Algebraic & Geometric Topology 18 (2018) 1461–1480

Noncharacterizing slopes for hyperbolic knots

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A nontrivial slope r on a knot K in S^3 is called a characterizing slope if whenever the result of r–surgery on a knot K' is orientation-preservingly homeomorphic to the result of r–surgery on K, then K' is isotopic to K. Ni and Zhang ask: for any hyperbolic knot K, is a slope $r = p/q$ with $|p| + |q|$ sufficiently large a characterizing slope? In this article, we prove that if we can take an unknot c so that $(0,0)$–surgery on $K \cup c$ results in S^3 and c is not a meridian of K, then K has infinitely many noncharacterizing slopes. As the simplest known example, the hyperbolic, two-bridge knot 8_6 has no integral characterizing slopes. This answers the above question in the negative. We also prove that any L-space knot never admits such an unknot c.

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1 Introduction

Let K be a knot in the oriented 3–sphere S^3. Denote by $K(p/q)$ the 3–manifold obtained by p/q–Dehn surgery on K which has the orientation induced from S^3. We call $p/q \in \mathbb{Q}$ a *characterizing slope* for K if whenever $K'(p/q)$ is orientation-preservingly homeomorphic to $K(p/q)$, then K' is isotopic to K. For the trivial knot, Gordon [10] conjectured that every nontrivial slope $p/q \in \mathbb{Q}$ is a characterizing slope. Kronheimer, Mrowka, Ozsváth and Szabó [19] proved this conjecture in the positive using Seiberg–Witten monopoles. See Ozsváth and Szabó [28; 31] for alternative proofs using Heegaard Floer homology. Furthermore, Ozsváth and Szabó [30] showed that for the trefoil knot and the figure-eight knot, every nontrivial slope is a characterizing slope.

On the other hand, it is known that many knots have noncharacterizing slopes. The first such example was given by Lickorish [21]. Some torus knots have noncharacterizing slopes. For instance, 21–surgeries on $T_{5,4}$ and $T_{11,2}$ produce the same oriented 3–manifold, and hence 21 is a noncharacterizing slope for both $T_{5,4}$ and $T_{11,2}$ [26]. However, Ni and Zhang [26] prove that for a torus knot $T_{r,s}$ with $r > s > 1$, a slope p/q is a characterizing slope if $p/q > 30(r^2-1)(s^2-1)/67$. Later, McCoy [23]

Published: 3 April 2018 DOI: 10.2140/agt.2018.18.1461

lowers the bound to $43(rs - r - s)/4$. See also McCoy [24]. This suggests that for a given knot K, sufficiently large slopes should be characterizing ones. For hyperbolic knots, Ni and Zhang ask the following:

Question 1.1 [26] *Let K be a hyperbolic knot. Is a slope $r = p/q$ with $|p| + |q|$ sufficiently large a characterizing slope of K?*

Remark 1.2 For any given hyperbolic knot K, there is a number $N_K > 0$ such that a slope p/q with $|p| + |q| > N_K$ has a special geometric meaning due to Thurston's hyperbolic Dehn surgery theorem; see Benedetti and Petronio [4], Boileau and Porti [5], Petronio and Porti [32] and Thurston [34; 35]. For such a slope p/q, the 3–manifold $K(p/q)$ is hyperbolic, and the surgery dual to K is the unique shortest closed geodesic in $K(p/q)$. Hence for any *finite* family of hyperbolic knots $\mathcal{K}$, there is a number $N_{\mathcal{K}} > 0$ such that any slope p/q with $|p| + |q| > N_{\mathcal{K}}$ is a characterizing slope for every knot $K \in \mathcal{K}$.

The purpose in this article is to answer Question 1.1 in the negative. To this end, we need to construct a hyperbolic knot with infinitely many noncharacterizing slopes. The theorem below gives a sufficient condition for a knot K to have infinitely many noncharacterizing slopes.

Theorem 1.3 *Let K be a knot in S^3. Suppose that we can take an unknot c disjoint from K so that $(0,0)$–surgery on $K \cup c$ results in S^3 and c is not a meridian of K. Then K has infinitely many noncharacterizing slopes.*

Note that the condition that $(0,0)$–surgery on $K \cup c$ results in S^3 implies that $|\mathrm{lk}(K,c)| = 1$ for homological reasons, where $\mathrm{lk}(K,c)$ denotes the linking number between K and c. Of course, if c is a meridian of K, then the result of $(0,0)$–surgery on $K \cup c$ is always S^3; see Lemma 2.4.

As shown by Theorem 2.5, if we find a link $K \cup c$ which satisfies the condition in Theorem 1.3, then we have infinitely many distinct knots, each of which has infinitely many noncharacterizing slopes. We apply this to present explicit examples.

First, for comparison, recall that every nontrivial slope is a characterizing slope for a trefoil knot and the figure-eight knot [30], which are genus-one fibered knots. If we drop one of these conditions, we have:

Example 1.4 (1) Let K be the hyperbolic, fibered knot 9_{42} in Rolfsen's table, which has genus two. Then every integer except possibly 2 is not a characterizing slope for K.

(2) Let K be the hyperbolic, genus-one pretzel knot $P(-3,3,5)$, which is not fibered. Then every integer except possibly 0 is not a characterizing slope for K.

A modification of the above examples leads us to demonstrate:

Theorem 1.5 *There exists a hyperbolic knot for which every integral slope is a noncharacterizing slope. In particular, every integral slope is not a characterizing slope for the hyperbolic, two-bridge knot 8_6 in Rolfsen's table.*

Such a phenomenon can occur for prime satellite knots and composite knots as well. More precisely, we are able to prove:

Theorem 1.6 (1) *Given a nontrivial knot k, there exists a prime satellite knot with companion knot k for which every integral slope is a noncharacterizing slope.*

(2) *Given a nontrivial knot k, there exists a composite knot with k a connected summand for which every integral slope is a noncharacterizing slope.*

Among known examples, the knot 8_6 is the simplest knot (with respect to crossing numbers) which has infinitely many noncharacterizing slopes. So we would like to ask:

Question 1.7 *Are there any knots of crossing number less than 8 that have infinitely many noncharacterizing slopes?*

It is natural to ask which knots K admit an unknot c that satisfies the condition in Theorem 1.3.

Theorem 1.8 *Let $K \cup c$ be a two-component link in S^3 with unknotted component c which is not a meridian of K. Suppose that $(0,0)$–surgery on $K \cup c$ results in S^3. Then K is not an L-space knot.*

In the last section, we will give further questions concerning characterizing and noncharacterizing slopes for knots.

Throughout the paper we will use $N(*)$ to denote a tubular neighborhood of $*$ and $\mathcal{N}(*)$ to denote the interior of $N(*)$ for notational simplicity.

