

The universal sl_3 –link homology

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We define the universal sl_3 –link homology, which depends on 3 parameters, following Khovanov’s approach with foams. We show that this 3–parameter link homology, when taken with complex coefficients, can be divided into 3 isomorphism classes. The first class is the one to which Khovanov’s original sl_3 –link homology belongs, the second is the one studied by Gornik in the context of matrix factorizations and the last one is new. Following an approach similar to Gornik’s we show that this new link homology can be described in terms of Khovanov’s original sl_2 –link homology.

[57M27](#); [57M25](#), [81R50](#), [18G60](#)

1 Introduction

In [8], following his own seminal work in [6] and Lee [11], Bar-Natan [2] and Turner’s [13] subsequent contributions, Khovanov classified all possible Frobenius systems of dimension two which give rise to link homologies via his construction in [6] and showed that there is a universal one, given by

$$\mathbb{Z}[X, a, b]/(X^2 - aX - b).$$

Working over \mathbb{C} , one can take a and b to be complex numbers and study the corresponding homology with coefficients in \mathbb{C} . We refer to the latter as the sl_2 –link homologies over \mathbb{C} , because they are all deformations of Khovanov’s original link homology whose Euler characteristic equals the Jones polynomial which is well known to be related to the Lie algebra sl_2 [8]. Using the ideas in [8; 11; 13], the authors and Turner showed in [12] that there are only two isomorphism classes of sl_2 –link homologies over \mathbb{C} . Given $a, b \in \mathbb{C}$, the isomorphism class of the corresponding link homology is completely determined by the number of distinct roots of the polynomial $X^2 - aX - b$. The original Khovanov sl_2 –link homology $\text{KH}(L, \mathbb{C})$ corresponds to the choice $a = b = 0$.

Bar-Natan [2] obtained the universal sl_2 –link homology in a different way, using a clever setup with cobordisms modulo relations. He shows how Khovanov’s original construction of the sl_2 –link homology [6] can be used to define a universal functor

which corresponds to $U_{0,0,1}^*(L, \mathbb{C})$. The case in which $f(X)$ has two distinct roots, one of which has multiplicity two, is new and had not been studied before to our knowledge, although Dunfield, Gukov and Rasmussen [3] and Gukov and Walcher [5] make conjectures which are compatible with our results. We prove that there is a degree-preserving isomorphism

$$U_{a,b,c}^*(L, \mathbb{C}) \cong \bigoplus_{L' \subseteq L} \text{KH}^{*-j(L')}(L', \mathbb{C}),$$

where $j(L')$ is a shift of degree $2 \text{lk}(L', L \setminus L')$. This isomorphism does not take into account the internal grading of the Khovanov homology.

We have tried to make the paper reasonably self-contained, but we do assume familiarity with the papers by Bar-Natan [1; 2], Gornik [4] and Khovanov [6; 7; 8].

2 The universal sl_3 -link homology

Let L be an oriented link in S^3 and D a diagram of L . In [7] Khovanov constructed a homological link invariant associated to sl_3 . The construction starts by resolving each crossing of D in two different ways, as in Figure 1.

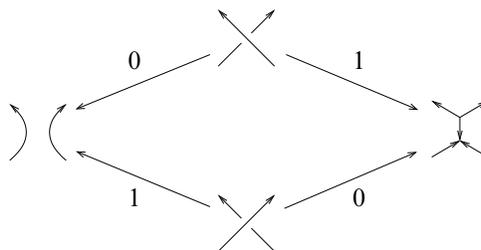


Figure 1: 0 and 1 resolutions of crossings

A diagram Γ obtained by resolving all crossings of D is an example of a *web*. A web is a trivalent planar graph where near each vertex all the edges are oriented “in” or “out” (see Figure 2). We also allow webs without vertices, which are oriented loops. Note that by definition our webs are closed; there are no vertices with fewer than 3 edges. Whenever it is necessary to keep track of crossings after their resolution we mark the corresponding edges as in Figure 3. A *foam* is a cobordism with singular arcs between two webs. A singular arc in a foam f is the set of points of f that have a neighborhood homeomorphic to the letter Y times an interval (see the examples in Figure 4). Interpreted as morphisms, we read foams from bottom to top by convention,



Figure 2: “In” and “out” orientations near a vertex

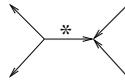


Figure 3: Marked edges corresponding to a crossing in D

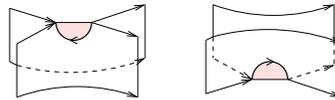


Figure 4: Basic foams from $\succ\zeta$ to \sphericalangle (left) and from \sphericalangle to $\succ\zeta$ (right)

and the orientation of the singular arcs is by convention as in Figure 4. Foams can have dots that can move freely on the facet to which they belong but are not allowed to cross singular arcs. Let $\mathbb{Z}[a, b, c]$ be the ring of polynomials in a, b, c with integer coefficients.

Definition 2.1 **Foam** is the category whose objects are (closed) webs and whose morphisms are $\mathbb{Z}[a, b, c]$ -linear combinations of isotopy classes of foams.

Foam is an additive category and, as we will show, each foam can be given a degree in such a way that **Foam** becomes a graded additive category, taking also a, b and c to have degrees 2, 4 and 6 respectively. For further details about this category, see Khovanov [7].

From all different resolutions of all the crossings in D we form a commutative hypercube of resolutions as in [7]. It has a web in each vertex and to an edge between two vertices, given by webs that differ only inside a disk \mathcal{D} around one of the crossings of D , we associate the foam that is the identity everywhere except inside the cylinder $\mathcal{D} \times I$, where it looks like one of the basic foams in Figure 4. An appropriate distribution of minus signs among the edges of the hypercube results in a chain complex of web diagrams analogous to the one in [2] which we call $\langle D \rangle$, with “column vectors” of webs as “chain objects” and “matrices of foams” as “differentials”. We borrow some of the notation from [2] and denote by $\mathbf{Kom}(\mathbf{Foam})$ the category of complexes in **Foam**.

In Section 2.1–Section 2.3 we first impose a set of local relations on **Foam**. We call this set of relations ℓ and denote by \mathbf{Foam}/ℓ the category **Foam** divided by ℓ . We prove that these relations guarantee the invariance of $\langle D \rangle$ under the Reidemeister moves up to homotopy in $\mathbf{Kom}(\mathbf{Foam}/\ell)$ in a pictorial way, which is analogous to Bar-Natan’s proof in [2]. Note that the category $\mathbf{Kom}(\mathbf{Foam}/\ell)$ is analogous to Bar-Natan’s category $\mathit{Kob}(\emptyset) = \mathit{Kom}(\mathit{Mat}(\mathit{Cob}_1^3(\emptyset)))$. Next we show that up to signs $\langle \rangle$ is functorial under oriented link cobordisms, ie defines a functor from **Link** to $\mathbf{Kom}/_{\pm h}(\mathbf{Foam}/\ell)$. Here **Link** is the category of oriented links in S^3 and ambient isotopy classes of oriented link cobordisms properly embedded in $S^3 \times [0, 1]$ and $\mathbf{Kom}/_{\pm h}(\mathbf{Foam}/\ell)$ is the homotopy category of $\mathbf{Kom}(\mathbf{Foam}/\ell)$ modded out by ± 1 . For the functoriality we need all relations in ℓ , including the ones which involve a, b , and c . In Section 2.4 we define a functor between \mathbf{Foam}/ℓ and $\mathbb{Z}[a, b, c]\text{-Mod}$, the category of graded $\mathbb{Z}[a, b, c]$ -modules. This induces a homology functor

$$U_{a,b,c}^*: \mathbf{Link} \rightarrow \mathbb{Z}[a, b, c]\text{-Mod}_{\mathbf{bg}},$$

where $\mathbb{Z}[a, b, c]\text{-Mod}_{\mathbf{bg}}$ is the category of bigraded $\mathbb{Z}[a, b, c]$ -modules.

The principal ideas in this section, as well as most homotopies, are motivated by the ones in Khovanov’s paper [7] and Bar-Natan’s paper [2].

2.1 Universal local relations

In order to construct the universal theory we divide **Foam** by the local relations $\ell = (3D, CN, S, \Theta)$ below.

$$\begin{array}{|c|} \hline \dots \\ \hline \end{array} = a \begin{array}{|c|} \hline \dots \\ \hline \end{array} + b \begin{array}{|c|} \hline \cdot \\ \hline \end{array} + c \begin{array}{|c|} \hline \\ \hline \end{array} \tag{3D}$$

$$-\text{cylinder} = \begin{array}{c} \text{cup} \\ \text{cup} \\ \text{cup} \end{array} + \begin{array}{c} \text{cup} \\ \text{cup} \\ \text{cup} \end{array} - a \left(\begin{array}{c} \text{cup} \\ \text{cup} \\ \text{cup} \end{array} + \begin{array}{c} \text{cup} \\ \text{cup} \\ \text{cup} \end{array} \right) - b \begin{array}{c} \text{cup} \\ \text{cup} \end{array} \tag{CN}$$

$$\begin{array}{c} \text{sphere} \\ \text{sphere} \end{array} = \begin{array}{c} \text{sphere} \\ \text{sphere} \end{array} = 0, \quad \begin{array}{c} \text{sphere} \\ \text{sphere} \end{array} = -1 \tag{S}$$

Note that the foams in the (S) relations are spheres and not theta-foams which we discuss next.

Recall from [7] that theta-foams are obtained by gluing three oriented disks along their boundaries (their orientations must coincide), as shown in Figure 5. Note the

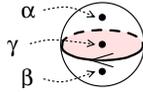


Figure 5: A theta foam

orientation of the singular circle. Let α, β, γ denote the number of dots on each facet. The (Θ) relation says that for α, β or $\gamma \leq 2$

$$\theta(\alpha, \beta, \gamma) = \begin{cases} 1 & (\alpha, \beta, \gamma) = (1, 2, 0) \text{ or a cyclic permutation} \\ -1 & (\alpha, \beta, \gamma) = (2, 1, 0) \text{ or a cyclic permutation} \\ 0 & \text{else} \end{cases} \quad (\Theta)$$

Reversing the orientation of the singular circle reverses the sign of $\theta(\alpha, \beta, \gamma)$. Note that when we have three or more dots on a facet of a foam we can use the (3D) relation to reduce to the case where it has less than three dots.

