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REDUCTION OF THE CODIMENSION OF TOTALLY REAL SUBMANIFOLDS OF A COMPLEX SPACE FORM

By

U-Hang KI and Hisao NAKAGAWA

Iutroduction.

A submanifold M of a Kaehlerian manifold $\bar{M}$ is said to be *totally real* if each tangent space to M is mapped to the normal space by the complex structure of $\bar{M}$. Many subjects for totally real submanifolds were studied from various different points of view, as ones of which Naitoh [9] and Naitoh and Takeuchi [10] classified an n-dimensional totally real submanifold with parallel second fundamental form in $\boldsymbol{P}_n\boldsymbol{C}$, and Ohnita [11] and Urbano [16] showed recently that the second fundamental form of such a submanifold of non-negative curvature is parallel, independently. Besides, the study for 3-dimensional totally real submanifols of S^6 by Mashimo [8] is also interesting.

In this paper the reduction of Allendoerfer type for the codimension of totally real submanifolds of a complex space form is treated with. As for all sorts of studies mentioned above, it is important that the dimension of the submanifolds is half of that of the ambient space. The purpose of this article is to show that the fact is essential, namely, to verify the following

THEOREM. *Let $\bar{M}$ be a complex space form of complex dimension m, and M an n-dimensional totally real submanifold of $\bar{M}$. If the induced f-structure in the normal bundle is parallel, then there exists a totally geodesic comlex space form M_0 of complex dimension n of $\bar{M}$ in which M is totally real.*

In the first section, preliminaries about totally real submanifolds of a complex space form are prepared for and the theorem is proved in the case where the ambient space is complex Euclidean. In § 2, a $(2m+1)$-dimensional anti-de Sitter space H_1^{2m+1} and the Sasakian structure on such a manifold are recalled. Lorentzian submanifolds of H_1^{2m+1} are treated with in the next section and the theorem is proved in § 4 provided that the ambient space is hyperbolic. In the last section the proof in $\boldsymbol{P}_m\boldsymbol{C}$ will be sketched.

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1. Totally real submanifolds of a complex space form.

A complete and simply connected Kaehlerian manifold of constant holomorphic sectional curvature is called a *complex space form*. By $M_m(c)$ a complex space form with constant holomorphic sectional curvature c and of complex dimension m is denoted. The complex space form consists of a complex projective space $\boldsymbol{P}_m\boldsymbol{C}$, a complex Euclidean one $\boldsymbol{C}^m$ or a complex hyperbolic one $\boldsymbol{H}_m\boldsymbol{C}$, according as $c>0$, $c=0$ or $c<0$. Let J and g be an almost complex structure and a Hermitian metric which are equipped with in $M_m(c)$. Let M be a real n-dimensional Riemannian submanifold immersed isometrically in $M_m(c)$. By the same symbol g the Riemannian metric induced on M from that of the ambient space is denoted.

Let $\mathcal{X}(M)$ and $\mathcal{X}^{\perp}(M)$ be the set of all vector fields tangent to M and the set of all vector fields normal to M, respectively. Manifolds and submanifolds which are discussed in this paper will be assumed to be connected and the smoothness of any geometric objects is also assumed to be of class C^∞. Liner mappings P and F of $\mathcal{X}(M)$ into $\mathcal{X}(M)$ and $\mathcal{X}^{\perp}(M)$ are defined as follows: for any vector field X in $\mathcal{X}(M)$, PX is the tangent part of JX and FX is the normal one of JX. Namely, $JX=PX+FX$. Similarly, other two operators p and f of $\mathcal{X}^{\perp}(M)$ into $\mathcal{X}(M)$ and $\mathcal{X}^{\perp}(M)$ are defined as follows: for any normal field ξ, $p\xi$ and $f\xi$ are given by the tangent part and the normal one of $J\xi$, respectively. That is, $J\xi=p\xi+f\xi$. Then it is well known [18] that the following relations between these operators hold true: for any tangent vector fields X and Y, and any normal fields ξ and η,

$$(1.1)\quad \begin{cases} g(PX, Y)+g(X, PY)=0, \\ g(FX, \eta)+g(X, p\eta)=0, \\ g(f\xi, \eta)+g(\xi, f\eta)=0, \end{cases}$$

$$(1.2)\quad \begin{cases} P^2=-I-pF,\ FP+fF=0, \\ Pp+pf=0,\ f^2=-I-Fp, \end{cases}$$

where I denotes the identity mapping.