Acknowledgements We would like to thank the referee for a careful reading and useful comments.

Baker was partially supported by a grant from the Simons Foundation (#209184 to Kenneth L Baker). Motegi was partially supported by JSPS KAKENHI Grant Number JP26400099 and Joint Research Grant of Institute of Natural Sciences at Nihon University for 2016.

2 Noncharacterizing slopes and twist families of surgeries

In this section, we establish the general principle Theorem 2.1 and its extension Theorem 2.5, from which Theorem 1.3 follows. Throughout this article, for two oriented 3–manifolds M and N, by $M \cong N$ we mean that M is orientation-preservingly homeomorphic to N.

Theorem 2.1 *Let $k \cup c$ be a two-component link in S^3 such that c is unknotted. Suppose that $(0,0)$–surgery on $k \cup c$ results in S^3. Let K be the knot in S^3 which is surgery dual to c, the image of c, in the surgered S^3, and let k_n be the knot obtained from k by twisting n times along c. Then $K(n) \cong k_n(n)$ for all integers n.*

Moreover, if c is not a meridian of k, then $K \not\cong k_n$ for all but finitely many integers n.

Proof Since $(0,0)$–surgery on $k \cup c$ is S^3, a homology calculation shows that $|\mathrm{lk}(k,c)| = 1$. Performing $(-1/n)$–surgery along c takes the knot k with the surgery slope 0 to a knot k_n with a surgery slope $n = 0 + n(\mathrm{lk}(k,c))^2$; ie an n–twist along c converts a knot-slope pair $(k,0)$ into another knot-slope pair (k_n, n). Thus we obtain a twist family of knot-slope pairs $\{(k_n, n)\}$. Let V be the solid torus $S^3 - \mathcal{N}(c)$ which contains k in its interior. Observe that $V(k;0) \cong V(k_n;n)$ for all n.

Let (μ_c, λ_c) be a preferred meridian-longitude pair of $c \subset S^3$, oriented with the right-handed orientation (so that if c is oriented in the same direction as λ_c in $\mathcal{N}(c)$, then $\mathrm{lk}(\mu_c, c) = 1$). Note that λ_c represents the 0–slope on $N(c)$, and λ_c bounds a meridian disk of the solid torus V. Let c_n be the surgery dual to the $(-1/n)$–surgery on c (ie a core of the filled solid torus) with meridian μ_n, the $(-1/n)$–surgery slope of c in ∂V. These curves μ_n are each longitudes of V and satisfy $[\mu_n] = -[\mu_c] + n[\lambda_c] \in H_1(\partial V)$; hence $[\mu_0] = -[\mu_c]$.

Since k wraps algebraically once in V, a preferred longitude of $k \subset V \subset S^3$ is homologous to μ_c in $V - \mathcal{N}(k)$. Hence μ_c is null-homologous in $V(k;0)$.

Let K be the surgery dual to c with respect to λ_c–surgery. (Adapting the above notation, K may be regarded as c_∞.) Since $(0,0)$–surgery on $k \cup c$ results in S^3, we have that K is a knot in this surgered S^3 with exterior $S^3 - \mathcal{N}(K) = V(k;0)$ and meridian λ_c. Because μ_c is null-homologous in $V(k;0)$, it is the boundary of a Seifert surface for K.

With right-handed orientation, a preferred meridian-longitude pair for K in S^3 is given by $(\lambda_c, -\mu_c)$. Thus $[\mu_n] = -[\mu_c] + n[\lambda_c] = n[\lambda_c] + (-[\mu_c])$ corresponds to a slope n with respect to the preferred meridian-longitude pair $(\lambda_c, -\mu_c)$. Therefore, $k_n(n) = K(n)$ for all integers n.

If c is not a meridian of k, then since $\mathrm{lk}(k,c) \neq 0$, any disk bounded by c intersects k more than once. Then it follows from [18] that there are only finitely many n such that k_n is isotopic to K. □

Remark 2.2 Gompf and Miyazaki [9] had previously utilized the mirror of the knot K associated to k as described in Theorem 2.1 for a satellite construction of ribbon knots that generalizes the connected sum of a knot and its mirror.

Let $k \cup c$ be a link as in Theorem 2.1; ie c is unknotted and the result of $(0,0)$–surgery on $k \cup c$ is S^3. In Theorem 2.1, K denotes the surgery dual to c. Similarly we denote by C the surgery dual to k. Thus we have the surgery dual link $C \cup K$ to $k \cup c$ in the surgered S^3.

Lemma 2.3 *Let $k \cup c$ be a link as in Theorem 2.1 with surgery dual link $C \cup K$. Then C is unknotted in S^3.*

Proof After 0–surgery on c, we have that k becomes some knot in $c(0) = S^1 \times S^2$. Since a nontrivial surgery (corresponding to the 0–surgery) on $k \subset S^1 \times S^2$ yields S^3, due to Gabai [8, Corollary 8.3], it turns out that k (as a knot in $S^1 \times S^2$) is an S^1–fiber in some product structure $\mathcal{P}$ of $S^1 \times S^2$, and intersects an S^2–fiber in $\mathcal{P}$ exactly once. As usual, we may isotope an S^2–fiber in $\mathcal{P}$ to $S = S^2 \times \{0\}$ in the original product structure; the knot k is simultaneously isotoped to a knot intersecting S in a single point. Then a further ambient isotopy, possibly with "light bulb" moves which are accomplished by an ambient isotopy of the type illustrated in Figure 1 (see [33, page 257]), enables us to deform k to an S^1–fiber in the original product structure of $c(0) = S^1 \times S^2$. Thus the surgery dual C to k in $(k \cup c)(0,0) = S^3$ is an unknot while the surgery dual K to c is not necessarily unknotted in this S^3. Figure 2 illustrates such a situation. □

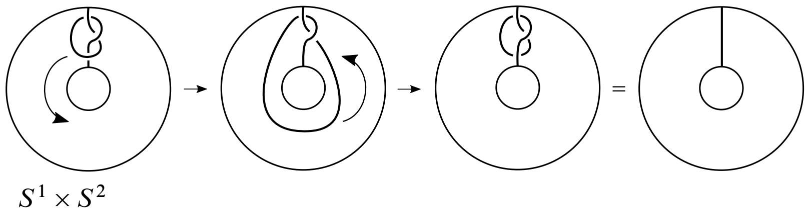

Figure 1: The "light bulb" move in $S^1 \times S^2$

In the special case where c is a meridian of k, we have:

Lemma 2.4 *Let $k \cup c$ be a two-component link in S^3 such that c is a meridian of k. Then $(0,0)$–surgery on $k \cup c$ results in S^3 with its surgery dual link $C \cup K$, for which C is a meridian of K, and K is isotopic to k in S^3.*