A closed foam f can be viewed as a morphism from the empty web to itself which by the relations (3D, CN, S, Θ) is an element of $\mathbb{Z}[a, b, c]$. It can be checked, as in [7], that this set of relations is consistent and determines uniquely the evaluation of every closed foam f , denoted $C(f)$. Define a q -grading on $\mathbb{Z}[a, b, c]$ by $q(1) = 0$, $q(a) = 2$, $q(b) = 4$ and $q(c) = 6$. As in [7] we define the q -grading of a foam f with d dots by

$$q(f) = -2\chi(f) + \chi(\partial f) + 2d,$$

where χ denotes the Euler characteristic.

Definition 2.2 \mathbf{Foam}/ℓ is the quotient of the category \mathbf{Foam} by the local relations ℓ . For webs Γ, Γ' and for families $f_i \in \mathbf{Hom}_{\mathbf{Foam}}(\Gamma, \Gamma')$ and $c_i \in \mathbb{Z}[a, b, c]$ we impose $\sum_i c_i f_i = 0$ if and only if $\sum_i c_i C(g' f_i g) = 0$ holds, for all $g \in \mathbf{Hom}_{\mathbf{Foam}}(\emptyset, \Gamma)$ and $g' \in \mathbf{Hom}_{\mathbf{Foam}}(\Gamma', \emptyset)$.

Lemma 2.3 We have the following relations in \mathbf{Foam}_ℓ :

(4C)

(RD)

(DR)

(SqR)

Proof Relations (4C) and (RD) are immediate and follow from (CN) and (Θ) . Relations (DR) and (SqR) are proved as in [7] (see also Lemma 2.9) \square

The following equality, and similar versions, which corresponds to an isotopy, we will often use in the sequel

(1)

where \circ denotes composition of foams.

In Figure 6 we also have a set of useful identities which establish the way we can exchange dots between faces. These identities can be used for the simplification of foams and are an immediate consequence of the relations in ℓ .

2.2 Invariance under the Reidemeister moves

In this subsection we prove invariance of $\langle \rangle$ under the Reidemeister moves. The main result of this section is the following:

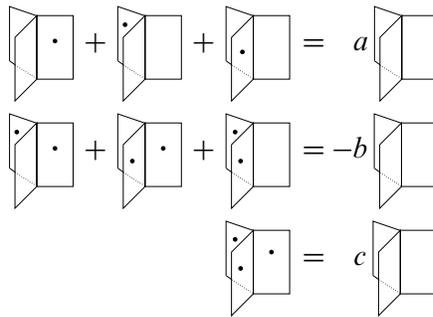


Figure 6: Exchanging dots between faces. The relations are the same regardless of which edges are marked and the orientation on the singular arcs.

Theorem 2.4 $\langle D \rangle$ is invariant under the Reidemeister moves up to homotopy, in other words it is an invariant in $\mathbf{Kom}_{/h}(\mathbf{Foam}_{/\ell})$.

Proof To prove invariance under the Reidemeister moves we work diagrammatically.

Reidemeister I Consider diagrams D and D' that differ only in a circular region as in the figure below.



We give the homotopy between complexes $\langle D \rangle$ and $\langle D' \rangle$ in Figure 7. By relation (S)

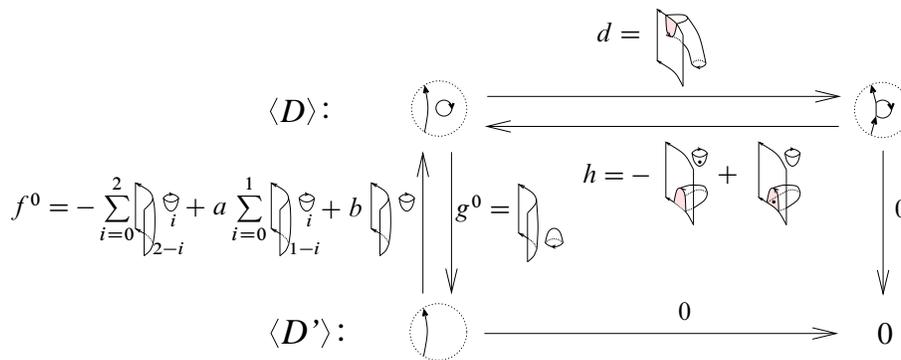


Figure 7: Invariance under Reidemeister I

we have $g^0 f^0 = \text{Id}(\cdot)$. To see that $df^0 = 0$ holds, one can use the dot exchange relations in Figure 6. The equality $dh = \text{id}(\cdot)$ follows from (DR) (note the orientations on the singular circles). To show that $f^0 g^0 + hd = \text{Id}_{\langle D \rangle 0}$, apply (RD) to hd and

then cancel all terms which appear twice with opposite signs. What is left is the sum of 6 terms which is equal to $\text{Id}_{\langle D \rangle^0}$ by (CN). Therefore $\langle D' \rangle$ is homotopy-equivalent to $\langle D \rangle$.

Reidemeister IIa Consider diagrams D and D' that differ in a circular region, as in the figure below.



We leave to the reader the task of checking that the diagram in Figure 8 defines a homotopy between the complexes $\langle D \rangle$ and $\langle D' \rangle$:

- g and f are morphisms of complexes (use only isotopies);
- $g^1 f^1 = \text{Id}_{\langle D' \rangle^1}$ (use (RD));
- $f^0 g^0 + hd = \text{Id}_{\langle D \rangle^0}$ and $f^2 g^2 + dh = \text{Id}_{\langle D \rangle^2}$ (use isotopies);
- $f^1 g^1 + dh + hd = \text{Id}_{\langle D \rangle^1}$ (use (DR)).

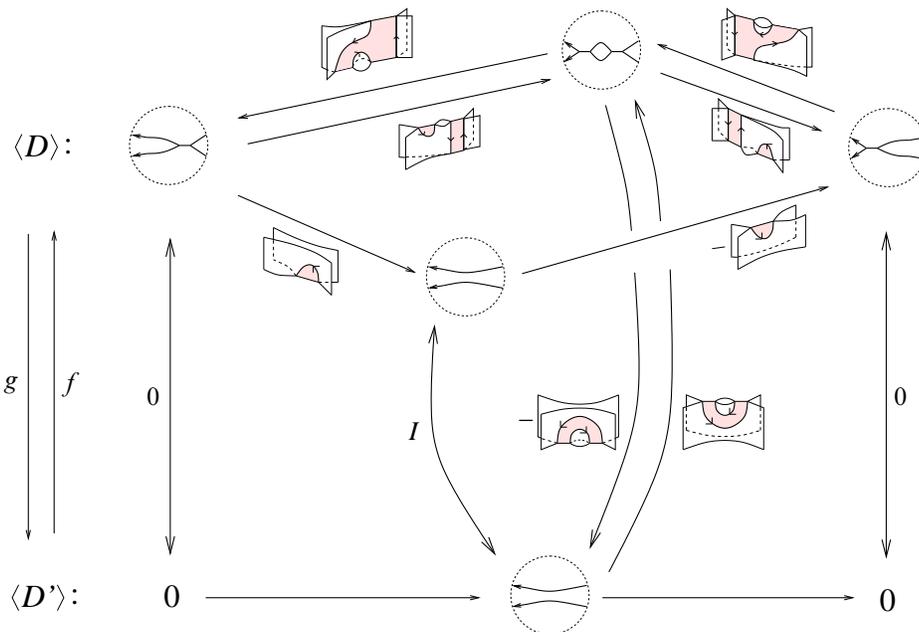


Figure 8: Invariance under Reidemeister IIa

Reidemeister IIb Consider diagrams D and D' that differ only in a circular region, as in the figure below.



Again, checking that the diagram in Figure 9 defines a homotopy between the complexes $\langle D \rangle$ and $\langle D' \rangle$ is left to the reader:

- g and f are morphisms of complexes (use only isotopies);
- $g^1 f^1 = \text{Id}_{\langle D' \rangle^1}$ (use (RD) and (S));
- $f^0 g^0 + hd = \text{Id}_{\langle D \rangle^0}$ and $f^2 g^2 + dh = \text{Id}_{\langle D \rangle^2}$ (use (RD) and (DR));
- $f^1 g^1 + dh + hd = \text{Id}_{\langle D \rangle^1}$ (use (DR), (RD), (4C) and (SqR)).

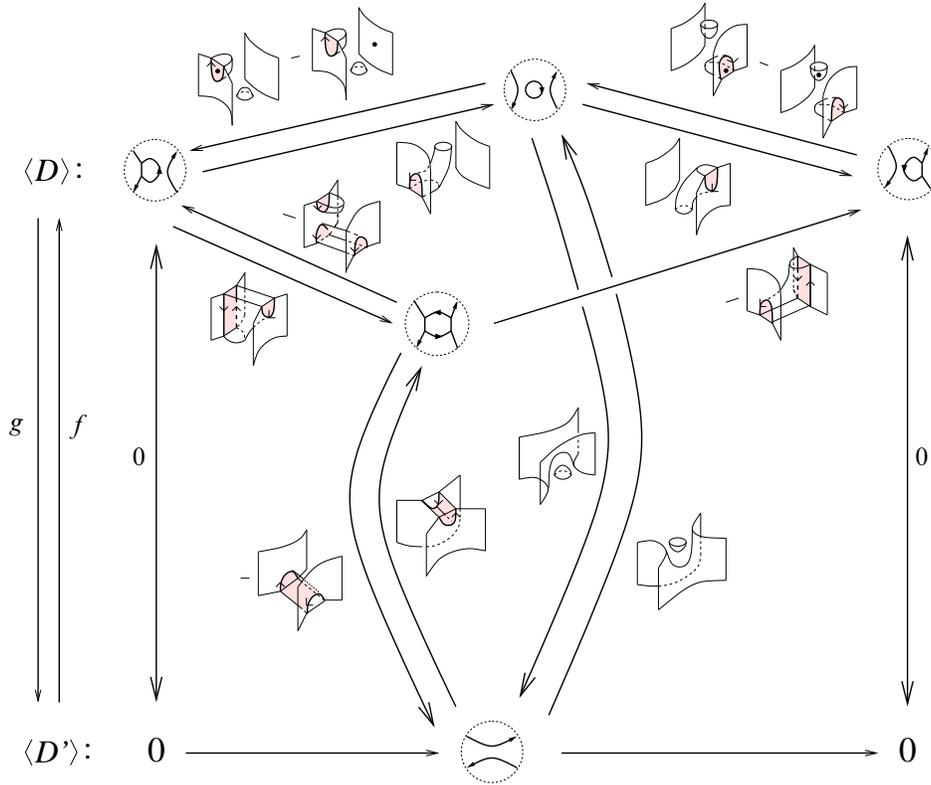


Figure 9: Invariance under Reidemeister IIb

Reidemeister III Consider diagrams D and D' that differ only in a circular region, as in the figure below.