An n-dimensional submanifold M of $M_m(c)$ is said to be *totally real* if $JM_x \subset M_x^{\perp}$ for any point x in M, where $M_x(=T_xM)$ and $M_x^{\perp}$ denote the tangent space of M at x and the normal one to M at x, respectively. The submanifold M is assumed to be always totally real at the rest of this paper. Accordingly, JX is the normal field to M, provided that X is tangent to M. This implies that $m\geqq n$ and the operator P is zero, and moreover F is the linear isomorphism. The orthogonal complement of JM_x in $M_x^{\perp}$ is invariant under the operator f and hence the orthogonal sum decomposition $M_x^{\perp}=FM_x+fM_x^{\perp}$ is obtained. This yields that the linear transformation f satisfies $f^3+f=0$ and its rank is equal to $2m-n$. A non-null tensor field ϕ of type (1,1) on a Riemannian manifold N is called an *f-structure of rank r on N* if it satisfies $\phi^3+\phi=0$ and rank $\phi=r$. It means that f becomes the f-structure of rank $2m-n$ in the normal bundle of M.

Let ∇ and $\overline{\nabla}$ be the Riemannian connections in $M_m(c)$ and M respectively, and $\nabla^{\perp}$ the normal connection in the normal bundle of M. Let β be the second fundamental form on M and B the shape operator of $\mathcal{X}^{\perp}(M)\times\mathcal{X}(M)$ into $\mathcal{X}(M)$ defined by $g(B_\xi X,Y)=g(\beta(X,Y),\xi)$. Since M is totally real, it follows that $FZ=JZ$, which means that for any vector fields tangent to M

$$g(\beta(X,Y),\ FZ)=g(\beta(X,Y),\ JZ)=g(\nabla_X Y,JZ)=-g(Y,\overline{\nabla}_X(JZ))$$
$$=g(JY,\nabla_X Z)=g(FY,\beta(X,Z)).$$

This implies that $g(\beta(X,Y),\ FZ)$ is symmetric with respect to X, Y and Z.

The covariant derivatives $\nabla_X P$, $\nabla_X F$, $\nabla_X p$ and $\nabla_X f$ of P, F, p and f are defiend by

$$\nabla_X p(Y)=\nabla_X(PY)-P\nabla_X Y,\ \ \nabla_X F(Y)=\nabla_X^{\perp}(FY)-F\nabla_X Y,$$
$$\nabla_X p(\xi)=\nabla_X(p\xi)-p\nabla_X^{\perp}\xi,\ \ \nabla_X f(\xi)=\nabla_X^{\perp}(f\xi)-f\nabla_X^{\perp}\xi.$$

Then it follows from the Gauss and Weingarten formulas that the following equations hold true:

$$p\beta(X,Y)=-B_{FX}Y,\ \ \nabla_X F(Y)=f\beta(X,Y),\tag{1.3}$$
$$\nabla_X p(\xi)=B_{f\xi}X,\ \ \nabla_X f(\xi)=-F(B_\xi X)-\beta(X,p\xi).$$

The f-structure f in the normal bundle is said to be *parallel* if it satisfies $\nabla f=0$. The operators F and p are also said to be *parallel*, provided that the covariant differentials of F and p are 0 respectively. For the parallelism of f one finds the following property, which was pointed out by the referee.