Proof This is essentially shown in [9, page 119] without a proof. So for completeness, we give a proof. Since c is a meridian of k, we may straighten k in $c(0) = S^1 \times S^2$ using light bulb moves and isotopies; the framing 0 of k is changed into some even integer, and the image K of c in $c(0) = S^1 \times S^2$ intersects $\{x\} \times S^2$ once for some $x \in S^1$. Then we see that $(0,0)$–surgery on $k \cup c$ results in S^3 with its surgery dual $C \cup K$ in which the dual C to k is a meridian of K in S^3. Let us see that K is isotopic to k. Since c is a meridian of k, the exterior $S^3 - \mathcal{N}(k \cup c)$ is the union of the 2–fold composing space X (ie [disk with 2–holes] $\times S^1$) and a knot space E which is homeomorphic to $S^3 - \mathcal{N}(k)$. Note that a regular fiber t of X which lies in $\partial N(c)$ intersects a meridian μ_c exactly once, and a regular fiber t of X which lies in $\partial N(k)$ coincides with a meridian μ_k. The former condition implies $X \cup N(c) \cong S^1 \times S^1 \times [0,1]$. After $(0,0)$–surgery on $k \cup c$, we obtain the dual link $C \cup K$ in this surgered S^3. Observe that the regular fiber t which lies in $\partial N(K)$ coincides with a meridian μ_K, and the regular fiber t of X which lies in $\partial N(C)$ intersects a meridian μ_C exactly once. The latter condition implies that $X \cup N(C) \cong S^1 \times S^1 \times [0,1]$. Hence

$$S^3 - \mathcal{N}(K) = (k \cup c)(0,0) - \mathcal{N}(K) = E \cup (X \cup N(C)) \cong E \cong S^3 - \mathcal{N}(k).$$

Thus Gordon and Luecke [11, Theorem 1] show K is isotopic to k. □

In the proof of Theorem 2.1, we observe that $(k \cup c)\big(0, -\frac{1}{n}\big) \cong (C \cup K)\big(\frac{1}{0}, n\big)$, $(k \cup c)\big(0, -\frac{1}{n}\big) \cong k_n(n)$ and $(C \cup K)\big(\frac{1}{0}, n\big) \cong K(n)$. Starting with m–surgery instead of 0–surgery on k, the argument in the proof of Theorem 2.1 leads us to the following extension. Recall that the surgery dual C to k is unknotted in S^3 by Lemma 2.3. In what follows, K_m denotes the knot obtained from K by twisting m times along the unknot C.

Theorem 2.5 *Let $k \cup c$ be a link as in Theorem 2.1 with surgery dual link $C \cup K$, where K is dual to c and C is dual to k. Then for any integers m and n,*

$$K_m(n+m) \cong k_n(m+n).$$

Moreover, if c is not a meridian of k, then each family $\{K_m\}$ and $\{k_n\}$ contains infinitely many distinct knots, each of which has only finitely many integral characterizing slopes.

Proof Observe that $S^3 - \mathcal{N}(k \cup c) = S^3 - \mathcal{N}(C \cup K)$ and the meridian-longitude pairs (μ_k, λ_k) for k and (μ_c, λ_c) for c become meridian-longitude pairs $(\lambda_k, -\mu_k)$ for C and $(\lambda_c, -\mu_c)$ for K. The latter correspondence was shown in the proof of Theorem 2.1. For the former correspondence, by definition, λ_k becomes a meridian of C, the surgery dual to k. Observe also that μ_k is homologous to λ_c (because $|\mathrm{lk}(k,c)| = 1$), which bounds a disk of the filled solid torus after 0–surgery on c. Thus μ_k is a preferred longitude of C. Now the orientation convention gives the desired result. We note here that the above observation shows that $(0,0)$–surgery on $C \cup K$ yields S^3 with surgery dual $k \cup c$. In particular, $|\mathrm{lk}(K,C)| = 1$.

Then we have the surgery relation

$$K_m(n+m) \cong (C \cup K)\left(-\tfrac{1}{m}, n\right) \cong (k \cup c)\left(m, -\tfrac{1}{n}\right) \cong k_n(m+n)$$

as claimed.

Following Lemma 2.4, if C is a meridian of K, then c is a meridian of k. Thus if c is not a meridian of k, then C is not a meridian of K either. Since $|\mathrm{lk}(k,c)| = 1$ and $|\mathrm{lk}(K,C)| = 1$, the wrapping numbers of k about c and K about C are at least 2. Then [18, Theorem 3.2] implies that each twist family of knots $\{k_n\}$ and $\{K_m\}$ partitions into infinitely many distinct knot types containing finitely many members. Therefore, since $K_m(n+m) \cong k_n(m+n)$, each knot in these two families has only finitely many characterizing slopes. □

Proof of Theorem 1.3 By the assumption, the link $K \cup c$ satisfies the condition in Theorem 2.5, where K should be read as k; ie notationally K and k are exchanged. Then we have $k_m(m) \cong K(m)$ by putting $n = 0$. Since $\{k_m\}$ contains infinitely many distinct knots, K has infinitely many noncharacterizing slopes. □

3 Alexander polynomials of knots in twist families

We take $\Delta_{A\cup B}(x,y)$ to be the *symmetrized* multivariable Alexander polynomial of the *oriented* two-component link $A\cup B$, where x corresponds to the oriented meridian μ_A of A and y corresponds to the oriented meridian μ_B of B. Due to the symmetrization,

$$\Delta_{A\cup B}(x,y)=\Delta_{A\cup B}(x^{-1},y^{-1})=\Delta_{-A\cup -B}(x,y).$$

However, in general, $\Delta_{A\cup B}(x,y)\neq\Delta_{A\cup -B}(x,y)$.

Recall that if $k\cup c$ is a link in S^3 such that c is unknotted and $(0,0)$–surgery on $k\cup c$ yields S^3 with surgery dual link $C\cup K$, then C is also unknotted and $|\mathrm{lk}(K,C)|=1$.

Proposition 3.1 *Assume $k\cup c$ is an oriented two-component link with $\mathrm{lk}(k,c)=1$ such that c is an unknot. Further assume $(0,0)$–surgery on $k\cup c$ results in S^3 with surgery dual $C\cup K$, where K is dual to c and C is dual to k, oriented so that $\mathrm{lk}(K,C)=1$. Then $\Delta_{K\cup C}(x,y)=\Delta_{k\cup c}(x,y^{-1})$, or equivalently, $\Delta_{k\cup c}(x,y)=\Delta_{K\cup C}(x,y^{-1})$.*

Proof Let us write μ_J and λ_J for the meridian and preferred longitude of an oriented knot J in S^3 which we view as oriented curves in $\partial\mathcal{N}(J)$ such that $\mathrm{lk}(J,\mu_J)=1$ and λ_J is homologous to J. Let $X=S^3-\mathcal{N}(k\cup c)$ be the exterior of the link $k\cup c$. Since the linking number of $k\cup c$ is 1, in $H_1(X;\mathbb{Z})$, we have that $[\mu_k]=[\lambda_c]$ and $[\mu_c]=[\lambda_k]$. Furthermore, these homologies are realized by oriented Seifert surfaces Σ_c and Σ_k that are each punctured once by k and c, respectively. In particular, restricting to X, we have that $\partial\Sigma_c=\lambda_c-\mu_k$ and $\partial\Sigma_k=\lambda_k-\mu_c$.

