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SMASH PRODUCTS AND COMODULES OF LINEAR MAPS

By

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Let G be a finite group and A be a G-graded algebra over a commutative ring k. Consider the G-graded right A-module $U=\bigoplus_{\sigma\in G} A(\sigma)$ where $A(\sigma)=A$ has grading shifted by σ. Năstăsescu and Rodinò [5] proved that

(1) $$\mathrm{End}_{A\text{-}gr}(U)*G\cong\mathrm{End}_A(U),\quad\text{and}\quad A\#k[G]^*\cong\mathrm{End}_{A\text{-}gr}(U)$$

where $\mathrm{End}_{A\text{-}gr}(U)$ denotes the algebra of graded A-endomorphisms of U, and $*$ means crossed product, [5], Theorems 1.2 and 1.3. The proofs are given by some explicit matrix computations relying on a graded isomorphism $\mathrm{End}_A(U)\cong M_n(A)$, $n=|G|$, [5], Prop. 1.1. The first isomorphism of (1) has recently been generalized to

(2) $$\mathrm{End}_{A\text{-}gr}(U)*G\cong\mathrm{END}_A(U),\qquad [2],\ \text{Thm. } 3.3,$$

for not necessarily finite groups G. The purpose of this paper is to give Hopf algebraic versions of (1) and (2). Write $H=k[G]$. First note that the above crossed products are also smash products. Furthermore, a G-graded k-module is the same as an H-comodule, and the A-isomorphism

$$U\xrightarrow{\sim}H\otimes A,\quad a(\sigma)\longmapsto\sigma^{-1}\otimes a(\sigma),\qquad a(\sigma)\in A(\sigma),$$

is H-colinear where $H\otimes A$ has coaction $\alpha: H\otimes A\to H\otimes A\otimes H$ defined by

(3) $$\alpha(h\otimes a)=\sum h_{(1)}\otimes a_{(0)}\otimes h_{(2)}a_{(1)},\qquad h\in H,\ a\in A.$$

Now let H be any Hopf algebra over k and set $U=H\otimes A$ for a right H-comodule algebra A. Let $\mathrm{End}_A^H(U)$ be the algebra of right A-linear maps $U\to U$ which are colinear with respect to (3). We shall generalize (1), for H finite over k, to

(4) $$\mathrm{End}_A^H(U)\#H\cong\mathrm{End}_A(U)\quad\text{and}\quad A\#H^*\cong\mathrm{End}_A^H(U).$$

It was pointed out in [5] that (1) implies the duality theorems of Cohen and Montgomery [4]. Correspondingly, (4) may be viewed as an improvement of the duality result for finite Hopf algebras [3], Cor. 2.7. Note that the second

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isomorphism of (4) gives a natural interpretation for an arbitrary smash product by a finite Hopf algebra.

Comodules of the form $\mathrm{HOM}_A(M, N)$ seem not have been considered yet for Hopf algebras others than $k[G]$. We introduce them here for arbitrary, projective Hopf algebras in section 2. We can then generalize (2) (and the first isomorphism of (4)) to

$$\mathrm{End}_A^H(U)\#H \cong \mathrm{END}_A(U)$$

for projective Hopf algebras. This turns out to be a special case of Theorem 2.4 which also includes [2], Thm. 3.6 (1), and shows that the finiteness conditions assumed there are not necessary.

Throughout the following, H denotes a Hopf algebra over a commutative ring k, and A a right H-comodule algebra. Recall that a Hopf A-module is a right A-module M supplied with a right H-comodule structure $\alpha: M \to M \otimes H$ such that

$$\alpha(ma) = \sum m_{(0)} a_{(0)} \otimes m_{(1)} a_{(1)}, \qquad m \in M, \ a \in A. \tag{5}$$

In case $H = A$, the descent theorem for Hopf H-modules says that the H-(co)-linear map

$$M^H \otimes H \longrightarrow M, \qquad m \otimes h \longmapsto mh,$$

is an isomorphism ([1], Thm. 3.1.8). Here $M^H = \{m \in M \mid \alpha(m) = m \otimes 1\}$. If H is finite over k, a right H-module M is a Hopf H-module iff M is a left H^*-module satisfying

$$g(mh) = \sum (g_{(1)} m)(g_{(2)} h), \qquad g \in H^*, \ m \in M, \ h \in H.$$

As usual, $H^* = \mathrm{Hom}_k(H, k)$ denotes the dual Hopf algebra (for H finite over k), and H is viewed as a left H^*-module by $gh = \sum h_{(1)} \langle g, h_{(2)} \rangle$. For a left H-module algebra B the smash product algebra $B\#H$ is $B \otimes H$ with multiplication defined by

$$(b' \otimes h)(b \otimes h') = \sum b'(h_{(1)} b) \otimes h_{(2)} h',$$

for $b, b' \in B$, $h, h' \in H$. The antipode and counit of a Hopf algebra will be denoted by λ and ε, respectively. We write $\otimes = \otimes_k$.

