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HOPF ALGEBRAS GENERATED BY A COALGEBRA

By

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Abstract. The concept of a free Hopf algebra generated by a coalgebra was introduced by Takeuchi to provide an example of a Hopf algebra with a non-bijective antipode. In general, this free Hopf algebra is not generated as an algebra by the coalgebra. In this paper, we construct a class of Hopf algebras, including $SL_q(2)$, which are generated as algebras by a coalgebra and which satisfy a useful universality condition.

Introduction

The paper is presented in three parts. First, a class of Hopf algebras which are generated as algebras by a coalgebra is constructed. Next, the universality of this class of Hopf algebras is addressed. Finally, relevant examples to this discussion are considered, including $SL_q(2)$.

Most of the important preliminaries can be found in [1] and [2]. In particular, following [1], we will use the superscripts "*op*" and "*cop*" to refer to the opposite algebra and opposite coalgebra, respectively. We will also make use of the well-known fact that the tensor algebra of a coalgebra (C, Δ, ε), denoted $(T(C), \bar{\mu}, \bar{\eta}, \bar{\Delta}, \bar{\varepsilon})$, is a bialgebra. For a reference, see [3].

1. The Construction

LEMMA 1.1. *Suppose that (C, Δ, ε) is a coalgebra, $(B, \mu_B, \eta_B, \Delta_B, \varepsilon_B)$ is a bialgebra, and $f : C \to B$ is a coalgebra map. Then, there exists a unique bialgebra map $\bar{f} : T(C) \to B$ extending f.*

PROOF. By the universality of $T(C)$, we know that f induces a unique algebra map $\bar{f} : T(C) \to B$. It remains to show that $\bar{f}$ is a coalgebra map, which requires $\varepsilon_B \circ \bar{f} = \bar{\varepsilon}$ and $\bar{f} \otimes \bar{f} \circ \bar{\Delta} = \Delta_B \circ \bar{f}$. Identify C with its image in $T(C)$,

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and we have $(\varepsilon_B \circ \bar{f})(c) = \varepsilon_B(\bar{f}(c)) = \varepsilon_B(f(c)) = \varepsilon(c) = \bar{\varepsilon}(c)$ and $(\Delta_B \circ \bar{f})(c) = \Delta_B(\bar{f}(c)) = \Delta_B(f(c)) = (f \otimes f)(\Delta(c)) = (\bar{f} \otimes \bar{f})(\Delta(c)) = (\bar{f} \otimes \bar{f})(\bar{\Delta}(c)) = (\bar{f} \otimes \bar{f} \circ \bar{\Delta})(c)$. □

We now proceed with the construction. Let (C, Δ, ε) be a coalgebra, and let $S : C \to C^{cop}$ be any coalgebra map. In other words, S is a coalgebra antimorphism on C. Then, by Lemma 1.1, S induces a bialgebra map $\bar{S} : T(C) \to T(C)^{op\,cop}$, and we have the commutative diagram

$$\begin{tikzcd}
C \arrow[r, hook, "i"] \arrow[d, "S"'] \arrow[dr, dotted] & T(C) \arrow[d, "\bar{S}"] \\
C^{cop} \arrow[r, hook, "i"] & T(C)^{op\,cop}.
\end{tikzcd}$$

The effect is that S has been extended to $\bar{S}$ in such a way that $\bar{S}(xy) = \bar{S}(y)\bar{S}(x)$, for all $x, y \in T(C)$ and with the property that $\bar{\varepsilon} \circ \bar{S} = \bar{\varepsilon}$ and $\bar{S} \otimes \bar{S} \circ \bar{\Delta} = \bar{\Delta}^{op} \circ \bar{S}$.

Next, let $I = I(S)$ be the two-sided ideal of $T(C)$ generated by elements of the form

$$\sum_{(x)} x'\bar{S}(x'') - \bar{\varepsilon}(x)1 \quad \text{and} \quad \sum_{(x)} \bar{S}(x')x'' - \bar{\varepsilon}(x)1 \quad \forall x \in i(C).$$