Since K is the surgery dual to c with respect to 0–surgery on c, and C is the surgery dual to k with respect to 0–surgery on k, we have $X=S^3-\mathcal{N}(K\cup C)$. Upon surgery, the punctured Seifert surfaces Σ_k and Σ_c respectively cap off to oriented Seifert surfaces Σ_K and Σ_C for K and C. Using these surfaces to orient K and C and thus their meridians and longitudes, we obtain that $(\mu_K,\lambda_K)=(\lambda_c,-\mu_c)$ and $(\mu_C,\lambda_C)=(\lambda_k,-\mu_k)$. Therefore, $[\mu_K]=[\mu_k]$ and $[\mu_C]=[\mu_c]$ in $H_1(X;\mathbb{Z})$. However, since $[\lambda_C]=-[\mu_k]=-[\mu_K]$, we find that $\mathrm{lk}(K,C)=-1$. To orient K and C so that $\mathrm{lk}(K,C)=1$, we must flip the orientation on C, say. Then for this correctly oriented C, we have $[\mu_C]=-[\mu_c]$. Hence $\Delta_{K\cup C}(x,y)=\Delta_{k\cup c}(x,y^{-1})$. $\square$

We recall also the following twisting formula for Alexander polynomials.

Proposition 3.2 [1, Theorem 2.1] *Let $k \cup c$ be an oriented two-component link such that c is an unknot and $\omega = \mathrm{lk}(k, c) > 0$. Denote by k_n a knot obtained from k by an n–twist along c. Then $\Delta_{k_n}(t) = \Delta_{k \cup c}(t, t^{n\omega})$.*

Propositions 3.1 and 3.2 lead us some symmetry among Alexander polynomials of k_n and K_n.

Corollary 3.3 *Let $k \cup c$ be a link as in Theorem 2.1 with surgery dual link $C \cup K$, where K is dual to c and C is dual to k. Then for the twist families of knots $\{k_n\}$ and $\{K_n\}$, we have $\Delta_{k_n}(t) = \Delta_{K_{-n}}(t)$. In particular, $\Delta_k(t) = \Delta_K(t)$.*

Proof We may orient k and c so that $\mathrm{lk}(k, c) = 1$. Then Propositions 3.1 and 3.2 show that $\Delta_{k_n}(t) = \Delta_{k \cup c}(t, t^n) = \Delta_{K \cup C}(t, t^{-n}) = \Delta_{K_{-n}}(t)$. In particular, putting $n = 0$, we have $\Delta_k(t) = \Delta_K(t)$. □

4 Examples

In this section, we will provide examples which satisfy the condition in Theorem 2.1, and hence Theorem 2.5. Example 1.4 follows from Examples 4.1 and 4.3. A slight modification gives a nonhyperbolic example, Example 4.5, that demonstrates Theorem 1.6. We will make a further modification of the first example to present Example 4.6 which implies Theorem 1.5.

Let us take a two-component link $k \cup c$ in S^3 with $|\mathrm{lk}(k, c)| = 1$ as in Figure 2. To perform 0–surgery on the unknot c, we first remove $N(c)$ and glue it back to $V = S^3 - \mathcal{N}(c)$ so that a meridian of $N(c)$ is identified with a meridian of V, a preferred longitude of c. Then the union of meridian disks of $N(c)$ and V forms a nonseparating 2–sphere S in $c(0) = S^1 \times S^2$. The second picture from the left of Figure 2 describes $c(0) = S^1 \times S^2$ in which the bottom 2–sphere and the top 2–sphere are identified (without twisting) to result in the nonseparating 2–sphere S. From the second to the seventh picture, since the total space is $S^1 \times S^2$ rather than S^3, we do not put extra labels to corresponding components. We apply light bulb moves from the third to the fourth and from the fifth to the sixth picture of Figure 2. In the second picture from the right of Figure 2, the straight knot is the image of k in $c(0) = S^1 \times S^2$, and 0–surgery on this knot gives S^3. This S^3 resulting from $(0, 0)$–surgery on $k \cup c$ is shown with the surgery dual link $C \cup K \subset S^3$ in the rightmost picture of Figure 2. Thus $k \cup c$ satisfies the condition in Theorem 2.1, and $K(n) \cong k_n(n)$ does hold for all integers n.

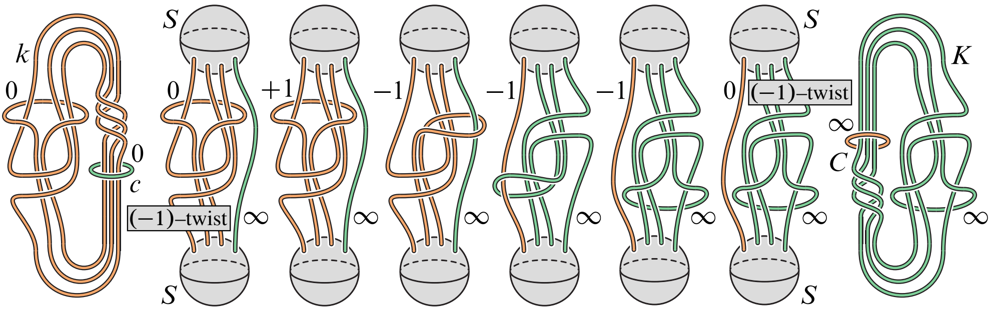

Figure 2: (0, 0)–surgery on $k \cup c$ results in S^3 with its surgery dual $C \cup K$.

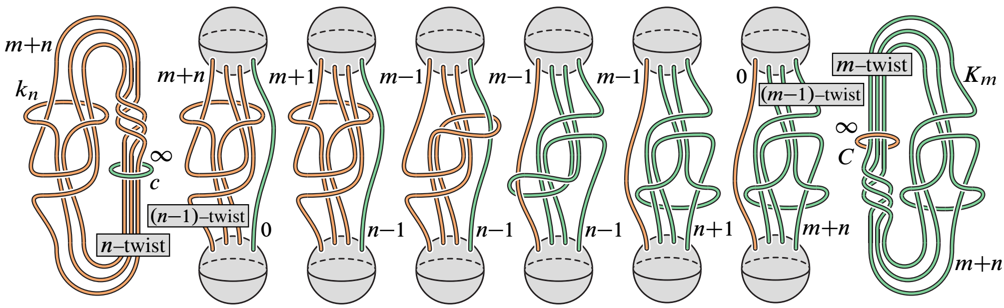

Figure 3: $(m{+}n)$–surgery on the knot k_n is equivalent to $(m+n)$–surgery on K_m.