1. Let M be a left H- and right A-module such that

$$(hm)a = h(ma), \qquad h \in H, \ m \in M, \ a \in A. \tag{6}$$

For $h \in H$ and $\phi \in \mathrm{End}_A(M)$ define $h\phi \in \mathrm{End}_A(M)$ by

(7) $$(h\phi)(m)=\sum h_{(1)}\phi(\lambda(h_{(2)})m), \qquad m\in M.$$

Then $\mathrm{End}_A(M)$ is a left H-module algebra [6], and

$$\mathrm{End}_A(M)\#H \longrightarrow \mathrm{End}_A(M), \qquad \phi\otimes h \longmapsto \phi h,$$

is a homomorphism of k-algebras, where $(\phi h)(m)=\phi(hm)$. Assume that M has also a right H-comodule structure $\alpha: M\to M\otimes H$ satisfying

(8) $$\alpha(hm)=\sum h_{(1)}m_{(0)}\otimes h_{(2)}m_{(1)}, \qquad h\in H,\ m\in M.$$

Let $\mathrm{End}_A^H(M)$ be the k-algebra of A-linear and H-colinear maps $M\to M$.

LEMMA 1.1. *$\mathrm{End}_A^H(M)$ is an H-submodule algebra of $\mathrm{End}_A(M)$.*

The easy proof is left to the reader.

In the following we consider $M=H\otimes A=U$ with H-comodule structure defined by (3); U is naturally a left H- and right A-module satisfying (5), (6) and (8).

LEMMA 1.2. *Suppose the antipode λ of H is bijective. Then*

(9) $$\chi: \mathrm{End}_A^H(U) \longrightarrow \mathrm{Hom}_k(H, A), \qquad \chi(\phi)(h)=(\varepsilon\otimes 1)\phi(h\otimes 1),$$

is an isomorphism of k-modules.

PROOF. Define $\mathrm{Hom}_k(H, A)\to\mathrm{End}_A^H(U)$, $v\mapsto\tilde{v}$, by

$$\tilde{v}(h\otimes a)=\sum h_{(2)}\lambda^{-1}(v(h_{(1)})_{(1)})\otimes v(h_{(1)})_{(0)}a,$$

for $h\in H$, $a\in A$. It is easy to see that $\tilde{v}$ is H-colinear. Clearly $\chi(\tilde{v})=v$. Let $\phi\in\mathrm{End}_A^H(U)$, $h\in H$, and write $\phi(h\otimes 1)=\sum h_i\otimes a_i$. The colinearity of ϕ implies for $v=\chi(\phi)$

$$\sum a_{i(0)}\otimes h_i a_{i(1)}=\sum v(h_{(1)})\otimes h_{(2)}.$$

Therefore

$$\begin{aligned}\phi(h\otimes 1)&=\sum h_i a_{i(2)}\lambda^{-1}(a_{i(1)})\otimes a_{i(0)}\\&=\sum h_{(2)}\lambda^{-1}(v(h_{(1)})_{(1)})\otimes v(h_{(1)})_{(0)}. \quad\square\end{aligned}$$

REMARK 1. If the comodule structure of A is trivial then (9) is an algebra map where $\mathrm{Hom}_k(H, A)$ has the opposite convolution product. (The bijectivity of λ is not needed in this case.)

Suppose now that H is finite over k. For $a\in A$ and $g\in H^*$ define a^0, g^0: $U\to U$ by

$$a^0(h\otimes b)=\sum h\lambda^{-1}(a_{(1)})\otimes a_{(0)}b\,,$$

$$g^0(h\otimes b)=g^0(h)\otimes b=\sum h_{(2)}\langle g,\,h_{(1)}\rangle\otimes b\,,$$

for $h\in H$, $b\in A$. It is not difficult to see that a^0 and g^0 are H-colinear. Furthermore, $(aa')^0=a^0a'^0$, while $(gg')^0=g'^0g^0$. Note that $g^0(h)=gh$ if H is cocommutative.