PROPOSITION 1.1. *Let M be a submanifold of $M_m(c)$. If there exists a totally geodesic submanifold M_0 of $M_m(c)$, in which M is totally real, then the f-structure in the normal bundle is parallel.*

PROOF. For any point x of M, the assumption gives the direct sum decomposition of the normal space $M_x^{\perp}$ of M in the ambient space $\overline{M}$

$$M_x^{\perp}=JM_x\oplus T_x(M_0)^{\perp}.$$

Since M is the totally geodesic submanifold of M_0, it follows $B_\xi=0$ for a normal vector ξ of $T_x(M_0)^{\perp}$, and moreover since $T_x(M_0)^{\perp}$ is J-invariant, it follows that $p\xi=(J\xi)^T=0$, where $(J\xi)^T$ denotes the tangent part of $J\xi$. It implies that

$$\nabla_X f(\xi)=-F(B_\xi X)-\beta(X,p\xi)=0\quad\text{for any } \xi \text{ in } T_x(M_0)^{\perp}$$

by the last equation of (1.3).

Next, for any vector field Y it follows

$$g(F(B_{JY}X),\ JZ)=g(J(B_{JY}X),\ JZ)=g(B_{JY}X,Z)=g(\beta(X,Z),\ FY)$$
$$=g(\beta(X,Y),\ FZ),$$

because $g(\beta(X,Y),\ FZ)$ is symmetric with respect to X, Y and Z, and thus we have $F(B_{JY}X)=\beta(X,Y)$. Consequently it follows from (1.2) and (1.3) that

$$\nabla_X f(JY)=-F(B_{JY}X)-\beta(X,pJY)=-\beta(X,Y)+\beta(X,Y)=0.$$

These imply that $\nabla f=0$.

LEMMA 1.2. *Let M be a totally real submanifold in $M_m(c)$. If the f-structure f in the normal bundle is parallel, then so do operators F and p.*

PROOF. The first equation of (1.3) implies $Fp\beta(X,Y)+FB_{FY}X=0$. By means of (1.2) the assumption of the parallelism of the structure f yields $f^2\beta(X,Y)=0$ and it follows that $f\beta(X,Y)=0$ by the property of $f^3+f=0$, which means that F is parallel. This is equivalent to $\nabla p=0$ in [1].

REMARK. (1) Lemma 1.2 holds true in the case where the ambient space is only a Kaehlerian manifold. In general, it is also seen in [1] that $\Delta F=0$ is equivalent to $\nabla p=0$ in a CR-submanifold in $M_m(c)$.

(2) Pak [14] proved that in a totally real submanifold of $\boldsymbol{P}_m\boldsymbol{C}$, if the second fundamental form is parallel, then so do the operators F, p and f.

The distribution $\mathcal{D}$ in the normal bundle of M is said to be *parallel* with respect to the normal connection, if for any normal field ξ in $\mathcal{D}$ the vector field $\nabla_X{}^{\perp}\xi$ is contained in $\mathcal{D}$ for any vector field X in $\mathcal{X}(M)$.

PROPOSITION 1.3. *Let M be an n-dimensional totally real submanifold in $\boldsymbol{C}^m$. If the f-structure f in the normal bundle is parallel, then there exists a totally geodesic submanifold $M_0=\boldsymbol{C}^n$ in which M is totally real.*

PROOF. Let $\mathcal{D}$ be the distribution consisting of subspaces FM_x in $M_x{}^{\perp}$ at each point x in M. Then it is of dimension n and $\mathcal{D}(x)=\mathrm{Ker}_x f$. From Lemma 1.2 it follows that the operator F is parallel and hence $\nabla_X{}^{\perp}(FY)=F\nabla_X Y$ in $\mathcal{D}$, which means that $\mathcal{D}$ is parallel with respect to the normal connection. The first normal space M_x^1 at x is defined by the orthogonal complement of the following subspace $\{\xi\in M_x : B_\xi=0\}$ in $M_x{}^{\perp}$. Accordingly the first normal space M_x^1 is identified with the linear subspace in $M_x{}^{\perp}$ spanned by $\beta(u,v)$ for any vectors u and v in M_x. The parallelism of the operator p together with the property $\mathcal{D}(x)=\mathrm{Ker}_x f$ implies that the first normal space M_x^1 is contained in the subspace $\mathcal{D}(x)$.