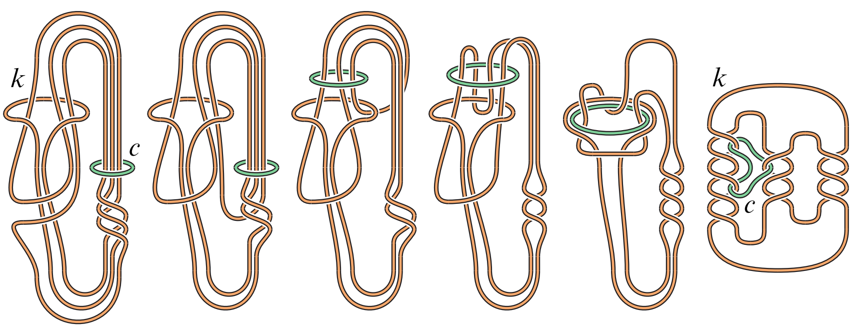

Figure 4: The knot $k = k_0$ is isotoped into a presentation as the pretzel knot $P(-5, 3, -3)$. The twisting circle c is carried along with the isotopy.

Furthermore, orienting $k \cup c$ so that $\mathrm{lk}(k, c) = 1$, one may calculate[1] the multivariable Alexander polynomial of $k \cup c$ to be

$$\Delta_{k \cup c}(x, y) = -(x^{-1} - 2 + x)y^{-1} + 1 - (x^{-1} - 2 + x)y.$$

[1]For a computer-assisted calculation, one may first use PLink within SnapPy [6] to obtain a Dowker–Thistlethwaite code (DT code) for the link. Then the KnotTheory package [3] for Mathematica can produce the multivariable Alexander polynomial from the DT code.

Hence by Proposition 3.2, we have

$$(\star)\qquad \Delta_{k_n}(t)=\Delta_{k\cup c}(t,t^n)=-(t^{-1}-2+t)t^{-n}+1-(t^{-1}-2+t)t^n.$$

In particular, since the Alexander polynomial of k_n varies depending on n, we have that c is not a meridian of k.

Let us generalize this following Theorem 2.5. Let K_m be a knot obtained from K by an m–twist along C. Then Theorem 2.5 asserts that $K_m(n+m)\cong k_n(m+n)$ for any integers m and n. Figure 3 demonstrates this fact pictorially.

Let us choose an integer m arbitrarily. Observe that in this example, we have $K_m=k_m$; see Figure 3. Hence if $k_n=K_m$ for some integer n, then $k_n=k_m$. Thus $\Delta(k_n)\doteq\Delta(k_m)$, and $(\star)$ implies that $n=\pm m$. Thus at most k_m and k_{-m} can be isotopic to K_m. Since $K_m(n+m)\cong k_n(m+n)$ for all integers m and n, we have the following:

- For a given integer m, every integral slope except possibly 0 and $2m$ fails to be a characterizing slope for K_m.
- If furthermore $K_{-m}\neq K_m$, then 0 will fail to be a characterizing slope as well.

Example 4.1 (genus-one, nonfibered knot) Let us choose $m=0$ in the above. Then $K_0(n)=k_n(n)$ for all integers n, and as mentioned above, every nonzero integral slope fails to be a characterizing slope for K_0. In Figure 4, we identify $K_0=k_0$ as the pretzel knot $P(-5,3,-3)$, which is known to be hyperbolic by [27]. The pretzel knot $P(-5,3,-3)$ is a genus-one knot, but it is not fibered. Seifert's algorithm easily produces a genus-one Seifert surface of $P(-5,3,-3)$.

Remark 4.2 Notably, the (mirror of the) knot $P(-5,3,-3)$ was the basic example of the first two families of nonstrongly invertible knots with a small Seifert fibered space surgery [22]. Indeed, (-1)–surgery on $P(-5,3,-3)$ is the Seifert fibered space $S^2\left(-\frac{2}{5},\frac{3}{4},-\frac{1}{3}\right)$.

Since $P(-5,3,-3)$ is the knot K_0, and $K_0(n)=k_n(n)$ for all integers n, we have $K_0(-1)=k_{-1}(-1)=K_{-1}(-1)$. Thus (-1)–surgery on K_{-1} is the same Seifert fibered space. SnapPy recognizes the complement of K_{-1} as the mirror of the census manifold $o9_{34801}$. Furthermore, SnapPy reports this manifold as asymmetric, implying that K_{-1} is neither strongly invertible nor cyclically periodic, and hence cannot be embedded in a genus-2 Heegaard surface; see [7, Lemma 7.4].

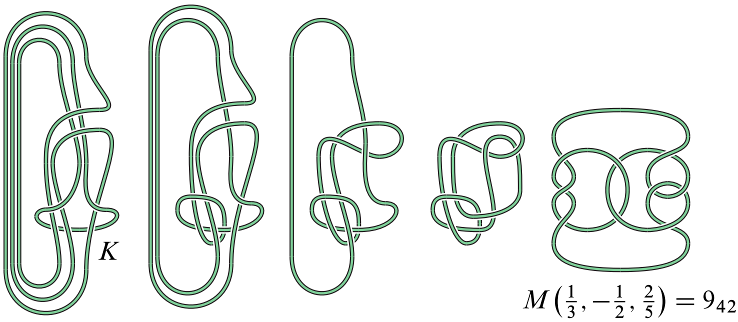

Figure 5: The knot K_1 in Figure 3 is isotoped into a presentation as the 9 crossing Montesinos knot $M\left(\frac{1}{3}, -\frac{1}{2}, \frac{2}{5}\right)$ which may be recognized as the knot 9_{42} in Rolfsen's table [33].

Example 4.3 (fibered, genus-two knot) By choosing $m = 1$ instead of 0, we obtain a knot K_1 for which we have $K_1(n+1) = k_n(1+n)$ for all integers n. As we mentioned, every integral slope other than 0 or 2 are noncharacterizing slope for K_1. In Figure 5, we recognize the knot K_1 as the 9–crossing Montesinos knot $M\left(\frac{1}{3}, -\frac{1}{2}, \frac{2}{5}\right)$ which is the knot 9_{42} in Rolfsen's table [33]. Following [27], K_1 is a hyperbolic knot. The knot 9_{42} is a fibered knot, but it has genus two [13, Theorem 3.2].