THEOREM 1.3. *Let H be a finitely generated and projective Hopf algebra over k, A a right H-comodule algebra, and $U=H\otimes A$ with comodule structure defined by* (3). *Then*

$$\mathrm{End}_A^H(U)\#H\longrightarrow\mathrm{End}_A(U)\,,\qquad \phi\otimes h\longmapsto\phi h\,,\tag{10}$$

and

$$A\#H^*\longrightarrow\mathrm{End}_A^H(U)\,,\qquad a\otimes g\longmapsto a^0\lambda(g)^0\,,\tag{11}$$

are isomorphisms of k-algebras.

PROOF. That (10) is bijective is a special case of Theorem 2.4 below. It may be worth, however, to give here a separate proof for the finite case. We claim that the right H-module $\mathrm{End}_A(U)$ is a Hopf module satisfying

$$\mathrm{End}_A^H(U)=\mathrm{End}_A(U)^H\,.\tag{12}$$

It suffices to exhibit a corresponding left H^*-module structure. View A and U as left H^*-modules in the natural way. Then

$$gu=\sum g_{(1)}h\otimes g_{(2)}a\,,\qquad \text{for } u=h\otimes a,\ g\in H^*.$$

Now $\mathrm{End}_A(U)$ becomes a left H^*-module by the formula (7), (with h replaced by $g\in H^*$). That $g\phi$ is A-linear follows in the present case from $g(ua)=\sum(g_{(1)}u)(g_{(2)}a)$, $u\in U$, $a\in A$. Furthermore, we have $g(hu)=\sum(g_{(1)}h)(g_{(2)}u)$ for $h\in H$, $u\in U$, and this implies $g(\phi h)=\sum(g_{(1)}\phi)(g_{(2)}h)$, as is easy to see. Thus $\mathrm{End}_A(U)$ is a Hopf H-module. If $\phi\in\mathrm{End}_A(U)$ is H^*-linear, then clearly $g\phi=\varepsilon(g)\phi$ for all $g\in H^*$. Conversely, if the latter holds, then

$$g(\phi(u))=\sum(g_{(1)}\phi)(g_{(2)}u)=\sum\varepsilon(g_{(1)})\phi(g_{(2)}u)=\phi(gu)\,,$$

so that ϕ is H^*-linear. Hence (12) holds, and (10) is an isomorphism by the descent theorem for Hopf modules.

The composite of (11) and (9) gives the map

$$A\otimes H^*\longrightarrow\mathrm{Hom}_k(H,\,A)\,,\qquad a\otimes g\longmapsto(h\mapsto a\langle\lambda(g),\,h\rangle)\,.$$

This is bijective, since H is finite over k. Hence (11) is an isomorphism by Lemma 1.2. That (11) is an algebra map follows from

$$g^0 a^0 = \sum (\lambda^{-1}(g_{(2)})a)^0 g^0_{(1)},$$

which may be verified by evaluating on elements $h \otimes 1$.

2. We assume throughout the following that H is projective over k. As before, A denotes a right H-comodule algebra. We want to define comodules $\mathrm{HOM}_A(M, N)$ which generalize those defined for graded modules.

Fix Hopf A-modules M and N. For $\phi \in \mathrm{Hom}_A(M, N)$ define $\alpha(\phi) \in \mathrm{Hom}_A(M, N \otimes H)$ by

$$\alpha(\phi)(m) = \sum \phi(m_{(0)})_{(0)} \otimes \phi(m_{(0)})_{(1)} \lambda(m_{(1)}), \qquad m \in M. \tag{13}$$

(That $\alpha(\phi)$ is A-linear follows from (5).) Evidently,

$$(1 \otimes \varepsilon)\alpha(\phi) = \phi, \qquad \phi \in \mathrm{Hom}_A(M, N). \tag{14}$$

LEMMA 2.1. *Let $\phi \in \mathrm{Hom}_A(M, N)$. Then ϕ is H-colinear if and only if $\alpha(\phi)(m) = \phi(m) \otimes 1$ for all $m \in M$.*

PROOF. "$\Rightarrow$": This is obvious. "$\Leftarrow$": We have, by (13) with m replaced by $m_{(0)}$,

$$\begin{aligned} \sum \phi(m_{(0)}) \otimes m_{(1)} &= \sum \phi(m_{(0)})_{(0)} \otimes \phi(m_{(0)})_{(1)} \lambda(m_{(1)}) m_{(2)} \\ &= \sum \phi(m)_{(0)} \otimes \phi(m)_{(1)}. \quad \square \end{aligned}$$