By means of the reduction theorem of Erbacher [4], there exists a $2n$-dimensional totally geodesic submanifold M_0 of $\boldsymbol{C}^m$ in which M is a submanifold. This means that M_0 is a complex Euclidean space equipped with the complex structure $J|\boldsymbol{C}^n$, which is denoted by the same J. Then $JM_x=FM_x=\mathcal{D}(x)$, which implies that M is totally real in $M_0=\boldsymbol{C}^n$.

This concludes the proof.

2. Odd dimensional anti-de Sitter spaces.

In this section, the Sasakian structure on a $(2m+1)$-dimensional anti-de Sitter space is recalled [15]. In an $(m+1)$-dimensional complex Euclidean space $\boldsymbol{C}^{m+1}$ with standard basis, a Hermitian form F is defined by

$$F(z,w)=-z_0\overline{w}_0+\sum_{k=1}^{m}z_k\overline{w}_k,$$

where $z=(z_0, z_1,\cdots, z_m)$ and $w=(w_0, w_1,\cdots, w_m)$ are in $\boldsymbol{C}^{m+1}$. The space$(\boldsymbol{C}^{m+1}, F)$ is called an $(m+1)$-dimensional *complex Minkowski space*, which is denoted by $\boldsymbol{C}_1^{m+1}$. A non-degenerate symmetric bilinear form g of a real vector space V is called a *scalar product* and the *index* of the scalar product g is by definition the largest number that is the dimension of a subspace W of V on which $g|W$ is negative definite. A metric tensor g on a smooth manifold M is by definition a symmetric non-degenerate (0,2) tensor field on M of constant index r, and a smooth manifold M furnished with a metric tensor field g is called a *semi-Riemannian manifold with semi-Riemannian metric g of index r*. The common value r of the index g on a semi-Riemannian manifold M is called the *index of M*. The scalar product given by Re $F(z,w)$ is a semi-Riemannian metric of index 2 on $\boldsymbol{C}_1^{m+1}$. Let H_1^{2m+1} be a real hypersurface in $\boldsymbol{C}_1^{m+1}$ denoted by

$$\boldsymbol{H}_1^{2m+1}=\{z\in\boldsymbol{C}_1^{m+1}:F(z,z)=-1\},$$

and let G be a semi-Riemannian metric on H_1^{2m+1} induced from the complex Lorentzian metric Re F in $\boldsymbol{C}_1^{m+1}$. Then (H_1^{2m+1}, G) is the Lorentzian manifold with constant curvature -1, which is called the *anti-de Sitter space* of constant curvature -1 ([3] and [17]). For any point z in H_1^{2m+1} the tangent space $T_z H_1^{2m+1}$ can be identified through parallel translation in $\boldsymbol{C}_1^{m+1}$ with the subspace $\{w\in\boldsymbol{C}_1^{m+1}: \text{Re } F(z,w)=0\}$ of $\boldsymbol{C}_1^{m+1}$. In particular iz is a point in $\boldsymbol{H}_1^{2m+1}$ where i denotes the imaginary unit, and moreover it is contained in the tangent space $T_zH_1^{2m+1}$. For any point z in $\boldsymbol{H}_1^{2m+1}$, $\xi_z=z$ can be regarded as a unit normal time-like vector to $\boldsymbol{H}_1^{2m+1}$ up to translation. For the almost complex structure J in $\boldsymbol{C}_1^{m+1}$ $E_z=-J\xi_z=-iz$ is the unit time-like vector of H_1^{2m+1}, i.e. $E\in\mathcal{X}(H_1^{2m+1})$ and $G(E,E)=-1$.