Now let us show that 0–slope is also a noncharacterizing slope for K_1. Since $K_1(0) \cong k_{-1}(0)$, it is sufficient to see that $K_1 \neq k_{-1}$. Recall that $K_m = k_m$ for any m. Alexander polynomials distinguish k_1 from k_n for all $n \neq \pm 1$; see ($\star$). The Jones polynomial[2] will however distinguish $k_1 = K_1$ and k_{-1}:

$$V_{k_1}(q) = q^{-3} - q^{-2} + q^{-1} - 1 + q - q^2 + q^3,$$

while

$$V_{k_{-1}}(q) = q^{-1} + q^{-3} - q^{-6} - q^{-8} + q^{-9} - q^{-10} + q^{-11}.$$

(As noted in Remark 4.2, SnapPy also identifies the complement of $K_{-1} = k_{-1}$ as distinct from the complement of $K_1 = 9_{42}$, thereby distinguishing these knots.) Hence all integers except possibly 2 are noncharacterizing slopes for the hyperbolic knot $K_1 = 9_{42}$.

Question 4.4 *Is 0 a characterizing slope for $P(-3,3,5)$? Is 2 a characterizing slope for 9_{42}?*

Next we provide examples of nonhyperbolic knots such that all integral slopes are noncharacterizing slopes, from which Theorem 1.6 follows.

[2] Kodama's software KNOT [17] was used confirm the Jones polynomials of knots.

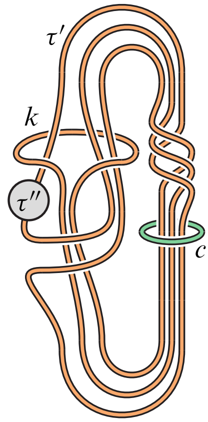

Figure 6: The sum of 1–string tangles τ' and τ'' is the connected sum $k = k' \# k''$.

Example 4.5 (nonhyperbolic example) Given any nontrivial knot k'', let us take a two-component link $k \cup c$ as in Figure 6, where k is a connected sum of a knot k' (which is k in Figure 2, the closure of the 1–string tangle τ') and the nontrivial knot k'' (the closure of the 1–string tangle τ'').

Then as in Figure 2, we see that $(0,0)$–surgery on $k \cup c$ gives S^3 with the surgery dual $C \cup K$. Actually, we follow the isotopy and light bulb moves as indicated in Figure 2 to obtain the sixth figure, in which k is almost an S^1 fiber, but it has the connected summand k'' (ie the knotted arc τ''). Then we apply further light bulb moves to k so that it becomes an S^1 fiber; K becomes a satellite knot with k'' as a companion knot; see Lemma 2.4. Then by Theorem 2.5, $K_m(n+m) \cong k_n(m+n)$ for all integers m and n.

It is easy to observe that k_n is a connected sum $k'_n \# k''$, where k'_n is a knot obtained from k' by an n–twist along c. For instance, $k_0 = P(-5,3,-3) \# k''$ and $k_1 = 9_{42} \# k''$. Since k'_n is nontrivial for all integers n by $(\star)$, k_n is not prime for all integers n.

On the other hand, we show that K_m is prime for all integers m. (We note that, by construction, K_m has k'' as a companion knot for every integer m.) In the following, we fix an integer m arbitrarily. First we observe that $k_n(m+n)$ is obtained by gluing $E(k'_n)$ and $E(k'')$ along their boundary tori. Recall that the exterior $E(k_n)$ may be expressed as the union of the 2–fold composing space X (ie [disk with 2–holes]$\times S^1$) and two knot spaces $E(k'_n)$ and $E(k'')$. We note that ∂X consists of $\partial E(k_n)$, $\partial E(k'_n)$ and $\partial E(k'')$, and a regular fiber in $\partial X \cap \partial E(k_n)$ is a meridian of k_n. Since the surgery slope $m+n$ is integral, the corresponding Dehn filling of X results in $S^1 \times S^1 \times [0,1]$, and $k_n(m+n)$ can be viewed as the union of $E(k'_n)$ and $E(k'')$.

Hence $K_m(n+m) \cong k_n(m+n) = E(k'_n) \cup E(k'')$ for all integers n. It should be noted here that $E(k'')$ is independent of n, but the topological type of $E(k'_n)$ depends on n.

Now assume for a contradiction that K_m is not prime and express $K_m = t_1 \# \cdots \# t_p$, where t_i is a prime knot for $1 \le i \le p$. Then $E(K_m)$ is the union of the p–fold composing space $Y =$ [disk with p–holes] $\times S^1$ and p knot spaces $E(t_1), \ldots, E(t_p)$, where a regular fiber in $\partial Y \cap \partial E(K_m)$ is a meridian of K_m. Since the surgery slope $n+m$ is integral, the corresponding Dehn filling of Y results in $(p-1)$–fold composing space $Y' =$ [disk with $(p-1)$–holes] $\times S^1$. Hence $K_m(n+m)$ is expressed as the union $Y' \cup E(t_1) \cup \cdots \cup E(t_p)$. If necessary, decomposing each $E(t_i)$ further by essential tori, we obtain a torus decomposition of $K_m(n+m)$ in the sense of Jaco, Shalen and Johannson [14; 15]. Note that identifications of Y' and $E(t_i)$ $(1 \le i \le p)$ depends on n, but the topological type of $E(t_i)$ $(1 \le i \le p)$ does not depend on n. To be precise, let us focus on the case of $n = 0, 1$. Then $K_m(n+m) \cong k_n(m+n) = E(k'_n) \cup E(k'')$, and $E(k'_n)$ admits a hyperbolic structure in its interior: $E(k'_0)$ is the exterior of the hyperbolic knot $P(-5, 3, -3)$ and $E(k'_1)$ is the exterior of the hyperbolic knot 9_{42}. If $E(k'')$ is neither hyperbolic nor Seifert fibered, we decompose $E(k'')$ by essential tori to obtain a torus decomposition of $K_m(n+m) \cong k_n(m+n)$ in the sense of Jaco, Shalen and Johannson. Since $E(k'_0) \not\cong E(k'_1)$, uniqueness of the torus decomposition of $K_m(n+m)$ shows that some $E(t_i)$ changes according as $n = 0, 1$. This is a contradiction. It follows that K_m is a prime knot.

Since the knot K_m is prime, while k_n is not prime for all integers m and n, we have $\{K_m\} \cap \{k_n\} = \varnothing$. Thus every integral slope fails to be a characterizing slope for a prime satellite knot K_m (with a given knot k'' a companion knot) for any integer m, establishing Theorem 1.6(1). Similarly, every integral slope fails to be a characterizing slope for a composite knot k_n (with a given knot k'' a connected summand) for any integer n. This establishes Theorem 1.6(2).

