckmann–Hilton argument and higher operads*, Adv. Math. 217 (2008) 334–385 MR

[3] **M A Batanin**, **C Berger**, *The lattice path operad and Hochschild cochains*, from "Alpine perspectives on algebraic topology" (C Ausoni, K Hess, J Scherer, editors), Contemp. Math. 504, Amer. Math. Soc., Providence, RI (2009) 23–52 MR

[4] **M Batanin**, **C Berger**, **M Markl**, *Operads of natural operations I: Lattice paths, braces and Hochschild cochains*, from "Operads 2009" (J-L Loday, B Vallette, editors), Sémin. Congr. 26, Soc. Math. France, Paris (2013) 1–33 MR

[5] **C Berger**, *Combinatorial models for real configuration spaces and E_n–operads*, from "Operads: proceedings of renaissance conferences" (J-L Loday, J D Stasheff, A A Voronov, editors), Contemp. Math. 202, Amer. Math. Soc., Providence, RI (1997) 37–52 MR

[6] **C Berger**, **B Fresse**, *Combinatorial operad actions on cochains*, Math. Proc. Cambridge Philos. Soc. 137 (2004) 135–174 MR

[7] **C Berger**, **I Moerdijk**, *Resolution of coloured operads and rectification of homotopy algebras*, from "Categories in algebra, geometry and mathematical physics" (A Davydov, M Batanin, M Johnson, S Lack, A Neeman, editors), Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007) 31–58 MR

[8] **M Brun**, **Z Fiedorowicz**, **R M Vogt**, *On the multiplicative structure of topological Hochschild homology*, Algebr. Geom. Topol. 7 (2007) 1633–1650 MR

[9] **B Day**, **R Street**, *Abstract substitution in enriched categories*, J. Pure Appl. Algebra 179 (2003) 49–63 MR

[10] **V A Dolgushev**, **D E Tamarkin**, **B L Tsygan**, *Proof of Swiss cheese version of Deligne's conjecture*, Int. Math. Res. Not. 2011 (2011) 4666–4746 MR

[11] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Progress in Mathematics 174, Birkhäuser, Basel (1999) MR

[12] **E Hoefel**, **M Livernet**, **A Quesney**, *On the deformation complex of homotopy affine actions*, preprint (2016) arXiv

[13] **E Hoefel**, **M Livernet**, **J Stasheff**, *A_∞–actions and recognition of relative loop spaces*, Topology Appl. 206 (2016) 126–147 MR

[14] **M Hovey**, *Model categories*, Math. Surveys Monogr. 63, Amer. Math. Soc., Providence, RI (1999) MR

[15] **M Kontsevich**, *Operads and motives in deformation quantization*, Lett. Math. Phys. 48 (1999) 35–72 MR

[16] **T Leinster**, *Higher operads, higher categories*, London Math. Soc. Lecture Note Ser. 298, Cambridge Univ. Press (2004) MR

[17] **M Markl**, **S Shnider**, **J Stasheff**, *Operads in algebra, topology and physics*, Math. Surveys Monogr. 96, Amer. Math. Soc., Providence, RI (2002) MR

[18] **J P May**, *The geometry of iterated loop spaces*, Lecture Notes in Math. 271, Springer (1972) MR

[19] **J E McClure**, **J H Smith**, *A solution of Deligne's Hochschild cohomology conjecture*, from "Recent progress in homotopy theory" (D M Davis, J Morava, G Nishida, W S Wilson, N Yagita, editors), Contemp. Math. 293, Amer. Math. Soc., Providence, RI (2002) 153–193 MR

[20] **J E McClure**, **J H Smith**, *Multivariable cochain operations and little n–cubes*, J. Amer. Math. Soc. 16 (2003) 681–704 MR

[21] **J E McClure**, **J H Smith**, *Cosimplicial objects and little n–cubes, I*, Amer. J. Math. 126 (2004) 1109–1153 MR

[22] **V Turchin**, *Hodge-type decomposition in the homology of long knots*, J. Topol. 3 (2010) 487–534 MR

[23] **A A Voronov**, *The Swiss-cheese operad*, from "Homotopy invariant algebraic structures" (J-P Meyer, J Morava, W S Wilson, editors), Contemp. Math. 239, Amer. Math. Soc., Providence, RI (1999) 365–373 MR

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Received: 18 July 2017 Revised: 18 November 2017