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SELFINJECTIVITY OF RINGS RELATIVE TO LAMBEK TORSION THEORY

By

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Throughout this note R stands for an associative ring with identity, modules are unitary modules and torsion theories are Lambek torsion theories. We use the prefix "τ–" to mean "relative to Lambek torsion theory".

In this note we call a ring R left τ-selfinjective if $\mathrm{Ext}_R^1(X, R)$ is torsion for every left R-module X. Our main aim is to characterize left τ-selfinjective rings R by a certain kind of linear compactness. Recall that a module X is called absolutely pure if $\mathrm{Ext}_R^1(-, X)$ vanishes on the finitely presented modules. Also, let us call a module X semicompact if $\varprojlim \pi_\lambda$ is an epimorphism for every inverse system of epimorphisms $\{\pi_\lambda : X \to Y_\lambda\}_{\lambda \in \Lambda}$ with the Y_λ torsionless. Then, as pointed out by Stenström [18], the argument of Matlis [13, Propositions 2 and 3] yields that a ring R is left selfinjective if and only if it is left absolutely pure and right semicompact. It is shown in [9] that $\mathrm{Ext}_R^1(R/I, R)$ is torsion for every left ideal I of R if and only if R is τ-absolutely pure and right τ-semicompact. However, since τ-epimorphisms are not necessarily set-theoretic surjections, Baer's lemma does not work. Namely, even if $\mathrm{Ext}_R^1(R/I, R)$ is torsion for every left ideal I of R, R is not necessarily left τ-selfinjective. So we need a rather strong notion of linear compactness to characterize left τ-selfinjective rings R.

We are also concerned with an arbitrary class of left R-modules $\mathscr{C}$ which contains ${}_RR$ and is closed under taking factor modules and extensions. We ask when every submodule X of $E({}_RR)$, the injective envelope of ${}_RR$, with $X \in \mathscr{C}$ is torsionless. In various situations, this problem has been considered by several authors (e.g., [3], [1], [16], [20], [2], [6], [7], [4], [15] and [8]). As a particular case, we study the class of all τ-finitely generated modules.

In the following, we denote by $\mathrm{Mod}\, R$ the category of left R-modules. Right R-modules are considered as left R^{op}-modules, where R^{op} denotes the opposite ring of R. Sometimes, we use the notation ${}_RX$(resp. X_R) to stress that

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the module X considered is a left (resp. right) R-module. For a module X we denote by $E(X)$ its injective envelope. We denote by $(\ )^*$ both the R-dual functors and for a module X we denote by $\varepsilon_X : X \to X^{**}$ the usual evaluation map. A module X is called torsionless (resp. reflexive) if ε_X is a monomorphism (resp. an isomorphism). For a module $X \in \operatorname{Mod} R$ we denote by $\tau(X)$ its Lambek torsion submodule. Namely, $\tau(X)$ is a submodule of X such that $\operatorname{Hom}_R(\tau(X), E({}_R R)) = 0$ and $X/\tau(X)$ is cogenerated by $E({}_R R)$. Then a module X is called torsion (resp. torsionfree) if $\tau(X) = X$ (resp. $\tau(X) = 0$). Note that torsionless modules are torsionfree. Finally, a submodule Y of a module X is called a dense (resp. closed) submodule of X if X/Y is torsion (resp. torsionfree).

1. Preliminaries

In this section, we collect several basic results which we need in later sections.

Note first that $\operatorname{Ker} \varepsilon_X \subset Y$ (resp. $\tau(X) \subset Y$) for every submodule Y of X with X/Y torsionless (resp. torsionfree). In particular, since torsionless modules are torsionfree, $\tau(X) \subset \operatorname{Ker} \varepsilon_X$ for every module X.

The first three lemmas are obvious.

LEMMA 1.1. *A module X is torsion if and only if $Y^* = 0$ for every (cyclic) submodule Y of X.* □

LEMMA 1.2. *For a module X the following are equivalent.*
(a) $\tau(X) = \operatorname{Ker} \varepsilon_X$.
(b) $\operatorname{Ker} \varepsilon_X$ is torsion.
(c) $X/\tau(X)$ *is torsionless.* □

LEMMA 1.3. *Let $\mu : X \to Y$ be a monomorphism. Then the following hold.*
(1) $\mu^* = 0$ *if and only if* $\varepsilon_Y \circ \mu = 0$.
(2) *If* $\operatorname{Ker} \varepsilon_Y$ *is torsion, so is* $\operatorname{Ker} \varepsilon_X$. □

LEMMA 1.4 ([7, Theorem A]). *For a ring R the following are equivalent.*
(a) $\tau(X) = \operatorname{Ker} \varepsilon_X$ *for every finitely presented* $X \in \operatorname{Mod} R$.
(a)$^{\mathrm{op}}$ $\tau(M) = \operatorname{Ker} \varepsilon_M$ *for every finitely presented* $M \in \operatorname{Mod} R^{\mathrm{op}}$. □

We call a ring R τ-absolutely pure if it satisfies the equivalent conditions in Lemma 1.4. Recall that a homomorphism $\pi : X \to Y$ is called a τ-epimorphism if $\operatorname{Cok}\pi$ is torsion. We call a module X τ-semicompact if $\varprojlim \pi_\lambda$ is a τ-epimorphism for every inverse system of τ-epimorphisms $\{\pi_\lambda : X \to Y_\lambda\}_{\lambda\in\Lambda}$ with the Y_λ torsionless (see [9] for details).