$$D = \text{[Diagram]} \quad D' = \text{[Diagram]}$$

We prove that $\langle D' \rangle$ is homotopy equivalent to $\langle D \rangle$ by showing that both complexes are homotopy equivalent to a third complex denoted $\langle Q \rangle$ (the bottom complex in

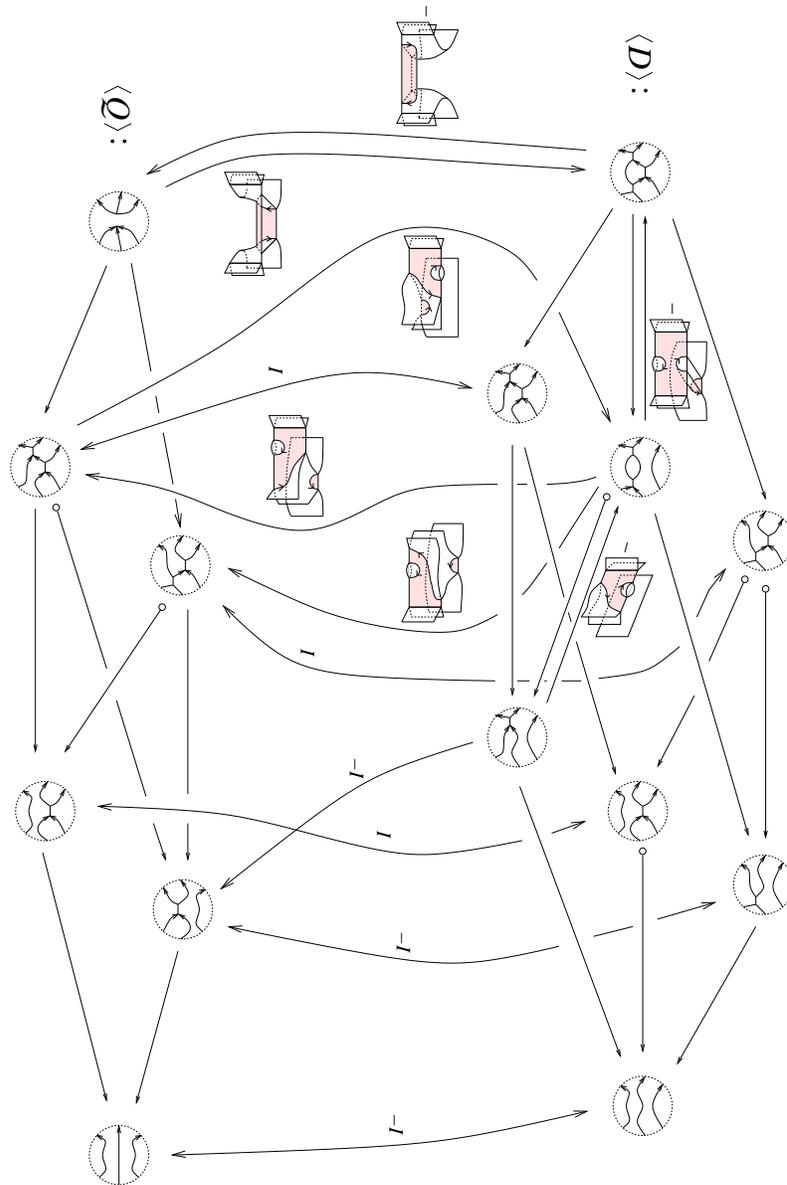


Figure 10: First step of invariance under Reidemeister III. A circle attached to the tail of an arrow indicates that the corresponding morphism has a minus sign.

Figure 10). Figure 10 shows that $\langle D \rangle$ is homotopy equivalent to $\langle Q \rangle$. By applying a symmetry relative to a horizontal axis crossing each diagram in $\langle D \rangle$ we obtain a homotopy equivalence between $\langle D' \rangle$ and $\langle Q \rangle$. It follows that $\langle D \rangle$ is homotopy equivalent to $\langle D' \rangle$. \square

By Theorem 2.4 we can use any diagram of L to obtain the invariant in $\mathbf{Kom}_{/h}(\mathbf{Foam}_{/\ell})$ and justifies the notation $\langle L \rangle$.

2.3 Functoriality

The construction and the results of the previous sections can be extended to the category of tangles, following Bar-Natan's approach in [2]. One can then prove the functoriality of $\langle \rangle$ as Bar-Natan does. Although we will not give the details of this proof some remarks are in order. In the first place, we have to consider tangle diagrams in a disk and foams inside a cylinder. To ensure additivity of the q -grading under lateral composition we need to add an extra term to the q -grading formula. For a foam between open webs with $|b|$ vertical boundary components and d dots we have

$$q(f) = -2\chi(f) + \chi(\partial f) + 2d + |b|.$$

Lemma 8.6 in [2] is fundamental in Bar-Natan's proof of the functoriality of the universal sl_2 -link homology. The analogue for sl_3 follows from the following two lemmas.

Lemma 2.5 *Let f be a closed foam. If $q(f) < 0$, then $f = 0$ holds. If $q(f) = 0$, then the evaluation of f gives an integer.*

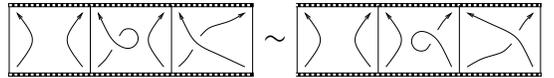
Proof Using (CN) and (RD) we can turn any closed foam into a $\mathbb{Z}[a, b, c]$ -linear combination of a disjoint union of spheres and theta foams. Since the grading remains unchanged by (CN) and (RD) it suffices to check the claims for spheres and theta foams, which is immediate from the (S) and (Θ) relations. \square

Lemma 2.6 *For a crossingless tangle diagram T we have that $\mathrm{Hom}_{\mathbf{Foam}_{/\ell}}(T, T)$ is zero in negative degrees and \mathbb{Z} in degree zero.*

Proof The set of singular points in every foam f from T to itself consists of a disjoint union of circles. Using (CN) and (RD) we can reduce f to a $\mathbb{Z}[a, b, c]$ -linear combination of disjoint unions of vertical disks and closed foams. Note that the q -degree of a dotted disc is always nonnegative. Therefore, if $q(f) < 0$, then the closed foams have to have negative q -degree as well and $f = 0$ has to hold by Lemma 2.5.

If $q(f) = 0$, then f is equal to a \mathbb{Z} -linear combination of undotted discs and closed foams of q -degree zero, so Lemma 2.5 shows that f is an integer multiple of the identity foam on T . \square

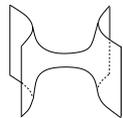
The proofs of the analogues of Lemmas 8.7–8.9 and Theorem 5 in [2] follow the same reasoning but use the homotopies of our Section 2.2. We illustrate this by showing that $\langle \rangle$ respects the movie move MM13 (actually it is the mirror of MM13 in [2]):



Going from left to right in homological degree 0 we find the composition

$$\langle \rangle \langle \rangle \rightarrow \langle \rangle \circ \langle \rangle \rightarrow \langle \rangle \langle \rangle$$

in both movies. The map for the movie on the left-hand side consists of the cobordism f^0 of Figure 7 between the left strand and the circle followed by a saddle cobordism between the circle and the right strand. For the movie on the right-hand side we have f^0 between the right strand and the circle followed by a saddle cobordism between the circle and the left strand. Both sides are equal to



Going from right to left and using the cobordism g^0 of Figure 7 we obtain the identity cobordism in both movies.

Without giving more details of this generalization, we state the main result. Let $\mathbf{Kom}_{/\pm h}(\mathbf{Foam}_{/\ell})$ denote the category $\mathbf{Kom}_{/h}(\mathbf{Foam}_{/\ell})$ modded out by ± 1 .

Proposition 2.7 $\langle \rangle$ defines a functor $\mathbf{Link} \rightarrow \mathbf{Kom}_{/\pm h}(\mathbf{Foam}_{/\ell})$.

2.4 Universal homology

Following Khovanov [7], we define a functor $C: \mathbf{Foam}_{/\ell} \rightarrow \mathbb{Z}[\mathbf{a}, \mathbf{b}, \mathbf{c}]\text{-Mod}$, which extends in a straightforward manner to the category $\mathbf{Kom}(\mathbf{Foam}_{/\ell})$.

Definition 2.8 For a closed web Γ , define $C(\Gamma) = \text{Hom}_{\mathbf{Foam}_{/\ell}}(\emptyset, \Gamma)$. From the q -grading formula for foams, it follows that $C(\Gamma)$ is graded. For a foam f between webs Γ and Γ' we define the $\mathbb{Z}[a, b, c]$ -linear map

$$C(f): \text{Hom}_{\mathbf{Foam}_{/\ell}}(\emptyset, \Gamma) \rightarrow \text{Hom}_{\mathbf{Foam}_{/\ell}}(\emptyset, \Gamma')$$

given by composition, whose degree equals $q(f)$.

Note that, if we have a disjoint union of webs Γ and Γ' , then $C(\Gamma \sqcup \Gamma') \cong C(\Gamma) \otimes C(\Gamma')$. Here, as in the sequel, the tensor product is taken over $\mathbb{Z}[a, b, c]$.

The following relations are a categorified version of Kuperberg’s skein relations [10] and were used and proved by Khovanov in [7] to relate his sl_3 –link homology to the quantum sl_3 –link invariant.

Lemma 2.9 (Khovanov–Kuperberg relations [7; 10]) *We have the following decompositions under the functor C :*

$$\begin{aligned}
 C(\circ\Gamma) &\cong C(\circ) \otimes C(\Gamma) && \text{(Circle Removal)} \\
 C(\curvearrowright\curvearrowleft) &\cong C(\longrightarrow)\{-1\} \oplus C(\longrightarrow)\{1\} && \text{(Digon Removal)} \\
 C(\text{square}) &\cong C(\text{cup}) \oplus C(\text{cap}) && \text{(Square Removal)}
 \end{aligned}$$

where $\{j\}$ denotes a positive shift in the q –grading by j .

Proof (Circle Removal) is immediate from the definition of $C(\Gamma)$. (Digon Removal) and (Square Removal) are proved as in [7]. Notice that (Digon Removal) and (Square Removal) are related to the local relations (DR) and (SqR) of Lemma 2.3. \square

Let $U_{a,b,c}^*(D)$ denote the bigraded homology of $C\langle D \rangle$ and $\mathbb{Z}[\mathbf{a}, \mathbf{b}, \mathbf{c}]\text{-Mod}_{\mathbf{bg}}$ the category of bigraded $\mathbb{Z}[a, b, c]$ –modules. Proposition 2.7 implies the following:

Proposition 2.10 $U_{a,b,c}^* : \mathbf{Link} \rightarrow \mathbb{Z}[\mathbf{a}, \mathbf{b}, \mathbf{c}]\text{-Mod}_{\mathbf{bg}}$ is a functor.