LEMMA 1.2. *I is a coideal of $T(C)$ such that $\bar{S}(I) \subseteq I$.*

PROOF. First, we prove that I is a coideal of $T(C)$. This requires that $\bar{\Delta}(I) \subseteq I \otimes T(C) + T(C) \otimes I$ and $\bar{\varepsilon}(I) = 0$. Note that $(\bar{S} \otimes \bar{S}) \circ \bar{\Delta} = \bar{\Delta}^{op} \circ \bar{S} \Leftrightarrow (\bar{S} \otimes \bar{S}) \circ \bar{\Delta}^{op} = \bar{\Delta} \circ \bar{S}$. It suffices to show the first coideal condition is true for the generators of I since $\bar{\Delta}$ is an algebra morphism. We have

$$\begin{aligned}
&\bar{\Delta}\left(\sum_{(x)} x'\bar{S}(x'') - \bar{\varepsilon}(x)1\right) \\
&\quad = \sum_{(x)} \bar{\Delta}(x')\bar{\Delta} \circ \bar{S}(x'') - \bar{\varepsilon}(x)\bar{\Delta}(1) \\
&\quad = \sum_{(x)} \bar{\Delta}(x') \cdot \bar{S} \otimes \bar{S} \circ \bar{\Delta}^{op}(x'') - \bar{\varepsilon}(x)1 \otimes 1 \\
&\quad = \sum_{(x)} x' \otimes x'' \cdot \bar{S}(x'''') \otimes \bar{S}(x''') - \bar{\varepsilon}(x)1 \otimes 1
\end{aligned}$$

$$= \sum_{(x)} x'\bar{S}(x'''') \otimes x''\bar{S}(x''') - \bar{\varepsilon}(x)1 \otimes 1$$

$$= \sum_{(x)} x'\bar{S}(x'''') \otimes [x''\bar{S}(x''') - \bar{\varepsilon}(x'')1 + \bar{\varepsilon}(x'')1] - \bar{\varepsilon}(x)1 \otimes 1$$

$$= \underbrace{\sum_{(x)} x'\bar{S}(x'''') \otimes \underbrace{[x''\bar{S}(x''') - \bar{\varepsilon}(x'')1]}_{\in I}}_{\in T(C)\otimes I} + \sum_{(x)} x'\bar{S}(x'''') \otimes \bar{\varepsilon}(x'')1 - \bar{\varepsilon}(x)1 \otimes 1$$

$$\equiv \sum_{(x)} x'\bar{S}(x'''') \otimes \bar{\varepsilon}(x'')1 - \bar{\varepsilon}(x)1 \otimes 1 \quad \operatorname{mod} I \otimes T(C) + T(C) \otimes I$$

$$\equiv \sum_{(x)} x'\bar{S}(x''') \otimes \bar{\varepsilon}(x'')1 - \bar{\varepsilon}(x)1 \otimes 1 \quad \operatorname{mod} I \otimes T(C) + T(C) \otimes I$$

$$\equiv \sum_{(x)} x'\bar{S}(x'') \otimes \bar{\varepsilon}(x'')1 - \bar{\varepsilon}(x)1 \otimes 1 \quad \operatorname{mod} I \otimes T(C) + T(C) \otimes I$$

$$= \sum_{(x)} x'\bar{\varepsilon}(x'')\bar{S}(x'') \otimes 1 - \bar{\varepsilon}(x)1 \otimes 1$$

$$= \sum_{(x)} x'\bar{S}(x'') \otimes 1 - \bar{\varepsilon}(x)1 \otimes 1$$

$$= \left[\sum_{(x)} x'\bar{S}(x'') - \bar{\varepsilon}(x)1 + \bar{\varepsilon}(x)1\right] \otimes 1 - \bar{\varepsilon}(x)1 \otimes 1$$

$$= \underbrace{\underbrace{\left[\sum_{(x)} x'\bar{S}(x'') - \bar{\varepsilon}(x)1\right]}_{\in I} \otimes 1}_{\in I\otimes T(C)} + \bar{\varepsilon}(x)1 \otimes 1 - \bar{\varepsilon}(x)1 \otimes 1$$

$$\equiv 0 \quad \operatorname{mod} I \otimes T(C) + T(C) \otimes I.$$

The proof uses the coassociative and counitary axioms and is similar for generators of the form $\sum_{(x)} \bar{S}(x')x'' - \bar{\varepsilon}(x)1$, and thus, $\bar{\Delta}(I) \subseteq I \otimes T(C) + T(C) \otimes I$. Using the fact that $\bar{\varepsilon}$ is an algebra morphism, it is easy to show that the second coideal condition holds for the generators of I and so, $\bar{\varepsilon}(I) = 0$.