LEMMA 1.5 ([8, Theorem 1.2]). *For a ring R the following are equivalent.*
(a) $\tau(X) = \operatorname{Ker}\varepsilon_X$ *for every finitely generated* $X \in \operatorname{Mod} R$.
(b) *R is τ-absolutely pure and right τ-semicompact.* □

LEMMA 1.6 (cf. [10, Theorem 1.1]). *Let $\pi : F \to X$ be an epimorphism with F finitely generated free and put $M = \operatorname{Cok}\pi^*$. Then the following hold.*
(1) $\operatorname{Cok}\varepsilon_X \cong \operatorname{Ext}^1_R(M,R)$.
(2) $(\operatorname{Ker}\varepsilon_X)^*$ *embeds in* $\operatorname{Cok}\varepsilon_M$.

PROOF. (1) Obvious.
(2) Let $\phi : F^* \to M$ denote the canonical epimorphism and put $Y = \operatorname{Cok}\phi^*$. Then $Y \cong \operatorname{Im}\varepsilon_X$ and by the part (1) $\operatorname{Ext}^1_R(Y,R) \cong \operatorname{Cok}\varepsilon_M$. Thus by Lemma 1.3(1) the exact sequence $0 \to \operatorname{Ker}\varepsilon_X \to X \to Y \to 0$ yields the desired embedding. □

LEMMA 1.7. *Let $0 \to X \xrightarrow{\mu} Y \to Z \to 0$ be an exact sequence with $\operatorname{Ker}\varepsilon_Z$ and $\operatorname{Cok}\mu^*$ torsion. Then, if $\operatorname{Cok}\varepsilon_Y$ is torsion, so is $\operatorname{Cok}\varepsilon_X$.*

PROOF. Since μ^{**} is monic, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\mu"] \arrow[d, "\varepsilon_X"'] & Y \arrow[r, "\pi"] \arrow[d, "\varepsilon_Y"] & Z \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & X^{**} \arrow[r, "\mu^{**}"] & Y^{**} \arrow[r, "\phi"] & W \arrow[r] & 0.
\end{tikzcd}$$

By Snake lemma we get an exact sequence $\operatorname{Ker}\alpha \to \operatorname{Cok}\varepsilon_X \to \operatorname{Cok}\varepsilon_Y$, so that it suffices to show that $\operatorname{Ker}\alpha$ is torsion. Since $\pi^{**}\circ\mu^{**} = 0$, $\pi^{**} = \beta\circ\phi$ for some $\beta : W \to Z^{**}$. Then $\beta\circ\alpha\circ\pi = \beta\circ\phi\circ\varepsilon_Y = \pi^{**}\circ\varepsilon_Y = \varepsilon_Z\circ\pi$, thus $\beta\circ\alpha = \varepsilon_Z$ because π is epic. Hence $\operatorname{Ker}\alpha \subset \operatorname{Ker}\varepsilon_Z$ and $\operatorname{Ker}\alpha$ is torsion. □

LEMMA 1.8. *Let $\pi : X \to Y$ be a τ-epimorphism. Then, if X is τ-semicompact, so is Y.*

Proof. Let $\{\pi_\lambda : Y \to Z_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of τ-epimorphisms with the Z_λ torsionless. For each $\lambda \in \Lambda$ we have an exact sequence $\operatorname{Cok} \pi \to \operatorname{Cok}(\pi_\lambda \circ \pi) \to \operatorname{Cok} \pi_\lambda \to 0$ and thus $\operatorname{Cok}(\pi_\lambda \circ \pi)$ is torsion, so that $\operatorname{Cok}(\varprojlim \pi_\lambda \circ \pi)$ is torsion. Next, since $\varprojlim \pi_\lambda \circ \pi = (\varprojlim \pi_\lambda) \circ \pi$, we have an epimorphism $\operatorname{Cok}(\varprojlim \pi_\lambda \circ \pi) \to \operatorname{Cok}(\varprojlim \pi_\lambda)$. Thus $\operatorname{Cok}(\varprojlim \pi_\lambda)$ is torsion. □

The next lemma has been shown in the proof of [9, Proposition 2.4]. However, for completeness, we include a proof.