Define the k-module $\mathrm{HOM}_A(M, N)$ to consist of all $\phi \in \mathrm{Hom}_A(M, N)$ for which there exists an element $\sum \phi_{(0)} \otimes \phi_{(1)} \in \mathrm{Hom}_A(M, N) \otimes H$ such that

$$\alpha(\phi)(m) = \sum \phi_{(0)}(m) \otimes \phi_{(1)}, \qquad m \in M.$$

Note that, since H is projective, $\mathrm{Hom}_A(M, N) \otimes H$ may be viewed as a submodule of $\mathrm{Hom}_A(M, N \otimes H)$, and we may simply write

$$\alpha(\phi) = \sum \phi_{(0)} \otimes \phi_{(1)}, \qquad \phi \in \mathrm{HOM}_A(M, N).$$

Clearly, $\mathrm{HOM}_A(M, N) = \mathrm{Hom}_A(M, N)$ if H is finite over k.

LEMMA 2.2. *Let $\phi \in \mathrm{HOM}_A(M, N)$. Then $\alpha(\phi) \in \mathrm{HOM}_A(M, N) \otimes H$, and $\mathrm{HOM}_A(M, N)$ is a right H-comodule. Furthermore, $\mathrm{END}_A(M) = \mathrm{HOM}_A(M, M)$ is a right H-comodule algebra.*

PROOF. Let $m \in M$. We have by definition of $\alpha(\phi_{(0)})$, and by (13) with m

replaced by $m_{(0)}$,

$$\begin{aligned}\sum\alpha(\psi_{(0)})(m)\otimes\psi_{(1)}&=\sum\psi_{(0)}(m_{(0)})_{(0)}\otimes\psi_{(0)}(m_{(0)})_{(1)}\lambda(m_{(1)})\otimes\psi_{(1)}\\&=\sum\psi(m_{(0)})_{(0)}\otimes\psi(m_{(0)})_{(1)}\lambda(m_{(2)})\otimes\psi(m_{(0)})_{(2)}\lambda(m_{(1)})\\&=\sum\psi_{(0)}(m)\otimes\psi_{(1)}\otimes\psi_{(2)}\,.\end{aligned}$$

Thus $(\alpha\otimes 1)\alpha(\psi)=(1\otimes\delta)\alpha(\psi)$ for δ the comultiplication of H. This also implies that $\alpha(\psi)$ lies in $\mathrm{HOM}_A(M, N)\otimes H$. For, $\mathrm{HOM}_A(M, N)$ is the pull back for a and the canonical map $\kappa:\mathrm{Hom}_A(M, N)\otimes H\to\mathrm{Hom}_A(M, N\otimes H)$, and $(-)\otimes H$ preserves finite limits since H is flat. Thus $\mathrm{HOM}_A(M, N)\otimes H$ is the pull back for $\alpha\otimes id_H$ and $\kappa\otimes id_H$, and $(\alpha\otimes 1)\alpha(\psi)\in\mathrm{Im}\,(\kappa\otimes 1)$ implies $\alpha(\psi)\in\mathrm{HOM}_A(M, N)\otimes H$.

Next let $\phi,\psi\in\mathrm{END}_A(M)$. The definition of $\sum\psi_{(0)}\otimes\psi_{(1)}$ implies

$$\sum\psi(m_{(0)})\otimes\lambda(m_{(1)})=\sum\psi_{(0)}(m)_{(0)}\otimes\lambda(\psi_{(0)}(m)_{(1)})\psi_{(1)}\,.$$

From this we conclude

$$\begin{aligned}\alpha(\phi\psi)(m)&=\sum(\phi\psi(m_{(0)}))_{(0)}\otimes(\phi\psi(m_{(0)}))_{(1)}\lambda(m_{(1)})\\&=\sum\phi(\psi_{(0)}(m)_{(0)})_{(0)}\otimes\phi(\psi_{(0)}(m)_{(0)})_{(1)}\lambda(\psi_{(0)}(m)_{(1)})\psi_{(1)}\\&=\sum\phi_{(0)}\psi_{(0)}(m)\otimes\phi_{(1)}\psi_{(1)}\,.\end{aligned}$$

Hence $\alpha(\phi\psi)=\alpha(\phi)\alpha(\psi)$, and this completes the proof.