Lastly, since $\bar{S}$ is an algebra antimorphism, it is enough to show that $\bar{S}(I) \subseteq I$ for generators of I.

$$\bar{S}\left(\sum_{(x)} x'\bar{S}(x'') - \bar{\varepsilon}(x)1\right) = \sum_{(x)} \bar{S}(\bar{S}(x''))\bar{S}(x') - \bar{\varepsilon}(x)\bar{S}(1)$$
$$= [\bar{\mu} \circ (\bar{S} \otimes id) \circ (\bar{S} \otimes \bar{S} \circ \bar{\Delta}^{op})](x) - \bar{\varepsilon}(x)1$$
$$= [\bar{\mu} \circ (\bar{S} \otimes id) \circ (\bar{\Delta} \circ \bar{S})](x) - \bar{\varepsilon} \circ \bar{S}(x)1$$
$$= \sum_{(\bar{S}(x))} \bar{S}(\bar{S}(x)')\bar{S}(x)'' - \bar{\varepsilon}(\bar{S}(x))1$$
$$= \sum_{(y)} \bar{S}(y')y'' - \bar{\varepsilon}(y)1, \quad \text{for } y = \bar{S}(x) \in i(C)$$
$$\equiv 0 \mod I.$$

Thus, $\bar{S}\left(\sum_{(x)} x'\bar{S}(x'') - \bar{\varepsilon}(x)1\right) \in I$, and likewise for generators of the other form. Therefore, $\bar{S}(I) \subseteq I$. □

We summarize the preceding results in the following theorem.

THEOREM 1.3. *Let C be a coalgebra, and $S : C \to C^{cop}$ be any coalgebra map. Then, $\mathscr{H}(C,S) = T(C)/I(S)$ is a Hopf algebra with antipode $\hat{S}$, the unique bialgebra morphism $\hat{S} : \mathscr{H}(C,S) \to \mathscr{H}(C,S)^{op\,cop}$ induced by $\bar{S}$.*

PROOF. As a consequence of Lemma 1.2, $I(S)$ can be factored out of $T(C)$, yielding a nontrivial quotient $(\mathscr{H}(C,S), \hat{\mu}, \hat{\eta}, \hat{\Delta}, \hat{\varepsilon})$ with the structure of a bialgebra. In fact, the induced $\hat{S}$ is the antipode for $\mathscr{H}(C,S)$. Consider the intersection of the kernels of $id * \hat{S} - \hat{\eta} \circ \hat{\varepsilon}$ and $\hat{S} * id - \hat{\eta} \circ \hat{\varepsilon}$. It is a subalgebra of $\mathscr{H}(C,S)$ which contains $i(C)$, and since $i(C)$ generates $\mathscr{H}(C,S)$ as an algebra, we have $id * \hat{S} = \hat{\eta} \circ \hat{\varepsilon} = \hat{S} * id$. □

2. The Universality of $\mathscr{H}(C,S)$

A natural question to ask is: If we begin with a pair (C,S) and construct $\mathscr{H}(C,S)$, in what categorical sense is $\mathscr{H}(C,S)$ free? The following result characterizes the universality of $\mathscr{H}(C,S)$.

THEOREM 2.1. *Given any pair (H,f), where H is a Hopf algebra and $f : C \to H$ is a coalgebra map satisfying $f \circ S = S_H \circ f$, there is a unique Hopf* *algebra morphism* $\hat{f} : \mathscr{H}(C,S) \to H$ *such that* $\hat{f} \circ \imath = f$. *In other words, we have the commutative diagram*

$$\begin{tikzcd} C \arrow[r, "\imath"] \arrow[dr, "f"'] & \mathscr{H}(C,S) \arrow[d, "\hat{f}"] \\ & H \end{tikzcd}$$

where $\imath = \pi \circ i$, *with* $i : C \to T(C)$ *denoting the canonical injection and* $\pi : T(C) \to \mathscr{H}(C,S)$ *denoting the canonical surjection.*

Proof. We have to show that we can lift f to $\mathscr{H}(C,S)$ in the following diagram:

(2.1)
$$\begin{tikzcd} C \arrow[r, hook, "i"] \arrow[dr, "f"'] & T(C) \arrow[r, two heads, "\pi"] \arrow[d, "\bar{f}"] & \mathscr{H}(C,S) \arrow[dl, "\hat{f}"] \\ & H & \end{tikzcd}$$

Beginning with the left side of (2.1), we use Lemma 1.1 to lift f to a bialgebra map $\bar{f} : T(C) \to H$. The assumption $f \circ S = S_H \circ f$ lifts to $\bar{f} \circ \bar{S} = S_H \circ \bar{f}$, where $\bar{S} : T(C) \to T(C)^{op\,cop}$ is the previously constructed bialgebra map. Thus, f induces a bialgebra map $\bar{f} : T(C) \to H$ satisfying $\bar{f} \circ \bar{S} = S_H \circ \bar{f}$.