Lemma 1.9. *Let X be a module with $\operatorname{Cok} \varepsilon_X$ torsion. Suppose $\operatorname{Cok} \mu^*$ is torsion for every monomorphism $\mu : M \to X^*$. Then X is τ-semicompact.*

Proof. Let $\{\pi_\lambda : X \to Y_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of τ-epimorphisms with the Y_λ torsionless. Since each π_λ^* is monic, so is $\varinjlim \pi_\lambda^*$. Thus $\operatorname{Cok}(\varprojlim \pi_\lambda^{**}) \cong \operatorname{Cok}((\varinjlim \pi_\lambda^*)^*)$ is torsion. Since $(\varprojlim \varepsilon_{Y_\lambda}) \circ (\varprojlim \pi_\lambda) = (\varprojlim \pi_\lambda^{**}) \circ \varepsilon_X$, $\varprojlim \varepsilon_{Y_\lambda}$ induces homomorphisms $\alpha : \operatorname{Im}(\varprojlim \pi_\lambda) \to \operatorname{Im}(\varprojlim \pi_\lambda^{**})$ and $\beta : \operatorname{Cok}(\varprojlim \pi_\lambda) \to \operatorname{Cok}(\varprojlim \pi_\lambda^{**})$. We have an epimorphism $\operatorname{Cok} \varepsilon_X \to \operatorname{Cok} \alpha$. Also, since $\varprojlim \varepsilon_{Y_\lambda}$ is monic, by Snake lemma we have a monomorphism $\operatorname{Ker} \beta \to \operatorname{Cok} \alpha$. Consequently, $\operatorname{Ker} \beta$ is torsion, so is $\operatorname{Cok}(\varprojlim \pi_\lambda)$. □

2. Strongly exact full subcategories

Throughout this section $\mathscr{C}$ stands for a class of modules in Mod R. We ask when every submodule X of $E({}_RR)$ with $X \in \mathscr{C}$ is torsionless. In various situations, this problem has been considered by several authors (e.g., [3], [1], [16], [20], [2], [6], [7], [4], [15] and [8]).

The next lemma is obvious (cf. Lemma 1.2).

Lemma 2.1. *Suppose $\mathscr{C}$ is closed under taking factor modules. Then the following are equivalent.*

(a) *Every submodule X of $E({}_RR)$ with $X \in \mathscr{C}$ is torsionless.*

(b) $\tau(X) = \operatorname{Ker} \varepsilon_X$ *for every* $X \in \mathscr{C}$. □

Lemma 2.2 (cf. [8, Theorem 1.2]). *Suppose ${}_RR \in \mathscr{C}$ and $\mathscr{C}$ is closed under taking factor modules and extensions. Then the following are equivalent.*

(a) $\tau(X) = \operatorname{Ker} \varepsilon_X$ *for every* $X \in \mathscr{C}$.

(b) $\operatorname{Ext}_R^1(X, R)$ *is torsion for every* $X \in \mathscr{C}$.

PROOF. (a) $\Rightarrow$ (b). Let $0 \to K \to F \to X \to 0$ be an exact sequence with F free and $X \in \mathscr{C}$. Let $\pi : K^* \to \mathrm{Ext}_R^1(X, R)$ denote the canonical epimorphism and let $h \in K^*$. It suffices to show $(\pi(h)R_R)^* = 0$. Let us form a push-out diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "h"'] & F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R \arrow[r, "\phi"] & Y \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

Then $\pi(h)R_R$ is a homomorphic image of $\mathrm{Cok}\,\phi^*$. Since $X \in \mathscr{C}$ and ${}_RR \in \mathscr{C}$, $Y \in \mathscr{C}$ and $\mathrm{Ker}\,\varepsilon_Y$ is torsion. Thus $\mathrm{Im}\,\phi \cap \mathrm{Ker}\,\varepsilon_Y = 0$ and $\phi^{**} \circ \varepsilon_R = \varepsilon_Y \circ \phi$ is monic. Hence ϕ^{**} is monic and $(\mathrm{Cok}\,\phi^*)^* = 0$.

(b) $\Rightarrow$ (a). Let $X \in \mathscr{C}$ and let Y be a submodule of $\mathrm{Ker}\,\varepsilon_X$. We have only to show $Y^* = 0$. By Lemma 1.3(1) the exact sequence $0 \to Y \to X \to X/Y \to 0$ yields an embedding $Y^* \to \mathrm{Ext}_R^1(X/Y, R)$ with $X/Y \in \mathscr{C}$, so that Y^* is torsion and $Y^* = 0$. $\square$

LEMMA 2.3 (cf. [8, Theorem 1.2]). *Suppose ${}_RR \in \mathscr{C}$ and $\mathscr{C}$ is closed under taking factor modules and finite direct sums. Then the following are equivalent.*

(a) $\tau(X) = \mathrm{Ker}\,\varepsilon_X$ *for every* $X \in \mathscr{C}$.

(b) $\mathrm{Cok}\,\mu^*$ *is torsion for every monomorphism* $\mu : Y \to X$ *in* $\mathrm{Mod}\,R$ *with* $X \in \mathscr{C}$.

PROOF. (a) $\Rightarrow$ (b). Let $\mu : Y \to X$ be a monomorhism in $\mathrm{Mod}\,R$ with $X \in \mathscr{C}$. Let $\pi : Y^* \to \mathrm{Cok}\,\mu^*$ denote the canonical epimorphism and let $h \in Y^*$. Form a push-out square:

$$\begin{tikzcd}
Y \arrow[r, "\mu"] \arrow[d, "h"'] & X \arrow[d] \\
R \arrow[r, "\phi"] & Z.
\end{tikzcd}$$

Then $\pi(h)R_R$ is a homomorphic image of $\mathrm{Cok}\,\phi^*$. Also, since ${}_RR \oplus X \in \mathscr{C}$ and Z is a factor module of ${}_RR \oplus X$, $Z \in \mathscr{C}$. Thus, as in the proof of (a) $\Rightarrow$ (b) in Lemma 2.2, $(\pi(h)R_R)^* = 0$ and $\mathrm{Cok}\,\mu^*$ is torsion.