We use the notation $C(L)$ for $C\langle D \rangle$ and $U_{a,b,c}^*(L)$ for $U_{a,b,c}^*(D)$.

3 Isomorphism classes

In this section we work over \mathbb{C} and take a, b, c to be complex numbers. Using the same construction as in the first part of this paper we can define $U_{a,b,c}^*(L, \mathbb{C})$, which is the universal sl_3 –homology with coefficients in \mathbb{C} . We show that there are three isomorphism classes of $U_{a,b,c}^*(L, \mathbb{C})$. Throughout this section we write $f(X) = X^3 - aX^2 - bX - c$. For a given choice of $a, b, c \in \mathbb{C}$, the isomorphism class of $U_{a,b,c}^*(L, \mathbb{C})$ is determined by the number of distinct roots of $f(X)$.

Remark We could work over \mathbb{Q} just as well and obtain the same results, except that in the proofs we would first have to pass to quadratic or cubic field extensions of \mathbb{Q} to guarantee the existence of the roots of $f(X)$ in the field of coefficients of the homology. The arguments we present for $U_{a,b,c}^*(L, \mathbb{C})$ remain valid over those quadratic or cubic extensions. The universal coefficient theorem then shows that our results hold true for the homology defined over \mathbb{Q} .

If $f(X) = (X - \alpha)^3$, then the isomorphism $\square \mapsto \square - \alpha\square$ induces an isomorphism between $U_{a,b,c}^*(L, \mathbb{C})$ and Khovanov's original sl_3 -link homology, which in our notation is equal to $U_{0,0,0}^*(L, \mathbb{C})$.

In the following two subsections we study the cases in which $f(X)$ has two or three distinct roots. We first work out the case for three distinct roots, because this case has essentially been done by Gornik [4]. Even in this case we define and prove everything precisely and completely. We have two good reasons for doing this. First of all we generalize Gornik's work to the arbitrary case of three distinct roots, whereas he, strictly speaking, only considers the particular case of the third roots of unity. Given the definitions and arguments for the general case, one easily recognizes Gornik's definitions and arguments for his particular case. Working one's way back is harder, also because Gornik followed the approach using matrix factorizations and not cobordisms. Secondly these general definitions and arguments are necessary for understanding the last subsection, where we treat the case in which $f(X)$ has only two distinct roots, which is clearly different from Gornik's.

3.1 Three distinct roots

In this subsection we assume that the three roots of $f(X)$, denoted $\alpha, \beta, \gamma \in \mathbb{C}$, are all distinct. First we determine Gornik's idempotents in the algebra $\mathbb{C}[X]/(f(X))$. By the Chinese Remainder Theorem we have the following isomorphism of algebras

$$\mathbb{C}[X]/(f(X)) \cong \mathbb{C}[X]/(X - \alpha) \oplus \mathbb{C}[X]/(X - \beta) \oplus \mathbb{C}[X]/(X - \gamma) \cong \mathbb{C}^3.$$

Definition 3.1 Let $Q_\alpha(X)$, $Q_\beta(X)$ and $Q_\gamma(X)$ be the idempotents in $\mathbb{C}[X]/(f(X))$ corresponding to $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in \mathbb{C}^3 under the isomorphism in the Chinese Remainder Theorem.

As a matter of fact it is easy to compute the idempotents explicitly:

$$Q_\alpha(X) = \frac{(X-\beta)(X-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}, \quad Q_\beta(X) = \frac{(X-\alpha)(X-\gamma)}{(\beta-\alpha)(\beta-\gamma)}, \quad Q_\gamma(X) = \frac{(X-\alpha)(X-\beta)}{(\gamma-\alpha)(\gamma-\beta)}.$$

By definition we get:

Lemma 3.2

$$\begin{aligned} Q_\alpha(X) + Q_\beta(X) + Q_\gamma(X) &= 1, \\ Q_\alpha(X)Q_\beta(X) &= Q_\alpha(X)Q_\gamma(X) = Q_\beta(X)Q_\gamma(X) = 0, \\ Q_\alpha(X)^2 &= Q_\alpha(X), \quad Q_\beta(X)^2 = Q_\beta(X), \quad Q_\gamma(X)^2 = Q_\gamma(X). \end{aligned}$$

Let Γ be a resolution of a link L and let $E(\Gamma)$ be the set of all edges in Γ . In [7] Khovanov defines the following algebra (in his case for $a = b = c = 0$).

Definition 3.3 Let $R(\Gamma)$ be the commutative \mathbb{C} -algebra with generators X_i , for $i \in E(\Gamma)$, modulo the relations

$$(2) \quad X_i + X_j + X_k = a, \quad X_i X_j + X_j X_k + X_i X_k = -b, \quad X_i X_j X_k = c,$$

for any triple of edges i, j, k which share a trivalent vertex.

The following definitions and results are analogous to Gornik's results in Sections 2 and 3 of [4]. Let $S = \{\alpha, \beta, \gamma\}$.

Definition 3.4 A *coloring* of Γ is defined to be a map $\phi: E(\Gamma) \rightarrow S$. Denote the set of all colorings by $S(\Gamma)$. An *admissible coloring* is a coloring such that

$$(3) \quad \begin{aligned} a &= \phi(i) + \phi(j) + \phi(k) \\ -b &= \phi(i)\phi(j) + \phi(j)\phi(k) + \phi(i)\phi(k) \\ c &= \phi(i)\phi(j)\phi(k), \end{aligned}$$

for any edges i, j, k incident to the same trivalent vertex. Denote the set of all admissible colorings by $AS(\Gamma)$.

Of course admissibility is equivalent to requiring that the three colors $\phi(i), \phi(j)$ and $\phi(k)$ be all distinct.

A simple calculation shows that $f(X_i) = 0$ in $R(\Gamma)$, for any $i \in E(\Gamma)$. Therefore, for any edge $i \in E(\Gamma)$, there exists a homomorphism of algebras from $\mathbb{C}[X]/(f(X))$ to $R(\Gamma)$ defined by $X \mapsto X_i$. Thus, we define the following:

Definition 3.5 For any coloring ϕ ,

$$Q_\phi(\Gamma) = \prod_{i \in E(\Gamma)} Q_{\phi(i)}(X_i) \in R(\Gamma).$$

Lemma 3.2 implies the following corollary.

Corollary 3.6 Let δ_ψ^ϕ be the Kronecker delta. Then

$$\sum_{\phi \in S(\Gamma)} Q_\phi(\Gamma) = 1, \quad Q_\phi(\Gamma) Q_\psi(\Gamma) = \delta_\psi^\phi Q_\phi(\Gamma),$$

Note that the definition of $Q_\phi(\Gamma)$ implies that

$$(4) \quad X_i Q_\phi(\Gamma) = \phi(i) Q_\phi(\Gamma).$$

The following lemma is our analogue of Gornik’s Theorem 3.

Lemma 3.7 For any nonadmissible coloring ϕ , we have

$$Q_\phi(\Gamma) = 0.$$

For any admissible coloring ϕ , we have

$$Q_\phi(\Gamma) R(\Gamma) \cong \mathbb{C}.$$

Therefore, we get a direct sum decomposition

$$R(\Gamma) \cong \bigoplus_{\phi \in AS(\Gamma)} \mathbb{C} Q_\phi(\Gamma).$$

Proof Let ϕ be any coloring and let $i, j, k \in E(\Gamma)$ be three edges sharing a trivalent vertex. By the relations in (2) and equation (4), we get

$$(5) \quad \begin{aligned} a Q_\phi(\Gamma) &= (\phi(i) + \phi(j) + \phi(k)) Q_\phi(\Gamma) \\ -b Q_\phi(\Gamma) &= (\phi(i)\phi(j) + \phi(j)\phi(k) + \phi(i)\phi(k)) Q_\phi(\Gamma) \\ c Q_\phi(\Gamma) &= \phi(i)\phi(j)\phi(k) Q_\phi(\Gamma). \end{aligned}$$

If ϕ is nonadmissible, then, by comparing (3) and (5), we see that $Q_\phi(\Gamma)$ vanishes.

Now suppose ϕ is admissible. Recall that $R_\phi(\Gamma)$ is a quotient of the algebra

$$(6) \quad \bigotimes_{i \in E(\Gamma)} \mathbb{C}[X_i]/(f(X_i)).$$

Just as in Definition 3.5 we can define the idempotents in the algebra in (6), which we also denote $Q_\phi(\Gamma)$. By the Chinese Remainder Theorem, there is a projection of the algebra in (6) onto \mathbb{C} , which maps $Q_\phi(\Gamma)$ to 1 and $Q_\psi(\Gamma)$ to 0, for any $\psi \neq \phi$. It is not hard to see that, since ϕ is admissible, that projection factors through the quotient $R(\Gamma)$, which implies the second claim in the lemma. \square

As in [7], the relations in Figure 6 show that $R(\Gamma)$ acts on $C(\Gamma)$ by the usual action induced by the cobordism which merges a circle and the relevant edge of Γ . Let us write $C_\phi(\Gamma) = Q_\phi(\Gamma)C(\Gamma)$. By Corollary 3.6 and Lemma 3.7, we have a direct sum decomposition

$$(7) \quad C(\Gamma) = \bigoplus_{\phi \in AS(\Gamma)} C_\phi(\Gamma).$$

Note that we have

$$(8) \quad z \in C_\phi(\Gamma) \iff \text{for all } i, X_i z = \phi(i)z$$

for any $\phi \in AS(\Gamma)$.

Let ϕ be a coloring of the arcs of L by α, β and γ . Note that ϕ induces a unique coloring of the unmarked edges of any resolution of L .

Definition 3.8 We say that a coloring of the arcs of L is *admissible* if there exists a resolution of L which admits a compatible admissible coloring. Note that if such a resolution exists, its coloring is uniquely determined by ϕ , so we use the same notation. Note also that an admissible coloring of ϕ induces a unique admissible coloring of L . If ϕ is an admissible coloring, we call the elements in $C_\phi(\Gamma)$ *admissible cochains*. We denote the set of all admissible colorings of L by $AS(L)$.

We say that an admissible coloring of L is a *canonical coloring* if the arcs belonging to the same component of L have the same color. If ϕ is a canonical coloring, we call the elements in $C_\phi(\Gamma)$ *canonical cochains*. We denote the set of canonical colorings of L by $S(L)$.