Next, consider the right side of (2.1). We have reduced the problem to lifting the bialgebra map $\bar{f}$ to a Hopf algebra map $\hat{f} : \mathscr{H}(C,S) \to H$. This requires that $I(S) \subseteq \ker \bar{f}$ and $\hat{f} \circ \bar{S} = S_H \circ \hat{f}$. Clearly, the former condition will hold if and only if $\bar{f}$ annihilates the generators of $I(S)$. Identify C with its image in $T(C)$, and we have

$$\begin{aligned} \bar{f}\left(\sum_{(x)} x'\bar{S}(x'') - \bar{\varepsilon}(x)1\right) &= \sum_{(x)} \bar{f}(x')\bar{f} \circ \bar{S}(x'') - \bar{\varepsilon}(x)\bar{f}(1) \\ &= \sum_{(x)} \bar{f}(x')S_H \circ \bar{f}(x'') - \bar{\varepsilon}(x)1_H \\ &= \sum_{(\bar{f}(x))} \bar{f}(x)'S_H(\bar{f}(x)'') - \varepsilon_H(\bar{f}(x))1_H \\ &= \sum_{(y)} y'S_H(y'') - \varepsilon_H(y)1_H, \quad \text{for } y = \bar{f}(x) \in H \\ &= 0 \end{aligned}$$

Similarly, $\bar{f}\left(\sum_{(x)} \bar{S}(x')x'' - \bar{\varepsilon}(x)1\right) = 0$, and so, $I(S) \subseteq \ker \bar{f}$. The latter condition is immediate. Hence, $\bar{f}$ induces a Hopf algebra map $\hat{f} : \mathscr{H}(C, S) \to H$, and the theorem follows. □

3. Examples of Hopf Algebras $\mathscr{H}(C_2, S)$

In this section, we present some examples, including $SL_q(2)$, obtained from our construction. The following definition is from [4].

DEFINITION 3.1. Let $C_n = C_n(\boldsymbol{C})$ be a coalgebra with basis $\{x_{ij}\}_{1 \leq i,j \leq n}$ over $\boldsymbol{C}$ and structure maps defined by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{ij}.$$

Following Takeuchi, we call C_n the $n \times n$ matric coalgebra since it is isomorphic to M_n^*, the dual of the $n \times n$ matrices with convolution product.

EXAMPLE 3.2. Consider the situation of Theorem 2.1 with $C = C_2$ and $H = SL_q(2)$:

$$\begin{tikzcd} C_2 \arrow[r, "\imath"] \arrow[dr, "f"'] & \mathscr{H}(C_2, S) \arrow[d, "\hat{f}"] \\ & SL_q(2) \end{tikzcd}$$

where f is the coalgebra map defined by $f(x_{11}) = a$, $f(x_{12}) = b$, $f(x_{21}) = c$, $f(x_{22}) = d$, and $S : C_2 \to C_2^{cop}$ is the coalgebra map defined by $S(x_{11}) = x_{22}$, $S(x_{12}) = -qx_{12}$, $S(x_{21}) = -q^{-1}x_{21}$, $S(x_{22}) = x_{11}$. The hypotheses of Theorem 2.1 are easily seen to be satisfied. Thus, there is a Hopf algebra map $\hat{f}$: $\mathscr{H}(C_2, S) \to SL_q(2)$, which we claim is a Hopf algebra isomorphism. Now, $\mathscr{H}(C_2, S) = T(C_2)/I(S)$ where $T(C_2) \cong \boldsymbol{C}\{x_{11}, x_{12}, x_{21}, x_{22}\}$, the free associative algebra on four generators. See [1] for the latter fact. In Kassel's notation, the generators of $I(S)$ can be written in abridged matrix form as

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \bar{S}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \bar{\eta} \circ \bar{\varepsilon}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \tag{3.1}$$

and

$$\bar{S}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \bar{\eta} \circ \bar{\varepsilon}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}. \tag{3.2}$$

In addition, $SL_q(2)$ is defined in [1] as the quotient of the free associative algebra $\boldsymbol{C}\{a,b,c,d\}$ by the two-sided ideal with generators given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

and

$$\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.4}$$

in abridged matrix form. We will construct a two-sided inverse for $\hat{f}$. There exists an algebra map $g : \boldsymbol{C}\{a,b,c,d\} \to \mathscr{H}(C_2, S)$ defined by $g(a) = x_{11}$, $g(b) = x_{12}$, $g(c) = x_{21}$, and $g(d) = x_{22}$. Notice that under g, expressions of the form (3.3) and (3.4) are mapped to (3.1) and (3.2), respectively, and these images are zero in $\mathscr{H}(C_2, S)$. Thus, g induces a Hopf algebra map $\hat{g} : SL_q(2) \to \mathscr{H}(C_2, S)$ with $\hat{f} \circ \hat{g} = id_{SL_q(2)}$ and $\hat{g} \circ \hat{f} = id_{\mathscr{H}(C_2,S)}$. Therefore, $\hat{f}$ is an isomorphism of Hopf algebras, and we have the following result.