(b) $\Rightarrow$ (a). Let $X \in \mathscr{C}$ and let Y be a submodule of $\mathrm{Ker}\,\varepsilon_X$. Let $\mu : Y \to X$ denote the inclusion. Then by Lemma 1.3(1) $Y^* \cong \mathrm{Cok}\,\mu^*$, so that Y^* is torsion and $Y^* = 0$. $\square$

LEMMA 2.4. *Suppose $\mathscr{C}$ is closed under taking factor modules and extensions. Let $\hat{\mathscr{C}}$ be the class of all modules $X \in \operatorname{Mod} R$ which can be embedded in some $Y \in \mathscr{C}$. Then the following hold.*

(1) *$\hat{\mathscr{C}}$ is closed under taking submodules, factor modules and finite direct sums.*

(2) *For an exact sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{Mod} R$ with $Z \in \mathscr{C}$, $X \in \hat{\mathscr{C}}$ implies $Y \in \hat{\mathscr{C}}$.*

PROOF. (1) Obvious.

(2) Let $\mu : X \to X'$ be a monomorphism with $X' \in \mathscr{C}$ and form a push-out diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} & & \| & & \\ 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

Then ν is monic with $Y' \in \mathscr{C}$. □

THEOREM 2.5. *Suppose ${}_R R \in \mathscr{C}$ and $\mathscr{C}$ is closed under taking factor modules and extensions. Let $\hat{\mathscr{C}}$ be the class of all modules $X \in \operatorname{Mod} R$ which can be embedded in some $Y \in \mathscr{C}$. Then the following are equivalent.*

(a) *Every submodule X of $E({}_R R)$ with $X \in \mathscr{C}$ is torsionless.*

(b) *$\tau(X) = \operatorname{Ker} \varepsilon_X$ for every $X \in \mathscr{C}$.*

(c) *$\tau(X) = \operatorname{Ker} \varepsilon_X$ for every $X \in \hat{\mathscr{C}}$.*

(d) *$\operatorname{Ext}_R^1(X, R)$ is torsion for every $X \in \mathscr{C}$.*

(e) *$\operatorname{Cok} \mu^*$ is torsion for every monomorphism $\mu : X \to Y$ in $\hat{\mathscr{C}}$.*

PROOF. (a) $\Leftrightarrow$ (b). By Lemma 2.1.

(b) $\Rightarrow$ (c). By Lemma 1.3(2).

(c) $\Rightarrow$ (b). Obvious.

(b) $\Leftrightarrow$ (d). By Lemma 2.2.

(c) $\Leftrightarrow$ (e). By Lemmas 2.4(1) and 2.3. □

PROPOSITION 2.6 (cf. [20, Theorem 2]). *Suppose $\mathscr{C}$ is closed under taking submodules and factor modules. Then the following are equivalent.*

(1) *Every submodule X of $E({}_R R)$ with $X \in \mathscr{C}$ is torsionless.*

(2) *$\tau(X) = \operatorname{Ker} \varepsilon_X$ for every $X \in \mathscr{C}$.*

(3) (a) *Every* $X \in \mathscr{C}$ *with* $X^* = 0$ *is torsion.*

(b) *For an exact sequence* $0 \to X \to Y \to Z \to 0$ *in* $\operatorname{Mod} R$ *with* $Y \in \mathscr{C}$, *if both* X *and* Z *are torsionless, so is* Y.

PROOF. (1)$\Leftrightarrow$(2). By Lemma 2.1.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (2). Let $X \in \mathscr{C}$ and $h \in (\operatorname{Ker} \varepsilon_X)^*$. It suffices to show $h = 0$. Let $\mu : \operatorname{Ker} \varepsilon_X \to X$ denote the inclusion and form the push-out of μ and h:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \varepsilon_X \arrow[r, "\mu"] \arrow[d] & X \arrow[r] \arrow[d, "f"] & \operatorname{Im} \varepsilon_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Im} h \arrow[r] & Y \arrow[r] & \operatorname{Im} \varepsilon_X \arrow[r] & 0.
\end{tikzcd}$$

Then Y is torsionless. Thus $f \circ \mu = 0$ because $\varepsilon_Y \circ f \circ \mu = f^{**} \circ \varepsilon_X \circ \mu = 0$, so that $\operatorname{Im} h = 0$. □

3. τ-Finitely generated modules

Recall that a module X is called τ-finitely generated if it contains a finitely generated dense submodule. In particular, every torsion module is τ-finitely generated. Throughout this section, we denote by $\mathscr{C}(R)$ the class of all τ-finitely generated $X \in \operatorname{Mod} R$ and by $\hat{\mathscr{C}}(R)$ the class of all $X \in \operatorname{Mod} R$ which can be embedded in some $Y \in \mathscr{C}(R)$.

Note that a module X is τ-finitely generated if and only if there exists a τ-epimorphism $\pi : F \to X$ with F finitely generated free, and that composites of τ-epimorphisms are also τ-epimorphisms. Thus the next lemma follows.