Example 4.6 (Proof of Theorem 1.5) Figure 7 shows a sequence of transformations relating $(m+n, \infty)$–surgery on a link $k_n \cup c$ to $(\infty, n+m)$–surgery on a link $C \cup K_m$. In particular, it gives two twist families of knots $\{k_n\}$ and $\{K_m\}$ such that $k_n(m+n) = K_m(n+m)$.

Replace $(m+n, \infty)$–surgery on $k_n \cup c$ by $(0, 0)$–surgery on $k_0 \cup c$, and follow isotopies and light bulb moves as indicated in Figure 7 to see that $(0, 0)$–surgery on $k_0 \cup c$ yields S^3 with surgery dual $C \cup K_0$, where K_0 is dual to c and C is dual to k_0.

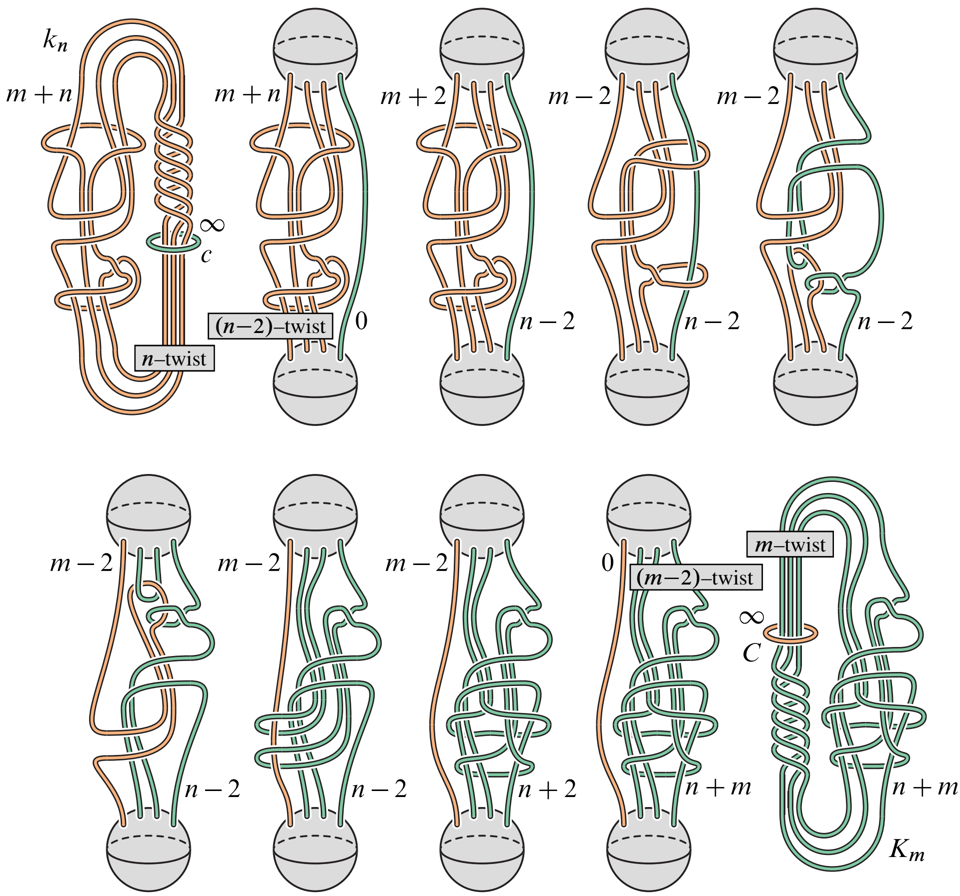

Figure 7: Two families of knots $\{k_n\}$ and $\{K_m\}$ such that $k_n(m+n) = K_m(n+m)$

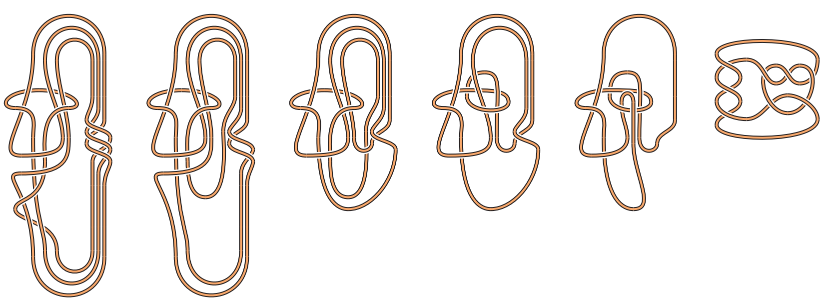

Figure 8: The knot k_1 of the family depicted in Figure 7 is isotoped into the 8–crossing Montesinos knot $M\left(3, \frac{1}{3}, \frac{1}{2}\right)$ which is the two-bridge knot $\frac{23}{10}$ and also the knot 8_6 in Rolfsen's table.

As Figure 8 demonstrates, the knot k_1 is the hyperbolic 8–crossing Montesinos knot $M\left(3, \frac{1}{3}, \frac{1}{2}\right)$. It is the knot 8_6 in Rolfsen's table, the two-bridge knot $\frac{23}{10}$. Following [27], (compare [25; 12]) it is a hyperbolic knot.

Using $n = 0$, we may calculate that

$$(\star\star)\quad \Delta_{k_0 \cup c}(x, y) = (x^{-1} - 2 + x)y^{-1} + (x^{-2} - 4x^{-1} + 5 - 4x + x^2) + (x^{-1} - 2 + x)y,$$

which is equal to $\Delta_{K_0 \cup C}(x, y^{-1})$ by Proposition 3.1. Note also that $\Delta_{k_0 \cup c}(x, y) = \Delta_{k_0 \cup c}(x, y^{-1})$; see $(\star\star)$. Hence $\Delta_{k_0 \cup c}(t, t^n) = \Delta_{k_0 \cup c}(t, t^{-n}) = \Delta_{K_0 \cup C}(t, t^n)$, and it follows from Proposition 3.2 that

$$\Delta_{k_n}(t) = \Delta_{K_n}(t) = (t^{-1} - 2 + t)t^{-n} + (t^{-2} - 4t^{-1} + 5 - 4t + t^2) + (t^{-1} - 2 + t)t^n.$$

Thus Alexander polynomials distinguish k_1 from K_m for all integers $m \neq \pm 1$.