Note that, for a fixed $\phi \in AS(L)$, the admissible cochain groups $C_\phi(\Gamma)$ form a sub-complex $C_{a,b,c}^*(L_\phi, \mathbb{C}) \subseteq C_{a,b,c}^*(L, \mathbb{C})$ whose homology we denote by $U_{a,b,c}^*(L_\phi, \mathbb{C})$. The following lemma shows that only the canonical cochain groups matter, as Gornik indicated in his remarks before his Main Theorem 2 in [4].

Theorem 3.9

$$U_{a,b,c}^*(L, \mathbb{C}) = \bigoplus_{\phi \in S(L)} U_{a,b,c}^*(L_\phi, \mathbb{C}).$$

Proof By (7) we have

$$U_{a,b,c}^*(L, \mathbb{C}) = \bigoplus_{\phi \in AS(L)} U_{a,b,c}^*(L_\phi, \mathbb{C}).$$

Let us now show that $U_{a,b,c}^*(L_\phi, \mathbb{C}) = 0$ if ϕ is admissible but noncanonical. Let

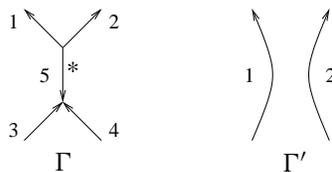
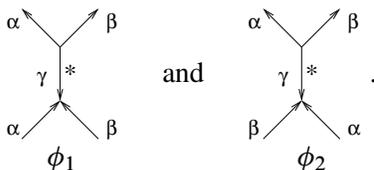
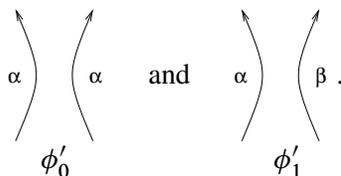


Figure 11: Ordering edges

Γ and Γ' be the diagrams in Figure 11, which are the boundary of the cobordism which defines the differential in $C_{a,b,c}^*(L, \mathbb{C})$, and order their edges as indicated. Up to permutation, the only admissible colorings of Γ are



Up to permutation, the only admissible colorings of Γ' are



Note that only ϕ_2 and ϕ'_0 can be canonical. We get

$$(9) \quad \begin{aligned} 0 &\leftarrow C_{\phi'_0}(\Gamma'), \\ C_{\phi_1}(\Gamma) &\cong C_{\phi'_1}(\Gamma'), \\ C_{\phi_2}(\Gamma) &\rightarrow 0. \end{aligned}$$

Note that the elementary cobordism has to map colorings to compatible colorings. This explains the first and the third line. Let us explain the second line. Apply the elementary cobordisms $\Gamma' \rightarrow \Gamma \rightarrow \Gamma'$ and use relation (RD) of Lemma 2.3 to obtain the linear map $C_{\phi'_{1,2}}(\Gamma') \rightarrow C_{\phi'_{1,2}}(\Gamma')$ given by

$$z \mapsto (\beta - \alpha)z.$$

Since $\alpha \neq \beta$, we see that this map is injective. Therefore the map $C_{\phi'_{1,2}}(\Gamma') \rightarrow C_{\phi_{1,2}}(\Gamma)$ is injective too. A similar argument, using the (DR) relation, shows that $C_{\phi_{1,2}}(\Gamma) \rightarrow C_{\phi'_{1,2}}(\Gamma')$ is injective. Therefore both maps are isomorphisms.

Next, let ϕ be admissible but noncanonical. Then there exists at least one crossing, denoted c , in L which has a resolution with a noncanonical coloring. Let $C_{a,b,c}^*(L_\phi^1, \mathbb{C})$

be the subcomplex of $C_{a,b,c}^*(L_\phi, \mathbb{C})$ defined by the resolutions of L in which c has been resolved by the 1-resolution. Let $C_{a,b,c}^*(L_\phi^0, \mathbb{C})$ be the complex obtained from $C_{a,b,c}^*(L_\phi, \mathbb{C})$ by deleting all resolutions which do not belong to $C_{a,b,c}^*(L_\phi^1, \mathbb{C})$ and all arrows which have a source or target which is not one of the remaining resolutions. Note that we have a short exact sequence of complexes

$$(10) \quad 0 \rightarrow C_{a,b,c}^*(L_\phi^1, \mathbb{C}) \rightarrow C_{a,b,c}^*(L_\phi, \mathbb{C}) \rightarrow C_{a,b,c}^*(L_\phi^0, \mathbb{C}) \rightarrow 0.$$

The isomorphism in (9) shows that the natural map

$$C_{a,b,c}^*(L_\phi^0, \mathbb{C}) \rightarrow C_{a,b,c}^{*+1}(L_\phi^1, \mathbb{C}),$$

defined by the elementary cobordisms which induce the connecting homomorphism in the long exact sequence associated to (10), is an isomorphism. By exactness of this long exact sequence we see that $U_{a,b,c}^*(L_\phi, \mathbb{C}) = 0$. \square

Lemma 3.10 *For any $\phi \in AS(\Gamma)$, we have $C_\phi(\Gamma) \cong \mathbb{C}$.*

Proof We use induction with respect to v , the number of trivalent vertices in Γ . The claim is obviously true for a circle. Suppose Γ has a digon, with the edges ordered as in Figure 12. Note that $X_1 = X_4 \in R(\Gamma)$ holds as a consequence of the relations in (2).

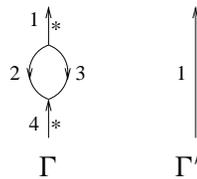
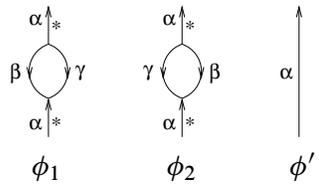


Figure 12: Ordering edges in digon

Let Γ' be the web obtained by removing the digon, as in Figure 12. Up to permutation, the only possible admissible colorings of Γ and the corresponding admissible coloring of Γ' are



The (Digon Removal) isomorphism in Lemma 2.9 yields

$$C_{\phi_1}(\Gamma) \oplus C_{\phi_2}(\Gamma) \cong C_{\phi'}(\Gamma') \oplus C_{\phi'}(\Gamma').$$

By induction, we have $C_{\phi'}(\Gamma') \cong \mathbb{C}$, so $\dim C_{\phi_1}(\Gamma) + \dim C_{\phi_2}(\Gamma) = 2$. For symmetry reasons this implies that $\dim C_{\phi_1}(\Gamma) = \dim C_{\phi_2}(\Gamma) = 1$, which proves the claim. To be a bit more precise, let $B_{\beta,\gamma}$ and $B_{\gamma,\beta}$ be the following two colored cobordisms:

$$B_{\beta,\gamma} = \begin{array}{c} \beta \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \gamma \end{array}, \quad B_{\gamma,\beta} = \begin{array}{c} \gamma \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \beta \end{array}.$$

Note that we have

$$B_{\beta,\gamma} + B_{\gamma,\beta} = 0 \quad \text{and} \quad \gamma B_{\beta,\gamma} + \beta B_{\gamma,\beta} = \text{id}_{C_{\phi'}(\Gamma')}$$

by  = 0, and  = ,

respectively. These two identities imply

$$(\gamma - \beta)B_{\beta,\gamma} = (\beta - \gamma)B_{\gamma,\beta} = \text{id}_{C_{\phi'}(\Gamma')}.$$

Therefore we conclude that $C_{\phi_1}(\Gamma)$ and $C_{\phi_2}(\Gamma)$ are nonzero, which for dimensional reasons implies $\dim C_{\phi_1}(\Gamma) = \dim C_{\phi_2}(\Gamma) = 1$.

Now, suppose Γ contains a square, with the edges ordered as in Figure 13 on the left. Let Γ' and Γ'' be the two corresponding webs under the (Square Removal) isomorphism

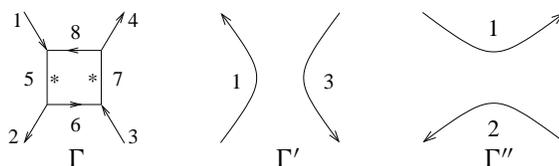
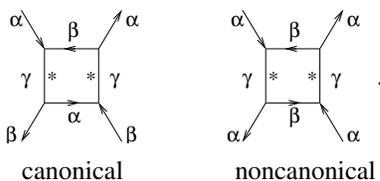


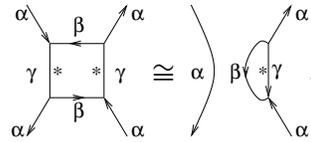
Figure 13: Ordering edges in square

in Lemma 2.9. Up to permutation there is only one admissible noncanonical coloring and one canonical coloring:



Let us first consider the canonical coloring. Clearly $C_{\phi}(\Gamma)$ is isomorphic to $C_{\phi''}(\Gamma'')$, where ϕ'' is the unique compatible canonical coloring, because there is no compatible coloring of Γ' . Therefore the result follows by induction.

Now consider the admissible noncanonical coloring. As proved in [Theorem 3.9](#) we have the following isomorphism:



By induction the right-hand side is one-dimensional, which proves the claim. \square

Thus we arrive at Gornik’s Main Theorem 2. Note that there are 3^n canonical colorings of L , where n is the number of components of L . Note also that the homological degrees of the canonical cocycles are easy to compute, because we know that the canonical cocycles corresponding to the oriented resolution without vertices have homological degree zero.

Theorem 3.11 *The dimension of $U_{a,b,c}^*(L, \mathbb{C})$ is 3^n , where n is the number of components of L .*

For any $\phi \in S(L)$, there exists a nonzero element $a_\phi \in U_{a,b,c}^i(L, \mathbb{C})$, unique up to a scalar, where

$$i = \sum_{(\epsilon_1, \epsilon_2) \in S \times S, \epsilon_1 \neq \epsilon_2} \text{lk}(\phi^{-1}(\epsilon_1), \phi^{-1}(\epsilon_2)).$$

3.2 Two distinct roots

In this section we assume that $f(X) = (X - \alpha)^2(X - \beta)$, with $\alpha \neq \beta$. We follow an approach similar to the one in the previous section. First we define the relevant idempotents. By the Chinese Remainder Theorem we have

$$\mathbb{C}[X]/(f(X)) \cong \mathbb{C}[X]/((X - \alpha)^2) \oplus \mathbb{C}[X]/(X - \beta).$$

Definition 3.12 Let Q_α and Q_β be the idempotents in $\mathbb{C}[X]/(f(X))$ corresponding to $(1, 0)$ and $(0, 1)$ in $\mathbb{C}[X]/((X - \alpha)^2) \oplus \mathbb{C}[X]/(X - \beta)$ under the above isomorphism.