LEMMA 3.1. *The class* $\mathscr{C}(R)$ *is closed under taking factor modules and extensions.* □

Since the class of all finitely generated $X \in \operatorname{Mod} R$ is also closed under taking factor modules and extensions, in the following we apply results in Section 2 to finitely generated modules as well as τ-finitely generated modules.

LEMMA 3.2. *Let* Q *be a maximal left quotient ring of* R. *Then the following are equivalent.*

(a) ${}_RQ$ *is torsionless.*

(b) $\operatorname{Ext}^1_R(X, R)$ *is torsion for every torsion* $X \in \operatorname{Mod} R$.

PROOF. Let $\mu : {}_RR \to {}_RQ$ denote the inclusion. Since μ is an essential monomorphism and $\varepsilon_Q \circ \mu = \mu^{**} \circ \varepsilon_R$, it follows that ${}_RQ$ is torsionless if and only if μ^{**} is monic.

(a) $\Rightarrow$ (b). Let $0 \to K \to F \to X \to 0$ be an exact sequence in $\operatorname{Mod} R$ with X torsion and F free, and let $\pi : K^* \to \operatorname{Ext}_R^1(X, R)$ denote the canonical epimorphism. Let $h \in K^*$ and form a push-out diagram:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "h"'] & F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R \arrow[r, "\phi"] & Y \arrow[r] & X \arrow[r] & 0. \end{tikzcd}$$

Then $\pi(h)R_R$ is a homomorphic image of $\operatorname{Cok} \phi^*$, so that it suffices to show $(\operatorname{Cok} \phi^*)^* = 0$. Since $\operatorname{Hom}_R(\phi, Q)$ is a bijection, $\mu = f \circ \phi$ for some $f : {}_RY \to {}_RQ$. Thus $\mu^* = \phi^* \circ f^*$ and we get an epimorphism $\operatorname{Cok} \mu^* \to \operatorname{Cok} \phi^*$. Since μ^{**} is monic, $(\operatorname{Cok} \mu^*)^* = 0$ and thus $(\operatorname{Cok} \phi^*)^* = 0$.

(b) $\Rightarrow$ (a). Since $\operatorname{Cok} \mu^*$ embeds in $\operatorname{Ext}_R^1({}_RQ/R, R)$, $\operatorname{Cok} \mu^*$ is torsion and thus μ^{**} is monic. $\square$

REMARK. Let Q be a maximal left quotient ring of R. It follows from [11, Proposition 2] and [19, Proposition 6] that every finitely generated submodule of ${}_RQ$ is torsionless if and only if $\operatorname{Ext}_R^1(X, R)$ is torsion for every finitely generated torsion $X \in \operatorname{Mod} R$. A slight modification of the proof above provides a direct proof of this fact. Also, it follows from Lemma 1.1 and [8, Lemma 5.2] that ${}_RQ$ is torsionless if and only if arbitrary direct products of copies of $(Q/R)_R$ are torsion.

PROPOSITION 3.3. *Let Q be a maximal left quotient ring of R. Then the following are equivalent.*

(1) $\tau(X) = \operatorname{Ker} \varepsilon_X$ *for every* $X \in \mathscr{C}(R)$.

(2) (a) $\tau(X) = \operatorname{Ker} \varepsilon_X$ *for every finitely generated* $X \in \operatorname{Mod} R$.

(b) ${}_RQ$ *is torsionless.*

PROOF. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\operatorname{Mod} R$ with X finitely generated and Z torsion. By Lemmas 3.1 and 2.2 it suffices to show that $\operatorname{Ext}_R^1(Y, R)$ is torsion. Since $\operatorname{Ext}_R^1(X, R)$ is torsion by Lemma 2.2 and $\operatorname{Ext}_R^1(Z, R)$ is torsion by Lemma 3.2, it follows that $\operatorname{Ext}_R^1(Y, R)$ is torsion. $\square$

Recall that a dense right ideal I of R is called a minimal dense right ideal of R if it is contained in every dense right ideal of R. Note that R has a minimal dense right ideal if and only if arbitrary direct products of torsion right modules are torsion.

COROLLARY 3.4. *Suppose R has a minimal dense right ideal. Then the following are equivalent.*

(a) $\tau(X) = \operatorname{Ker}\varepsilon_X$ *for every* $X \in \mathscr{C}(R)$.

(b) $\tau(X) = \operatorname{Ker}\varepsilon_X$ *for every finitely generated* $X \in \operatorname{Mod} R$.

PROOF. (a) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (a). Let Q be a maximal left quotient ring of R. Since ${}_RQ$ embeds in $E({}_RR)$, by Lemma 2.1 every finitely generated submodule of ${}_RQ$ is torsionless. Thus by [9, Proposition 5.6] ${}_RQ$ is torsionless and Proposition 3.3 applies. □

LEMMA 3.5. *Suppose R is τ-absolutely pure and left τ-semicompact. Then the following hold.*

(1) $\operatorname{Cok}\varepsilon_X$ *is torsion for every* $X \in \mathscr{C}(R)$.