For the orthogonal projection $\pi: T_zC_1^{2m+1} \to T_zH_1^{2m+1}$ the linear transformation ϕ of $\mathcal{X}(H_1^{2m+1})$ into itself is defined by the composition of J and π, and let ω be a 1-form on H_1^{2m+1} defiend by $JX=\phi X+\omega(X)\xi$ for any tangent vector field X on H_1^{2m+1}. Then it is seen in [15] that

$$(2.1)\quad \begin{cases} \omega(X)=-G(X,E),\ \omega(E)=1, \\ \phi E=0,\ \omega\circ\phi=0, \\ G(\phi X,Y)+G(X,\phi Y)=0, \\ \phi^2=-I+\omega\otimes E, \\ G(\phi X,\phi Y)=G(X,Y)+\omega(X)\omega(Y). \end{cases}$$

Let α be the second fundamental form for the hypersurface H_1^{2m+1} of C_1^{m+1} and A_ξ the shape operator with respect to ξ. Then H_1^{2m+1} is the totally umbilical hypersurface and $A=-I$. Let $\bar{D}$ be the semi-Riemannian connection of H_1^{2m+1}. By making use of the Gauss and Weingarten formulas, the following relations are obained:

$$(2.2)\quad \begin{cases} \bar{D}_U\phi(V)=-\omega(V)U-G(U,V)E, \\ \bar{D}_U\omega(V)=G(\phi U,V),\ d\omega(U,V)=G(\phi U,V), \\ \bar{D}_UE=-\phi U \end{cases}$$

for any vector fields U and V on H_1^{2m+1}.

Let N be an odd dimensional manifold equipped with the set (ϕ, E, ω, G), which consists of a tensor field ϕ of type (1,1), a unit time-like vector field E, a 1-form ω and a Lorentzian metric G. The set is called the *Sasakian structure* if it satisfies properties (2.1) and (2.2) [15]. The anti-de Sitter space H_1^{2m+1} admits the Sasakian structure.

3. Lorentzian submanifolds of H_1^{2m+1}.

Let H_1^{2m+1} be a $(2m+1)$-dimensional anti-de Sitter space of constant curvature -1 and (ϕ, E, ω, G) its admitting Sasakian structure on H_1^{2m+1}. Let N be a semi-Riemannian submanifold of H_1^{2m+1} tangent to the structure vector field E. By the same G the semi-Riemannian metric induced on N from that of H_1^{2m+1} is denoted. Each tangent space N_p is by definition a non-degenerate subspace of $T_pH_1^{2m+1}$, and hence a property of the linear space with indefinite scalar product gives the direct sum decomposition $T_pH_1^{2m+1}=N_p\oplus N_p\perp$, and the subspace $N_p\perp$ which is called the *normal space* at p to N is also non-degenerate. Its dimension is equal to $2m-n$, where dim $N=n+1$. The index of G restricted to N_p is called the *co-index*. In fact, the co-index of N is independent of the choice of the point p and it is easily seen that ind $H_1^{2m+1}=$ind $N+$co-ind N. Accordingly ind $N=1$ and co-ind $N=0$.

Now, by the similar definition to that of 2-sets (P, F) and (p, f) of operators defined for the submanifold M of the Kaehlerian manifold $\bar{M}$, operators P', F', p' and f' are defined as follows: for any vector field U in $\mathcal{X}(N)$ and any normal field τ in $\mathcal{X}^{\perp}(N)$, $\phi U=P'U+F'U$ and $\phi\tau=p'\tau+f'\tau$, where $P'U$ and $p'\tau$ are tangent parts of ϕU and $\phi\tau$ respectively, and $F'U$ and $f'\tau$ are also normal parts of ϕU and $\phi\tau$ respectively. Then it is easily seen that for any vector fields U and V on N and any normal fields τ and σ on N, the following relations hold true:

$$(3.1)\quad \begin{cases} G(P'U,V)+G(U,P'V)=0, \\ G(F'U,\sigma)+G(U,p'\sigma)=0, \\ G(f'\tau,\sigma)+G(\tau,f'\sigma)=0, \end{cases}$$

$$(3.2)\quad \begin{cases} F'^2+p'F'=-I+\omega\otimes E,\ F'P'+f'F'=0, \\ P'p'+p'f'=0,\ f'^2=-I-F'p', \\ P'E=F'E=0. \end{cases}$$