THEOREM 3.3. *With the coalgebra map S of Example* 3.2, *$\mathscr{H}(C_2, S)$ is isomorphic to $SL_q(2)$.*

EXAMPLE 3.4. Now, we will turn our attention to a slightly different question involving C_2. Example 3.2 suggests a general situation in which we can ask: Are there other coalgebra maps $S : C_2 \to C_2^{cop}$ which yield Hopf algebras $\mathscr{H}(C_2, S)$ that are not isomorphic to $SL_q(2)$? Since the dimension of C_2 is small, we can use Mathematica to search for solutions. Any coalgebra map $S : C_2 \to C_2^{cop}$ must be of the form:

$$S(x_{11}) = a_{11}x_{11} + a_{12}x_{12} + a_{13}x_{21} + a_{14}x_{22}$$

$$S(x_{12}) = a_{21}x_{11} + a_{22}x_{12} + a_{23}x_{21} + a_{24}x_{22}$$

$$S(x_{21}) = a_{31}x_{11} + a_{32}x_{12} + a_{33}x_{21} + a_{34}x_{22}$$

$$S(x_{22}) = a_{41}x_{11} + a_{42}x_{12} + a_{43}x_{21} + a_{44}x_{22}$$

where $a_{ij} \in \boldsymbol{C}$ for $1 \le i, j \le 4$. Moreover, since $S : C_2 \to C_2^{cop}$ is a coalgebra map, it must satisfy the abridged matrix relations:

$$S \otimes S \circ \Delta^{op} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \Delta \circ S \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \tag{3.5}$$

and

$$\varepsilon \circ S \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \varepsilon \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}. \tag{3.6}$$

The equations from (3.5) can be expanded out and written in terms of a basis for $C_2 \otimes C_2$, namely $\{x_{ij} \otimes x_{kl}\}_{1 \le i,j,k,l \le 2}$ to yield 64 equations upon equating coefficients. From (3.6), there are 4 additional equations. We use Mathematica to solve the 68 equations in 16 unknowns a_{ij}, $1 \le i, j \le 4$. In particular, this search found the coalgebra map S of Example 3.2 and Theorem 3.3 among the solutions. It can be expressed as

$$S \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{22} & -qx_{12} \\ -q^{-1}x_{21} & x_{11} \end{pmatrix}. \tag{3.7}$$

In addition, there were several other families of solutions, including a simple one given in abridged matrix form by

$$T \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} & qx_{21} \\ q^{-1}x_{12} & x_{22} \end{pmatrix}. \tag{3.8}$$

Notice that S is the quantum analogue to the inverse map and that T is the quantum analogue to the transpose map.

Moreover, $\mathscr{H}(C_2, S)$ and $\mathscr{H}(C_2, T)$ are not isomorphic. This can be seen by computing S^2 and T^2. We have

$$S^2 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} S(x_{22}) & -qS(x_{12}) \\ -q^{-1}S(x_{21}) & S(x_{11}) \end{pmatrix} = \begin{pmatrix} x_{11} & q^2x_{12} \\ q^{-2}x_{21} & x_{22} \end{pmatrix} \tag{3.9}$$

and

$$T^2 \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} T(x_{11}) & qT(x_{21}) \\ q^{-1}T(x_{12}) & T(x_{22}) \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}. \tag{3.10}$$

Equations (3.9) and (3.10) imply that S is of infinite order and T is of finite order, respectively. In addition, S^2 and T^2 do not have the same set of eigenvalues because T^2 has only real eigenvalues, and S^2 has some complex eigenvalues. This guarantees that $\mathscr{H}(C_2, S)$ and $\mathscr{H}(C_2, T)$ are not isomorphic because any isomorphism between them would have to preserve the eigenvalues for the antipodes and their powers. Example 3.4 shows that the construction of $\mathscr{H}(C, S)$ depends on both C and S.

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