(2) *Every* $X \in \mathscr{C}(R)$ *is τ-semicompact.*

PROOF. (1) Let $\pi : F \to X$ be a τ-epimorphism with F finitely generated free and put $M = \operatorname{Cok}\pi^*$. Since π^* is monic, $\operatorname{Cok}\pi^{**} \cong \operatorname{Ext}^1_R(M, R)$, so that by Lemmas 1.5 and 2.2 $\operatorname{Cok}\pi^{**}$ is torsion. Since F is reflexive, we have an epimorphism $\operatorname{Cok}\pi^{**} \to \operatorname{Cok}\varepsilon_X$ and thus $\operatorname{Cok}\varepsilon_X$ is torsion.

(2) Let Y be a finitely generated dense submodule of X. Then by [8, Corollary 1.5] Y is τ-semicompact and hence by Lemma 1.8 so is X. □

PROPOSITION 3.6. *Suppose* $\tau(X) = \operatorname{Ker}\varepsilon_X$ *for every* $X \in \mathscr{C}(R)$. *Then* $X^* \in \hat{\mathscr{C}}(R^{\mathrm{op}})$ *for every* $X \in \hat{\mathscr{C}}(R)$.

PROOF. Let $\pi : F \to Y$ be a τ-epimorphism with F finitely generated free. Then π^* is monic with $F^* \in \mathscr{C}(R^{\mathrm{op}})$, so that $Y^* \in \hat{\mathscr{C}}(R^{\mathrm{op}})$. Next, let $\mu : X \to Y$ be a monomorphism in $\operatorname{Mod} R$ with $Y \in \mathscr{C}(R)$. Since $Y^* \in \hat{\mathscr{C}}(R^{\mathrm{op}})$, by Lemma 2.4 (1) $\operatorname{Im}\mu^* \in \hat{\mathscr{C}}(R^{\mathrm{op}})$. Also, by Lemma 2.3 $\operatorname{Cok}\mu^*$ is torsion and $\operatorname{Cok}\mu^* \in \mathscr{C}(R^{\mathrm{op}})$. Thus by Lemma 2.4(2) $X^* \in \hat{\mathscr{C}}(R^{\mathrm{op}})$. □

THEOREM 3.7. *Suppose* $\tau(X) = \operatorname{Ker}\varepsilon_X$ *for every* $X \in \mathscr{C}(R)$ *and R is left τ-semicompact. Then the following hold.*

(1) *Both* $\operatorname{Ker}\varepsilon_X$ *and* $\operatorname{Cok}\varepsilon_X$ *are torsion for every* $X \in \hat{\mathscr{C}}(R)$.

(2) $(\ )^{**}$ *induces a mono-preserving endofunctor of* $\hat{\mathscr{C}}(R)$.

(3) *A module* $X \in \hat{\mathscr{C}}(R)$ *is reflexive if* $\operatorname{Ext}_R^i(-, X)$ *vanishes on the torsion modules for* $i = 0$ *and* 1.

Proof. Let $X \in \hat{\mathscr{C}}(R)$.

(1) By Theorem 2.5 $\operatorname{Ker}\varepsilon_X = \tau(X)$ is torsion. Next, let $0 \to X \xrightarrow{\mu} Y \to Z \to 0$ be an exact sequence in $\operatorname{Mod} R$ with $Y \in \mathscr{C}(R)$. Since $Z \in \mathscr{C}(R)$, $\operatorname{Ker}\varepsilon_Z$ is torsion. Also, by Lemma 2.3 $\operatorname{Cok}\mu^*$ is torsion. Thus, since by Lemma 3.5(1) $\operatorname{Cok}\varepsilon_Y$ is torsion, by Lemma 1.7 so is $\operatorname{Cok}\varepsilon_X$.

(2) By Lemma 2.4(1) $\operatorname{Im}\varepsilon_X \in \hat{\mathscr{C}}(R)$. Also, since $\operatorname{Cok}\varepsilon_X$ is torsion, $\operatorname{Cok}\varepsilon_X \in \mathscr{C}(R)$. Thus by Lemma 2.4(2) $X^{**} \in \hat{\mathscr{C}}(R)$. It then follows by Theorem 2.5 that the functor $(\ )^{**} : \hat{\mathscr{C}}(R) \to \hat{\mathscr{C}}(R)$ is mono-preserving.

(3) Suppose $\operatorname{Ext}_R^i(-, X)$ vanishes on the torsion modules for $i = 0$ and 1. Then $\operatorname{Hom}_R(\operatorname{Ker}\varepsilon_X, X) = 0$ implies $\operatorname{Ker}\varepsilon_X = 0$ and $\operatorname{Ext}_R^1(\operatorname{Cok}\varepsilon_X, X) = 0$ implies ε_X a splitting monomorphism. Finally, $\operatorname{Hom}_R(\operatorname{Cok}\varepsilon_X, X^{**}) = 0$ implies Cok $\varepsilon_X = 0$. □

Proposition 3.8. *Suppose* $\tau(X) = \operatorname{Ker}\varepsilon_X$ *for every* $X \in \mathscr{C}(R)$ *and* $\tau(M) = \operatorname{Ker}\varepsilon_M$ *for every* $M \in \mathscr{C}(R^{\mathrm{op}})$. *Then every* $X \in \hat{\mathscr{C}}(R)$ *is* τ*-semicompact.*