Again it is easy to compute the idempotents explicitly:

$$Q_\alpha = 1 - \frac{(X - \alpha)^2}{(\beta - \alpha)^2}, \quad Q_\beta = \frac{(X - \alpha)^2}{(\beta - \alpha)^2}.$$

By definition we get:

Lemma 3.13

$$Q_\alpha + Q_\beta = 1, \quad Q_\alpha Q_\beta = 0, \quad Q_\alpha^2 = Q_\alpha, \quad Q_\beta^2 = Q_\beta.$$

Throughout this subsection let $S = \{\alpha, \beta\}$. We define colorings of webs and admissibility as in Definition 3.4. Note that a coloring is admissible if and only if at each trivalent vertex the, unordered, incident edges are colored α, α, β . Let Γ be a web and ϕ a coloring. The definition of the idempotents $Q_\phi(\Gamma)$ in $R(\Gamma)$ is the same as in Definition 3.5. Clearly Corollary 3.6 also holds in this section. However, equation (4) changes. By the Chinese Remainder Theorem, we get

$$(11) \quad \begin{cases} (X_i - \beta)Q_\phi(\Gamma) = 0, & \text{if } \phi(i) = \beta \\ (X_i - \alpha)^2 Q_\phi(\Gamma) = 0, & \text{if } \phi(i) = \alpha. \end{cases}$$

Lemma 3.7 also changes. Its analogue becomes:

Lemma 3.14 For any nonadmissible coloring ϕ , we have

$$Q_\phi(\Gamma) = 0.$$

Therefore, we have a direct sum decomposition

$$R(\Gamma) \cong \bigoplus_{\phi \in AS(\Gamma)} Q_\phi(\Gamma)R(\Gamma).$$

For any $\phi \in AS(\Gamma)$, we have $\dim Q_\phi(\Gamma)R(\Gamma) = 2^m$, where m is the number of cycles in $\phi^{-1}(\alpha) \subseteq \Gamma$.

Proof First we prove that inadmissible colorings yield trivial idempotents. Let ϕ be any coloring of Γ and let i, j, k be three edges sharing a trivalent vertex. First suppose that all edges are colored by β . By equations (11) we get

$$aQ_\phi(\Gamma) = (X_i + X_j + X_k)Q_\phi(\Gamma) = 3\beta Q_\phi(\Gamma),$$

which implies that $Q_\phi(\Gamma) = 0$, because $a = 2\alpha + \beta$ and $\alpha \neq \beta$.

Next suppose $\phi(i) = \phi(j) = \beta$ and $\phi(k) = \alpha$. Then

$$aQ_\phi(\Gamma) = (X_i + X_j + X_k)Q_\phi(\Gamma) = (2\beta + X_k)Q_\phi(\Gamma).$$

Thus $X_k Q_\phi(\Gamma) = (2\alpha - \beta)Q_\phi(\Gamma)$. Therefore we get

$$0 = (X_k - \alpha)^2 Q_\phi(\Gamma) = (\alpha - \beta)^2 Q_\phi(\Gamma),$$

which again implies that $Q_\phi(\Gamma) = 0$.

Finally, suppose i, j, k are all colored by α . Then we have

$$((X_i - \alpha)^2 + (X_j - \alpha)^2 + (X_k - \alpha)^2)Q_\phi(\Gamma) = 0.$$

Using the relations in (2) we get

$$(X_i - \alpha)^2 + (X_j - \alpha)^2 + (X_k - \alpha)^2 = (\alpha - \beta)^2,$$

so we see that $Q_\phi(\Gamma) = 0$.

Now, let ϕ be an admissible coloring. Note that the admissibility condition implies that $\phi^{-1}(\alpha)$ consists of a disjoint union of cycles. To avoid confusion, let us remark that we do not take into consideration the orientation of the edges when we speak about cycles, as one would in algebraic topology. What we mean by a cycle is simply a piecewise linear closed loop. Recall that $R(\Gamma)$ is a quotient of the algebra

$$(12) \quad \bigotimes_{i \in E(\Gamma)} \mathbb{C}[X_i]/(f(X_i))$$

and that we can define idempotents, also denoted $Q_\phi(\Gamma)$, in the latter. Note that by the Chinese Remainder Theorem there exists a homomorphism of algebras which projects the algebra in (12) onto

$$(13) \quad \bigotimes_{\phi(i)=\alpha} \mathbb{C}[X_i]/((X_i - \alpha)^2) \otimes \bigotimes_{\phi(i)=\beta} \mathbb{C}[X_i]/(X_i - \beta),$$

which maps $Q_\phi(\Gamma)$ to 1 and $Q_\psi(\Gamma)$ to 0, for any $\psi \neq \phi$. Define $R_\phi(\Gamma)$ to be the quotient of the algebra in (13) by the relations $X_i + X_j = 2\alpha$, for all edges i and j which share a trivalent vertex and satisfy $\phi(i) = \phi(j) = \alpha$. Note that $X_i X_j = \alpha^2$ also holds in $R_\phi(\Gamma)$, for such edges i and j .

Suppose that the edges i, j, k are incident to a trivalent vertex in Γ and that they are colored α, α, β . It is easy to see that by the projection onto $R_\phi(\Gamma)$ we get

$$\begin{aligned} X_i + X_j + X_k &\mapsto a \\ X_i X_j + X_i X_k + X_j X_k &\mapsto -b \\ X_i X_j X_k &\mapsto c. \end{aligned}$$

Therefore the projection descends to a projection from $R(\Gamma)$ onto $R_\phi(\Gamma)$. Since $Q_\phi(\Gamma)$ is mapped to 1 and $Q_\psi(\Gamma)$ to 0, for all $\psi \neq \phi$, we see that the projection restricts to a surjection of algebras

$$Q_\phi(\Gamma)R(\Gamma) \rightarrow R_\phi(\Gamma).$$

A simple computation shows that the equality

$$(X_i + X_j)Q_\phi(\Gamma) = 2\alpha Q_\phi(\Gamma)$$

holds in $R(\Gamma)$, which implies that the surjection above is an isomorphism of algebras. This proves the final claim in the lemma. \square

As in (8), for any $\phi \in AS(\Gamma)$, we get

$$(14) \quad z \in C_\phi(\Gamma) \iff \begin{cases} (X_i - \beta)z = 0, & \text{for all } i \text{ such that } \phi(i) = \beta, \\ (X_i - \alpha)^2 z = 0, & \text{for all } i \text{ such that } \phi(i) = \alpha. \end{cases}$$

Let ϕ be a coloring of the arcs of L by α and β . Note that ϕ induces a unique coloring of the unmarked edges of any resolution of L . We define *admissible* and *canonical colorings* of L as in Definition 3.8.

Note, as before, that, for a fixed admissible coloring ϕ of L , the admissible cochain groups $C_\phi(\Gamma)$ form a subcomplex $C_{a,b,c}^*(L_\phi, \mathbb{C}) \subseteq C_{a,b,c}^*(L, \mathbb{C})$ whose homology we denote by $U_{a,b,c}^*(L_\phi, \mathbb{C})$. The following theorem is the analogue of Theorem 3.9.

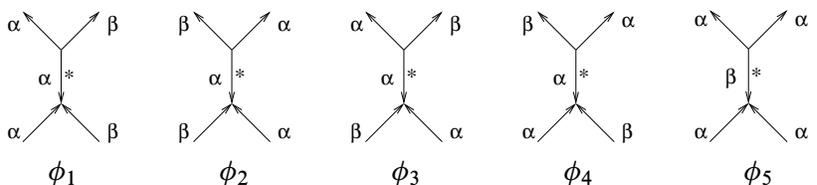
Theorem 3.15

$$U_{a,b,c}^*(L, \mathbb{C}) = \bigoplus_{\phi \in S(L)} U_{a,b,c}^*(L_\phi, \mathbb{C}).$$

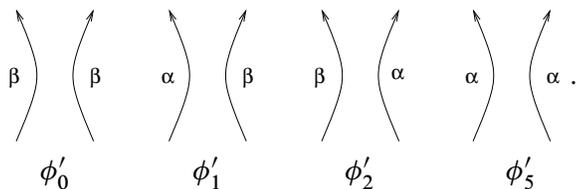
Proof By Lemma 3.14 we get

$$U_{a,b,c}^*(L, \mathbb{C}) = \bigoplus_{\phi \in AS(L)} U_{a,b,c}^*(L_\phi, \mathbb{C}).$$

Let us now show that $U_{a,b,c}^*(L_\phi, \mathbb{C}) = 0$ if ϕ is admissible but noncanonical. Let Γ and Γ' be the diagrams in Figure 11, which are the boundary of the cobordism which induces the differential in $C_{a,b,c}^*(L, \mathbb{C})$, and order their edges as indicated. The only admissible colorings of Γ are



The only admissible colorings of Γ' are



Note that only $\phi_3, \phi_4, \phi_5, \phi'_0$ and ϕ'_5 can be canonical. We get

$$\begin{aligned}
 (15) \quad & 0 \leftarrow C_{\phi'_0}(\Gamma'), \\
 & C_{\phi_1}(\Gamma) \cong C_{\phi'_1}(\Gamma'), \\
 & C_{\phi_2}(\Gamma) \cong C_{\phi'_2}(\Gamma'), \\
 & C_{\phi_3}(\Gamma) \rightarrow 0, \\
 & C_{\phi_4}(\Gamma) \rightarrow 0, \\
 & C_{\phi_5}(\Gamma) \leftrightarrow C_{\phi'_5}(\Gamma').
 \end{aligned}$$

Note that the last line in the list above only states that the cobordism induces a map from one side to the other or vice-versa, but not that it is an isomorphism in general. The second and third line contain isomorphisms. Let us explain the second line, the third being similar. Apply the elementary cobordism $\Gamma' \rightarrow \Gamma \rightarrow \Gamma'$ and use relation (RD) of [Lemma 2.3](#) to obtain the linear map $C_{\phi'_{1,2}}(\Gamma') \rightarrow C_{\phi'_{1,2}}(\Gamma')$ given by

$$z \mapsto (\beta - X_1)z.$$

Suppose $(X_1 - \beta)z = 0$. Then $z \in C_{\phi'_0}(\Gamma')$, because $X_1z = X_2z = \beta z$. This implies that $z \in C_{\phi'_{1,2}}(\Gamma') \cap C_{\phi'_0}(\Gamma') = \{0\}$. Thus the map above is injective, and therefore the map $C_{\phi'_{1,2}}(\Gamma') \rightarrow C_{\phi_{1,2}}(\Gamma)$ is injective. A similar argument, using the relation (DR), shows that $C_{\phi_{1,2}}(\Gamma) \rightarrow C_{\phi'_{1,2}}(\Gamma')$ is injective. Therefore both maps are isomorphisms.