We further calculate the Jones polynomials of k_1, K_1, and K_{-1} to be

$$V_{k_1}(q) = \frac{1}{q^7} - \frac{2}{q^6} + \frac{3}{q^5} - \frac{4}{q^4} + \frac{4}{q^3} - \frac{4}{q^2} + \frac{3}{q} - 1 + q,$$

$$V_{K_1}(q) = -\frac{1}{q^{15}} + \frac{1}{q^{14}} + \frac{1}{q^{11}} - \frac{1}{q^8} + \frac{1}{q^7} - \frac{3}{q^6} + \frac{3}{q^5} - \frac{4}{q^4} + \frac{5}{q^3} - \frac{4}{q^2} + \frac{3}{q},$$

$$\begin{aligned} V_{K_{-1}}(q) = -\frac{1}{q^{21}} + \frac{1}{q^{20}} + \frac{1}{q^{17}} - \frac{1}{q^{14}} + \frac{1}{q^{13}} - \frac{2}{q^{12}} + \frac{1}{q^{11}} - \frac{1}{q^{10}} \\ + \frac{1}{q^9} - \frac{1}{q^8} + \frac{1}{q^7} - \frac{1}{q^6} + \frac{2}{q^5} - \frac{3}{q^4} + \frac{4}{q^3} - \frac{3}{q^2} + \frac{2}{q}, \end{aligned}$$

to conclude that $k_1 \neq K_{\pm 1}$. Thus k_1 is an 8–crossing hyperbolic knot for which every integral slope is not a characterizing slope.

5 Further discussions

Let $K \cup c$ be a two-component link such that c is unknotted and not a meridian of K. If $(0, 0)$–surgery on $K \cup c$ yields S^3, then Theorem 1.3 assures that K has infinitely many noncharacterizing slopes. Moreover, by Theorem 2.5, each knot K_n (obtained from K by an n–twist along c) has also infinitely many noncharacterizing slopes.

Proposition 5.1 *The $(0, 0)$–surgery on $K_n \cup c$ results in S^3.*

Proof Note that $(K_n \cup c)(0, 0) \cong (K \cup c)(-n, 0)$. Since $(K \cup c)(0, 0) \cong S^3$, viewing $K \subset c(0) = S^1 \times S^2$, it is isotopic to an S^1–fiber in $c(0)$; see the proof of Lemma 2.3 and Figure 2 for an illustration. Hence, we see that $(K \cup c)(-n, 0)$ is also S^3. This then implies that $(0, 0)$–surgery on $K_n \cup c$ results in S^3 as well. □

Toward characterization of knots with infinitely many noncharacterizing slopes, we would like to ask:

Question 5.2 *Assume that K is a knot with infinitely many noncharacterizing slopes. Then can we take an unknot c so that c is not a meridian of K and $(0,0)$–surgery on $K \cup c$ yields S^3 ?*

Question 5.3 *For which knot K can we take an unknot c so that the link $K \cup c$ enjoys the following properties:*

- *c is not a meridian of K, and*
- *the result of the $(0,0)$–surgery on $K \cup c$ is S^3 ?*

Following Lemma 2.4, if c is a meridian of K, then $(0,0)$–surgery on $K \cup c$ always results in S^3 independent of the knot K. However, if c is not a meridian of K, the second condition in Question 5.3 imposes a strong restriction on K. It follows from Ni and Zhang [26] and McCoy [23] that if K is a torus knot $T_{r,s}$ with $r > s > 1$ (resp. $-r > s > 1$), then sufficiently positive (resp. negative) slopes are characterizing slopes. Hence any nontrivial torus knot does not admit an unknot c described in Question 5.3.

A knot K is an *L-space knot* if for some nonzero slope $p/q \in \mathbb{Q}$ the manifold $K(p/q)$ is an *L-space*, a rational homology 3–sphere for which $\operatorname{rk} \widehat{\mathrm{HF}}(K(p/q)) = |H_1(K(p/q))|$; see [29]. If $p/q > 0$, then K is called a *positive* L-space knot, and if $p/q < 0$, then it is called a *negative* L-space knot. Motivated by the fact that torus knots are fundamental examples of L-space knots, we can prove:

Theorem 1.8 *Let $K \cup c$ be a link with c a trivial knot. If c is not a meridian of K and the result of the $(0,0)$–surgery on $K \cup c$ is S^3, then K is not an L-space knot.*

Proof Since $(0,0)$–surgery on $K \cup c$ results in S^3, the linking number between K and c must be ± 1. Now suppose for a contradiction that $K = K_0$ is an L-space knot. Then $K_0(m)$ is an L-space for infinitely many integers m [29, Proposition 2.1]; more precisely, if K_0 is a positive (resp. negative) L-space knot, then $K_0(m)$ is an L-space for $m \geq 2g(K_0) - 1$ (resp. $m \leq -2g(K_0) + 1$); see [31]. By Theorem 2.5, $K_0(m) = k_m(m)$ for all integers m; hence the twist family $\{(k_m, m)\}$ contains infinitely many L-space surgeries. Recall that the linking number between k and C is also ± 1; see the proof of Theorem 2.5. Furthermore, it follows from [1, Proposition 1.10] that the k_m have the same Alexander polynomial for all $m \in \mathbb{Z}$, and $g(k_m)$ is constant for infinitely many integers m. On the other hand, since $|\mathrm{lk}(K, C)| = 1$, a recent work of Baker and Taylor [2] shows that $g(k_m) \to \infty$ when $|m| \to \infty$, a contradiction. $\square$

So we may expect a positive answer to the following:

Question 5.4 *Does an L-space knot have only finitely many noncharacterizing slopes?*

Although the construction given by Theorem 2.1 and Theorem 2.5 provides infinitely many knots with infinitely many noncharacterizing slopes, we still expect that these knots have characterizing slopes as well.

Question 5.5 *Does every knot K have a characterizing slope? More strongly, does every knot have infinitely many characterizing slopes?*

Our technique does not work directly for nonintegral slopes. So we would like to propose a modified version of Ni and Zhang's question:

Question 5.6 *For a hyperbolic knot K, is a* ***nonintegral*** *slope p/q with $|p| + |q|$ sufficiently large a characterizing slope?*

Remark 5.7 Lackenby [20] shows that for each atoroidal, homotopically trivial knot K in a 3–manifold Y with $H_1(Y; \mathbb{Q}) \neq \{0\}$, there exists a number $C(Y, K)$ such that p/q is a characterizing slope for K if $|q| > C(Y, K)$.

Ni and Zhang ask if every rational number is a noncharacterizing slope for some knot [26, Question 1.5]. We ask the opposite:

Question 5.8 *Is there a rational number r which is a characterizing slope for all knots?*

More strongly and specifically, we would like to ask:

Question 5.9 *Let r be a rational number which cannot be written in the form $m + \frac{1}{n}$ for any integers m and n. Then is r a characterizing slope for all knots?*

It should be noted here that Kawauchi [16] demonstrates that if r is written as $m + \frac{1}{n}$ for some nonzero integers m and n, then it is not a characterizing slope for some hyperbolic knot. More precisely, he demonstrates that for any integer $N > 1$ and any such an r, there are hyperbolic knots $K_1, \ldots, K_N$ whose r–surgery result in the same oriented 3–manifold.

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Received: 21 November 2016 Revised: 15 May 2017