Proof. Let $X \in \hat{\mathscr{C}}(R)$ and let $\mu : M \to X^*$ be a monomorphism. Then by Theorem 3.7(1) $\operatorname{Cok}\varepsilon_X$ is torsion. Also, since by Proposition 3.6 $X^* \in \hat{\mathscr{C}}(R^{\mathrm{op}})$, by Theorem 2.5 $\operatorname{Cok}\mu^*$ is torsion. Thus by Lemma 1.9 X is τ-semicompact. □

4. τ-Selfinjective rings

We call a ring R left τ-selfinjective if $\operatorname{Ext}_R^1(X, R)$ is torsion for every $X \in \operatorname{Mod} R$. We characterize left τ-selfinjective rings R by a certain kind of linear compactness.

For a module X and a set A, we denote by $X^{(A)}$ (resp. X^A) the direct sum (resp. direct product) of copies of X indexed by the elements of A.

Theorem 4.1. *For a ring* R *the following are equivalent.*

(1) R *is left* τ*-selfinjective.*

(2) (a) R *is* τ*-absolutely pure.*

(b) $\varprojlim \pi_\lambda$ *is a* τ*-epimorphism for every inverse system of* τ*-epimorphisms* $\{\pi_\lambda : F_\lambda \to M_\lambda\}_{\lambda \in \Lambda}$ *in* $\operatorname{Mod} R^{\mathrm{op}}$ *with the* F_λ *finitely generated free and the* M_λ *torsionless.*

PROOF. (1) $\Rightarrow$ (2). By Lemma 2.2 R is τ-absolutely pure. Next, let $\{\pi_\lambda : F_\lambda \to M_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of τ-epimorphisms in $\operatorname{Mod} R^{\mathrm{op}}$ with the F_λ reflexive and the M_λ torsionless. Since each π_λ^* is monic, so is $\varinjlim \pi_\lambda^*$. Thus by Theorem 2.5 $\operatorname{Cok}(\varprojlim \pi_\lambda^{**}) \cong \operatorname{Cok}((\varinjlim \pi_\lambda^*)^*)$ is torsion. Since $\varprojlim \varepsilon_{F_\lambda}$ is an isomorphism and $\varprojlim \varepsilon_{M_\lambda}$ is monic, $\operatorname{Cok}(\varprojlim \pi_\lambda)$ embeds in $\operatorname{Cok}(\varprojlim \pi_\lambda^{**})$, so that $\operatorname{Cok}(\varprojlim \pi_\lambda)$ is torsion.

(2) $\Rightarrow$ (1). By Lemmas 1.5 and 2.2 $\operatorname{Ext}_R^1(X, R)$ is torsion for every finitely generated $X \in \operatorname{Mod} R$. Next, let $0 \to K \xrightarrow{\mu} F \to X \to 0$ be an exact sequence in $\operatorname{Mod} R$ with $F = {}_R R^{(A)}$ free. Let Λ be the directed set of all nonempty finite subsets of A. For each $\lambda \in \Lambda$, put $F_\lambda = {}_R R^{(\lambda)}$ and let $j_\lambda : F_\lambda \to F$ denote the inclusion. Then $\varinjlim j_\lambda$ is an isomorphism. For each $\lambda \in \Lambda$, form the pull-buck of μ and j_λ:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\mu"] & F \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & K_\lambda \arrow[r, "\mu_\lambda"] \arrow[u, "i_\lambda"] & F_\lambda \arrow[r] \arrow[u, "j_\lambda"'] & X_\lambda \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

Since $\operatorname{Cok} \mu_\lambda^* \cong \operatorname{Ext}_R^1(X_\lambda, R)$ is torsion, we get an inverse system of τ-epimorphisms $\{\mu_\lambda^* : F_\lambda^* \to K_\lambda^*\}_{\lambda \in \Lambda}$ with the F_λ^* finitely generated free and the K_λ^* torsionless, so that $\operatorname{Cok}(\varprojlim \mu_\lambda^*)$ is torsion. Since $\varinjlim j_\lambda$ is an isomorphism, so is $\varprojlim j_\lambda^*$. Also, by the exactness of $\varinjlim$, $\varinjlim i_\lambda$ is an isomorphism, so is $\varprojlim i_\lambda^*$. Thus $\operatorname{Cok} \mu^* \cong \operatorname{Cok}(\varprojlim \mu_\lambda^*)$ and $\operatorname{Ext}_R^1(X, R) \cong \operatorname{Cok} \mu^*$ is torsion. □

LEMMA 4.2. *Suppose R is right τ-selfinjective. Then every $X \in \operatorname{Mod} R$ with $\operatorname{Cok} \varepsilon_X$ torsion is τ-semicompact.*

PROOF. By Theorem 2.5 and Lemma 1.9. □

LEMMA 4.3. *Let $F = {}_R R^{(A)}$ with A an infinite set. Then F is not τ-semicompact.*