The isomorphisms in (15) imply that $U_{a,b,c}^*(L_\phi, \mathbb{C}) = 0$ holds, when ϕ is admissible but noncanonical, as we explained in the proof of [Theorem 3.9](#). \square

Let $C_\phi(\Gamma)$ be a canonical cochain group. In this case it does not suffice to compute the dimensions of $C_\phi(\Gamma)$, for all ϕ and Γ , because we also need to determine the differentials. Therefore we first define a canonical cobordism in $C_\phi(\Gamma)$.

Definition 3.16 Let $\phi \in S(L)$. We define a cobordism $\Sigma_\phi(\Gamma): \emptyset \rightarrow \Gamma$ by gluing together the elementary cobordisms in [Figure 14](#) and multiplying by $Q_\phi(\Gamma)$. We call $\Sigma_\phi(\Gamma)$ the *canonical cobordism* in $C_\phi(\Gamma)$.

For any canonically colored web, we can find a way to build up the canonical cobordism using only the above elementary cobordisms with canonical colorings, except when we have several digons as in [Figure 15](#) where we might have to stick in two digons at a time to avoid getting webs with admissible noncanonical colorings.

There is a slight ambiguity in the rules above. At some point we may have several choices which yield different cobordisms, depending on the order in which we build

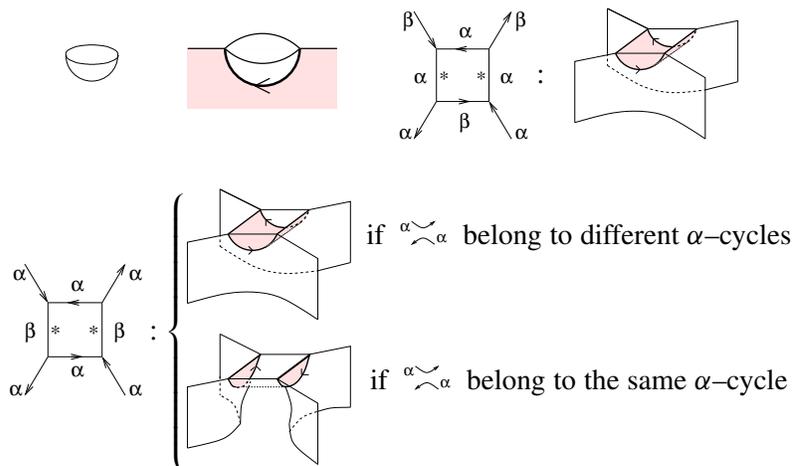


Figure 14: Elementary cobordisms



Figure 15: Several digons

them up. To remove this ambiguity we order all arcs of the link, which induces an ordering of all unmarked edges in any of its resolutions. By convention we first build up the square or digon which contains the lowest order edge.

With these two observations in mind, it is not hard to see that the rules defining $\Sigma_\phi(\Gamma)$ are consistent. One can check that the two different ways of defining it for the square-digon webs in Figure 16 yield the same cobordism indeed.

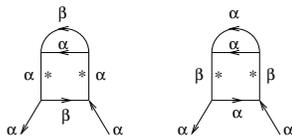
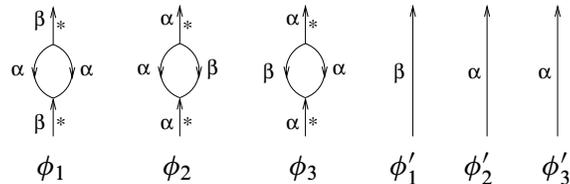


Figure 16: Square-digons with coloring

Recall that the definition of $R_\phi(\Gamma)$, which appears in the following lemma, can be found in the proof of Lemma 3.14, where we showed that $R_\phi(\Gamma) \cong Q_\phi(\Gamma)R(\Gamma)$.

Lemma 3.17 $C_\phi(\Gamma)$ is a free cyclic $R_\phi(\Gamma)$ -module generated by $\Sigma_\phi(\Gamma)$, for any $\phi \in S(\Gamma)$.

Proof We use induction with respect to v , the number of trivalent vertices in Γ . The claim is obviously true for a circle. Suppose Γ has a digon, with the edges ordered as in Figure 12. Note that $X_1 = X_4 \in R(\Gamma)$ holds as a consequence of the relations in (2). Let Γ' be the web obtained by removing the digon, as in Lemma 2.9. The possible canonical colorings of Γ and the corresponding canonical colorings of Γ' are



We treat the case of ϕ_1 first. Since the (Digon Removal) isomorphism in Lemma 2.9 commutes with the action of $X_1 = X_4$, we see that

$$C_{\phi_1}(\Gamma) \cong C_{\phi'_1}(\Gamma') \oplus C_{\phi'_1}(\Gamma').$$

By induction $C_{\phi'_1}(\Gamma')$ is a free cyclic $R_{\phi'_1}(\Gamma')$ -module generated by $\Sigma_{\phi'_1}(\Gamma')$. Note that the isomorphism maps

$$(\Sigma_{\phi'_1}(\Gamma'), 0) \quad \text{and} \quad (0, \Sigma_{\phi'_1}(\Gamma'))$$

to $\Sigma_{\phi_1}(\Gamma)$ and $X_2 \Sigma_{\phi_1}(\Gamma)$.

Since $\dim R_{\phi_1}(\Gamma) = 2 \dim R_{\phi'_1}(\Gamma')$, we conclude that $C_{\phi_1}(\Gamma)$ is a free cyclic $R_{\phi_1}(\Gamma)$ -module generated by $\Sigma_{\phi_1}(\Gamma)$.

Now, let us consider the case of ϕ_2 and ϕ_3 . The (Digon Removal) isomorphism in Lemma 2.9 yields

$$C_{\phi_2}(\Gamma) \oplus C_{\phi_3}(\Gamma) \cong C_{\phi'_2}(\Gamma') \oplus C_{\phi'_3}(\Gamma').$$

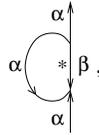
Note that $\phi'_2 = \phi'_3$ holds and by induction $C_{\phi'_2}(\Gamma') = C_{\phi'_3}(\Gamma')$ is a free cyclic $R_{\phi'_2}(\Gamma') = R_{\phi'_3}(\Gamma')$ -module. As in the previous case, by definition of the canonical generators, it is easy to see that the isomorphism maps

$$R_{\phi'_2}(\Gamma') \Sigma_{\phi'_2}(\Gamma') \oplus R_{\phi'_3}(\Gamma') \Sigma_{\phi'_3}(\Gamma')$$

to $R_{\phi_2}(\Gamma) \Sigma_{\phi_2}(\Gamma) \oplus R_{\phi_3}(\Gamma) \Sigma_{\phi_3}(\Gamma)$.

Counting the dimensions on both sides of the isomorphism, we see that this proves the claim in the lemma for $C_{\phi_2}(\Gamma)$ and $C_{\phi_3}(\Gamma)$.

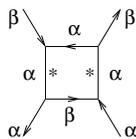
We could also have



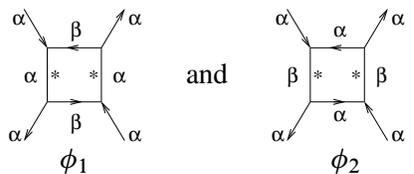
but the same arguments as above apply to this case.

If we have several digons as in [Figure 15](#), similar arguments prove the claim when we stick in two digons at a time.

Next, suppose Γ contains a square, with the edges ordered as in [Figure 13](#) left. Let Γ' and Γ'' be the two corresponding webs under the (Square Removal) isomorphism in [Lemma 2.9](#). There is a number of possible canonical colorings. Note that there is no canonical coloring which colors all external edges by β . To prove the claim for all canonical colorings it suffices to consider only two: the one in which all external edges are colored by α and the coloring



All other cases are similar. Suppose that all external edges are colored by α , then there are two admissible colorings:



Note that only ϕ_2 can be canonical. Clearly there are unique canonical colorings of Γ' and Γ'' , with both edges colored by α , which we denote ϕ' and ϕ'' . The isomorphism yields

$$C_{\phi_1}(\Gamma) \oplus C_{\phi_2}(\Gamma) \cong C_{\phi'}(\Gamma') \oplus C_{\phi''}(\Gamma'').$$

Suppose that the two edges in Γ' belong to the same α -cycle. We denote the number of α -cycles in Γ' by m . Note that the number of α -cycles in Γ'' equals $m + 1$. By induction $C_{\phi'}(\Gamma') = R_{\phi'}(\Gamma')\Sigma_{\phi'}(\Gamma')$ and $C_{\phi''}(\Gamma'') = R_{\phi''}(\Gamma'')\Sigma_{\phi''}(\Gamma'')$ are free cyclic modules of dimensions 2^m and 2^{m+1} respectively. Since ϕ_1 is noncanonical, we know that $C_{\phi_1}(\Gamma) \cong C_{\phi'}(\Gamma')$, using the isomorphisms in [\(15\)](#) and the results above about digon-webs. Therefore, we see that $\dim C_{\phi_1}(\Gamma) = 2^m$ and $\dim C_{\phi_2}(\Gamma) = 2^{m+1}$.

By construction, we have

$$\Sigma_{\phi''}(\Gamma'') \mapsto (*, \Sigma_{\phi_2}(\Gamma)).$$

The isomorphism commutes with the actions on the external edges and $R_{\phi_2}(\Gamma)$ is isomorphic to $R_{\phi''}(\Gamma'')$, so we get

$$R_{\phi_2}(\Gamma)\Sigma_{\phi_2}(\Gamma) \cong R_{\phi''}(\Gamma'')\Sigma_{\phi''}(\Gamma'').$$

For dimensional reasons, this implies that $C_{\phi_2}(\Gamma)$ is a free cyclic $R_{\phi_2}(\Gamma)$ -module generated by $\Sigma_{\phi_2}(\Gamma)$.