EXAMPLE. Let $H=k[G]$ for a group G. Hence A is a G-graded k-algebra, and $M=\bigoplus_\sigma M_\sigma$, $N=\bigoplus_\sigma N_\sigma$ are G-graded right A-modules. Let $\psi\in\mathrm{Hom}_A(M, N)$ and $m_\sigma\in M_\sigma$. Then

$$\alpha(\psi)(m_\sigma)=\sum_\rho\psi(m_\sigma)_\rho\otimes\rho\sigma^{-1}=\sum_\tau\psi(m_\sigma)_{\tau\sigma}\otimes\tau\,.$$

This shows $\psi\in\mathrm{HOM}_A(M, N)$ iff $\psi=\sum_\tau\psi_\tau\in\bigoplus_\tau H_\tau$ (see (14)) with

$$H_\tau=\{\psi_\tau\in\mathrm{Hom}_A(M, N)\,|\,\psi_\tau(M_\sigma)\subset N_{\tau\sigma},\ \sigma\in G\},$$

and in this case $\alpha(\psi)=\sum_\tau\psi_\tau\otimes\tau$. Hence our definition of $\mathrm{HOM}_A(M, N)$ coincides for $H=k[G]$ with the usual one for graded modules.

Suppose in the following that M is also a left H-module satisfying (6) and (8); hence $\mathrm{Hom}_A(M, N)$ is a right H-module with $(\psi h)(m)=\psi(hm)$.

LEMMA 2.3. *Let $\psi\in\mathrm{HOM}_A(M, N)$ and $h\in H$. Then*

$$\alpha(\psi h)=\sum\psi_{(0)}h_{(1)}\otimes\psi_{(1)}h_{(2)}\,.$$

In particular, $\phi h \in \mathrm{HOM}_A(M, N)$.

PROOF. From (8) (with h replaced by $h_{(1)}$) one obtains

$$\sum (h_{(1)}m)_{(0)} \otimes \lambda((h_{(1)}m)_{(1)})h_{(2)} = \sum hm_{(0)} \otimes \lambda(m_{(1)}) .$$

This implies

$$\begin{aligned}\alpha(\phi h)(m) &= \sum \phi(hm_{(0)})_{(0)} \otimes \phi(hm_{(0)})_{(1)}\lambda(m_{(1)}) \\ &= \sum \phi((h_{(1)}m)_{(0)})_{(0)} \otimes \phi((h_{(1)}m)_{(0)})_{(1)}\lambda((h_{(1)}m)_{(1)})h_{(2)} \\ &= \sum \phi_{(0)}(h_{(1)}m) \otimes \phi_{(1)}h_{(2)} . \quad \square\end{aligned}$$

THEOREM 2.4. *Let H be a projective Hopf k-algebra, A a right H-comodule algebra, and M, N Hopf A-modules. Suppose M is also a left H-module satisfying* (6) *and* (8). *Then*

$$\mathrm{Hom}_A^H(M, N) \otimes H \longrightarrow \mathrm{HOM}_A(M, N) , \qquad \phi \otimes h \longmapsto \phi h ,$$

is an isomorphism of right H-comodules, where $\mathrm{Hom}_A^H(M, N)$ *denotes the k-module of A-linear and H-colinear maps $M \to N$. Furthermore,*

$$\mathrm{End}_A^H(M) \# H \longrightarrow \mathrm{END}_A(M) , \qquad \phi \otimes h \longmapsto \phi h ,$$

is an isomorphism of right H-comodule algebras.

PROOF. $\mathrm{HOM}_A(M, N)$ is a Hopf H-module by Lemma 2.3, and

$$\mathrm{Hom}_A^H(M, N) = \mathrm{HOM}_A(M, N)^H$$

holds by Lemma 2.1. Hence the result follows from the descent theorem for Hopf H-modules.

REMARK 2. Assume that H is finite over k. Then $\mathrm{HOM}_A(M, N) = \mathrm{Hom}_A(M, N)$, and the corresponding H^*-module structure is

$$(g\phi)(m) = \sum g_{(1)}\phi(\lambda(g_{(2)})m)$$

for $g \in H^*$ and $\phi \in \mathrm{Hom}_A(M, N)$. In this case theorem 2.4 may be proved entirely in the same way as the bijectivity of (10).

Clearly, Theorem 2.4 applies to $M = U = H \otimes A$. More generally, one may consider $U(M) = H \otimes M$, for any Hopf A-module M, with comodule structure $h \otimes m \mapsto \sum h_{(1)} \otimes m_{(0)} \otimes h_{(2)}m_{(1)}$. Then

$$\mathrm{End}_A^H(U(M)) \# H \cong \mathrm{END}_A(U(M)) .$$

This shows for $H=k[G]$ that [2], Thm. 3.6 (1) holds without any finiteness conditions on G or M.

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