PROOF. Put $G = {}_R R^A$ and let $\mu : F \to G$ denote the inclusion. Then μ is not an essential monomorphism and $\operatorname{Cok} \mu$ is not torsion. Let Λ be the directed set of all nonempty finite subsets of A. For each $\lambda \in \Lambda$, put $G_\lambda = {}_R R^\lambda$ and let $\pi_\lambda : G \to G_\lambda$ denote the projection. Then $\varprojlim \pi_\lambda$ is an isomorphism, so that we get an inverse system of epimorphisms $\{\pi_\lambda \circ \mu : F \to G_\lambda\}_{\lambda \in \Lambda}$ with the G_λ torsionless such that $\operatorname{Cok}(\varprojlim \pi_\lambda \circ \mu) \cong \operatorname{Cok} \mu$ is not torsion. □

PROPOSITION 4.4. *Suppose R is right τ-selfinjective. Let $F = {}_RR^{(A)}$ with A an infinite set. Then* $\operatorname{Cok}\varepsilon_F$ *is not torsion. In particular, F is not reflexive.*

PROOF. By Lemmas 4.2 and 4.3. $\square$

PROPOSITION 4.5. *Suppose R is right τ-selfinjective and right τ-semicompact. Then for a module $X \in \operatorname{Mod} R$, $\operatorname{Cok}\varepsilon_X$ is torsion if and only if X is τ-semicompact.*

PROOF. By Lemma 4.2, [8, Theorem 1.2] and [9, Corollary 2.2]. $\square$

We end with making the following remarks on reflexive modules.

REMARKS. (1) As remarked in [9], a module $X \in \operatorname{Mod} R$ is reflexive if and only if $\operatorname{Cok}\,\varepsilon_X$ is torsion and X can be embedded as a closed submodule in a direct product of copies of ${}_RR$.

(2) Even if R is τ-absolutely pure and left and right τ-semicompact, a reflexive module $X \in \operatorname{Mod} R$ is not necessarily τ-semicompact. For example, let R be the ring of rational integers and let $F = {}_RR^{(A)}$ with A a countably infinite set. Then by Lemma 1.5 R is τ-absolutely pure and (left and right) τ-semicompact. Also, by Lemma 4.3 F is not τ-semicompact. On the other hand, it follows from a theorem of Specker [17] that F is reflexive.

(3) It follows from [14, Theorem 1] that in case R is a left and right PF ring, a module $X \in \operatorname{Mod} R$ is reflexive if and only if it is linearly compact. Proposition 4.5 above generalizes this fact (cf. also [12, Theorem 3] and [5, Corollary 2.6]).

References

[1] L. Bican: QF-3′ modules and rings, Comment. Math. Univ. Carolin. **14** (1973), 295–303.
[2] R. R. Colby and K. R. Fuller: QF-3′ rings and Morita duality, Tsukuba J. Math. 3 (1984), 183–188.
[3] R. R. Colby and E. A. Rutter, Jr.: Semiprimary QF-3 rings, Nagoya Math J. **32** (1968), 253–258.
[4] J. L. Gómez Pardo and P. A. Guil Asensio: Morita dualities associated with the R-dual functors, J. Pure and Appl. Algebra **93** (1994), 179–194.
[5] J. L. Gómez Pardo and P. A. Guil Asensio: Reflexive modules over QF-3′ rings, preprint.
[6] M. Hoshino: On dominant dimensions of noetherian rings, Osaka J. Math. **26** (1989), 275–280.
[7] M. Hoshino: On Lambek torsion theories, Osaka J. Math. **29** (1992), 447–453.
[8] M. Hoshino: On Lambek torsion theories III, Osaka J. Math. **32** (1995), 521–531.
[9] M. Hoshino and S. Takashima: On Lambek torsion theories II, Osaka J. Math. **31** (1994), 729–746.
[10] J. P. Jans: Duality in noetherian rings, Proc. Amer. Math. Soc. **12** (1961), 829–835.

[11] K. Masaike: On quotient rings and torsionless modules, Sci. Rep. Tokyo Kyoiku Daigaku A **11** (1971), 25–30.
[12] K. Masaike: Reflexive modules over QF-3 rings, Canad. Math. Bull. **35** (1992), 247–251.
[13] E. Matlis: Injective modules over Prüffer rings, Nagoya Math. J. **15** (1959), 57–69.
[14] B. J. Müller: Linear compactness and Morita duality, J. Algebra **16** (1970), 60–66.
[15] K. Ohtake: A generalization of Morita duality by localizations, preprint.
[16] H. Sato: On localizations of a 1-Gorenstein ring, Sci. Rep. Tokyo Kyoiku Daigaku A **13** (1977), 188–193.
[17] E. Specker: Additive Gruppen von Folgen ganzer Zahlen, Portugal. Math. **9** (1950), 131–140.
[18] B. Stenström: Coherent rings and FP-injective modules, J. London Math. Soc. **2** (1970), 323–329.
[19] T. Sumioka: On finite dimensional QF-3' rings, Proceedings of 10^{th} Symposium on Ring Theory, 99–105, Okayama Univ., Okayama, Japan 1978.
[20] T. Sumioka: On QF-3 and 1-Gorenstein rings, Osaka J. Math. **16** (1979), 395–403.

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