Now suppose that the two edges in Γ' belong to different α -cycles. This time we denote the number of α -cycles in Γ' and Γ'' by 2^{m+1} and 2^m respectively. We still have $C_{\phi_1}(\Gamma) \cong C_{\phi'}(\Gamma')$, so, by induction, we have $\dim C_{\phi_1}(\Gamma) = 2^{m+1}$ and, therefore, $C_{\phi_2}(\Gamma) = 2^m$. Consider the intermediate web Γ''' colored by ϕ''' as indicated and the map between Γ and Γ''' in Figure 17. By induction, $C_{\phi'''}(\Gamma''')$ is a free cyclic

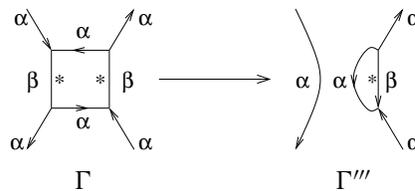


Figure 17

$R_{\phi'''}(\Gamma''')$ -module generated by $\Sigma_{\phi'''}(\Gamma''')$. By construction, we see that $\Sigma_{\phi_2}(\Gamma)$ is mapped to

$$X_6 \Sigma_{\phi'''}(\Gamma''') - X_1 \Sigma_{\phi'''}(\Gamma'''),$$

which is nonzero. Similarly we see that $X_1 \Sigma_{\phi_2}(\Gamma)$ is mapped to

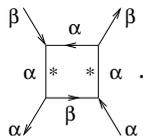
$$X_1 X_6 \Sigma_{\phi'''}(\Gamma''') - X_1^2 \Sigma_{\phi'''}(\Gamma''') = X_1 X_6 \Sigma_{\phi'''}(\Gamma''') - (2\alpha X_1 - \alpha^2) \Sigma_{\phi'''}(\Gamma''').$$

The latter is also nonzero and linearly independent from the first element. Since the map clearly commutes with the action of all elements not belonging to edges in the α -cycle of X_1 , the above shows that, for any nonzero element $Z \in R_{\phi_2}(\Gamma)$, the image of $Z \Sigma_{\phi_2}(\Gamma)$ in $C_{\phi'''}(\Gamma''')$ is nonzero. Therefore, we see that

$$\dim R_{\phi_2}(\Gamma)\Sigma_{\phi_2}(\Gamma) = \dim R_{\phi_2}(\Gamma) = 2^m.$$

For dimensional reasons we conclude that $\mathcal{Q}_{\phi_2}(\Gamma)C(\Gamma)$ is a free cyclic $R_{\phi_2}(\Gamma)$ -module generated by $\Sigma_{\phi_2}(\Gamma)$.

Finally, let us consider the canonical coloring



In this case $C_\phi(\Gamma)$ is isomorphic to $C_{\phi''}(\Gamma'')$, where ϕ'' is the unique compatible canonical coloring of Γ'' , because there is no compatible coloring of Γ' . Note that $R_\phi(\Gamma)$ is isomorphic to $R_{\phi''}(\Gamma'')$ and $\Sigma_\phi(\Gamma)$ is mapped to $\Sigma_{\phi''}(\Gamma'')$. Therefore the result follows by induction. \square

Finally we arrive at our main theorem in this subsection.

Theorem 3.18 *Let $j(L') = 2\text{lk}(L', L \setminus L')$. Then*

$$U_{a,b,c}^i(L, \mathbb{C}) \cong \bigoplus_{L' \subseteq L} KH^{i-j(L'),*}(L', \mathbb{C}).$$

Proof By [Theorem 3.15](#) we know that

$$U_{a,b,c}^i(L, \mathbb{C}) = \bigoplus_{\phi \in S(L)} U_{a,b,c}^i(L_\phi, \mathbb{C}).$$

Let $\phi \in S(L)$ be fixed and let L_α be the sublink of L consisting of the components colored by α . We claim that

$$(16) \quad U_{a,b,c}^i(L_\phi, \mathbb{C}) \cong KH^{i-j(L'),*}(L_\alpha, \mathbb{C}),$$

from which the theorem follows. First note that, without loss of generality, we may assume that $\alpha = 0$, because we can always apply the isomorphism $\square \mapsto \square - \alpha\square$. Let $C_\phi(\Gamma)$ be a canonical cochain group. By [Lemma 3.17](#), we know that $C_\phi(\Gamma)$ is a free cyclic $R_\phi(\Gamma)$ -module generated by $\Sigma_\phi(\Gamma)$. Therefore we can identify any $X \in R_\phi(\Gamma)$ with $X\Sigma_\phi(\Gamma)$. There exist isomorphisms

$$(17) \quad R_\phi(\Gamma) \cong \mathbb{C}[X_i \mid \phi(i) = \alpha] / (X_i + X_j, X_i^2) \cong A^{\otimes m},$$

where $A = \mathbb{C}[X] / (X^2)$. As before, the relations $X_i + X_j = 0$ hold whenever the edges i and j share a common trivalent vertex and m is the number of α -cycles in Γ . Note also that $X_i X_j = 0$ holds, if i and j share a trivalent vertex. The first isomorphism in (17) is immediate, but for the definition of the second isomorphism we have to make some choices. First of all we have to chose an ordering of the arcs of L . This ordering induces a unique ordering on all the unmarked edges of Γ , where we use Bar-Natan's [1] convention that, if an edge in Γ is the fusion of two arcs of L , we

assign to that edge the smallest of the two numbers. Now delete all edges colored by β . Consider a fixed α -cycle. In this α -cycle pick the edge i with the smallest number in our ordering. This edge has an orientation induced by the orientation of L .

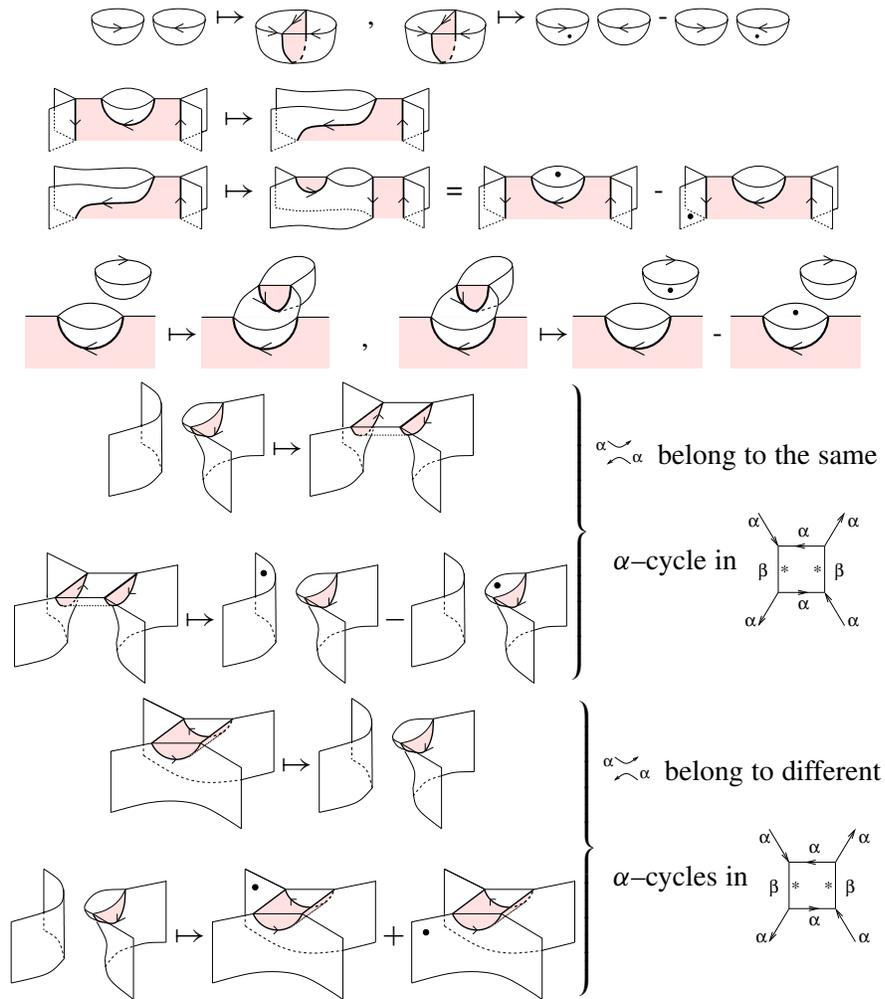


Figure 18: Behavior of canonical generators under elementary cobordisms

We identify the α -cycle with a circle, by deleting all vertices in the α -cycle, oriented according to the orientation of the edge i . If the circle is oriented clockwise we say that it is negatively oriented, otherwise we say that it is positively oriented. The circles corresponding to the α -cycles are ordered according to the order of their minimal

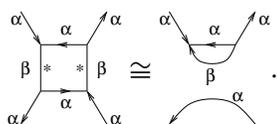
edges. They can be nested. As in Lee’s paper [11] we say that a circle is positively nested if any ray from that circle to infinity crosses the other circles in an even number of points, otherwise we say that it is negatively nested. The isomorphism in (17) is now defined as follows. Given the r -th α -cycle with minimal edge i we define

$$X_i \mapsto \epsilon 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1,$$

where X appears as the r -th tensor factor. If the orientation and the nesting of the α -cycle have the same sign, then $\epsilon = +1$, and if the signs are opposite, then $\epsilon = -1$. The final result, ie the claim of this theorem, holds true no matter which ordering of the arcs of L we begin with. It is easy to work out the behavior of the canonical generators with respect to the elementary cobordisms as can be seen in Figure 18. For the cobordisms shown in Figure 18 having one or more cycles there is also a version with one cycle inside the other cycle or a cycle inside a digon.

The two bottom maps in Figure 18 require some explanation. Both can only be understood by considering all possible closures of the bottoms and sides of their sources and targets. Since, by definition, the canonical generators are constructed step by step introducing the vertices of the webs in some order, we can assume, without loss of generality, that the first vertices in this construction are the ones shown. With this assumption the open webs at the top and bottom of the cobordisms in the figure are to be closed only by simple curves, without vertices, and the closures of these cobordisms, outside the bits which are shown, only use cups and identity cobordisms. Bearing this in mind, the claim implicit in the first map is a consequence of relation (1).

For the second map, recall that $\alpha \rightsquigarrow \alpha$ belong to different α -cycles. Therefore there are two different ways to close the webs in the target and source: two cycles side-by-side or one cycle inside another cycle. We notice that from Theorem 3.15 we have the isomorphism



We apply this isomorphism to the composite of the source foam and the elementary foam and to the target foam of the last map in Figure 18. Finally use equation (1) and relation (CN) to the former to see that both foams are isotopic.

Note that in $C_{a,b,c}^*(L_\phi, \mathbb{C})$ we only have to consider elementary cobordisms at crossings between two strands which are both colored by α . With the identification of $R_\phi(\Gamma)$ and $A^{\otimes m}$ as above, it is now easy to see that the differentials in $C_{a,b,c}^*(L_\phi, \mathbb{C})$ behave exactly as in Khovanov’s original sl_2 -theory for L_α .

The degree of the isomorphism in (16) is easily computed using the fact that in both theories the oriented resolution has homological degree zero. Therefore we get an isomorphism

$$U_{a,b,c}^i(L_\phi, \mathbb{C}) \cong \mathrm{KH}^{i-j(L'),*}(L_\alpha, \mathbb{C}). \quad \square$$

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