Let $\bar{D}$ and D be the semi-Riemannian connections on H_1^{2m+1} and N respectively, and $D^{\perp}$ the normal connection in the normal bundle of N. Let α be the second fundamental form of N and A the shape operator of $\mathcal{X}^{\perp}(N)\times\mathcal{X}(N)$ into $\mathcal{X}(N)$ defined by $G(A_\tau U,V)=G(\alpha(U,V),\tau)$. The Gauss and Weingarten formulas are also given by $\bar{D}_U V=D_U V+\alpha(U,V)$ and $\bar{D}_U\tau=-A_\tau U+D_U^{\perp}\tau$. For example, see [13]. For the vector field E the last equation of (2.2) gives

$$(3.3)\quad \begin{cases} D_U E=-P'U,\ \alpha(U,E)=-F'U, \\ \alpha(E,E)=0, \\ A_\tau E=p'\tau. \end{cases}$$

The covariant differentials DP', DF', Dp' and Df' of these operators are defined similarly. Then it follows from the Gauss and Weingarten formulas that the following relations are given:

$$(3.4)\quad \begin{cases} D_U P'(V)=A_{F'V}U+p'\alpha(U,V)-G(U,V)E-\omega(V)U, \\ D_U F'(V)=-\alpha(U,P'V)+f'\alpha(U,V), \\ D_U p'(\tau)=A_{f'\tau}U-P'A_\tau U, \\ D_U f'(\tau)=-F'A_\tau U-\alpha(U,p'\tau). \end{cases}$$

A submanifold N tangent to the structure vector field E in H_1^{2m+1} is said to be *totally real*, if ϕN_p is contained in the normal space $N_p^{\perp}$ at each point p in N. In this case, the operator P' satisfies $P'=0$, and therefore f' makes the f-structure in the normal bundle, because of (3.2). The structure f' is said to be *parallel*, provided that $Df'=0$. The others F' and p' are said to be *parallel*, if $DF'=0$ and $Dp'=0$ respectively. Similarly to Lemma 1.2, the following property holds true:

LEMMA 3.1. *Let N be a totally real submanifold tangent to E in H_1^{2m+1}. If the f-structure f' in the normal bundle is parallel, then so do the others F' and p'.*

PROOF. First of all, the operator F' is verified. Since N is totally real, the first equation of (3.4) gives

$$F'A_{F'V}U+F'p'\alpha(U,V)-\omega(V)F'U=0$$

for any vector fields U and V, which together with (3.2) and the assumption yields $f'^2\alpha(U,V)=0$. This means that $f'\alpha=0$, namely, it is equivalent to the fact that F' is parallel. The second assertion is easily given by the third equation of (3.4) and the total reality.

PROPOSTION 3.2. *Let N be an $(n+1)$-dimensional complete and totally real submanifold tangent to the structure vector field E in H_1^{2m+1}. If the f-structure f' in the normal bundle is parallel, then there exists a totally geodesic submanifold $N_0=H_1^{2n+1}$ in which N is totally real.*

PROOF. For the given distribution $\mathcal{D}$ consisting of each tangent space N_p at each point p in N, the distribution consisting of each subspace ϕN_p of the normal space is defined by $\phi\mathcal{D}$. By (3.2), $F'E=0$ and because of $G(F'U,F'V)=G(U,V)$ for any vector fields U and V on N orthogonal to E, the dimension of the subspace $F'N_p$ is equal to n for each point p in N, which implies dim $\phi\mathcal{D}=n$. Since the parallelism of the operator F' induces $D_U^{\perp}(F'V)=F'(D_UV)$ for any vector fields U and V in $\mathcal{X}(N)$, the distribution $\phi\mathcal{D}$ is parallel with respect to the normal connection.

On the other hand, the parallelism of the operator p' reduces to $A_{f'\tau}=0$ for any normal τ in $\mathcal{X}^{\perp}(N)$, which means that $f'N_p$ is the subspace of the subspace $\{\sigma\in N_p^{\perp}: A\sigma=0\}$ of the normal space $N_p^{\perp}$. Because the normal space $N_p^{\perp}$ has two kinds of orthogonal sum decompositions

$$N_p^{\perp}=F'N_p\oplus f'N_p^{\perp}=N_p^1\oplus\{\sigma\in N_p^{\perp}: A_\sigma=0\}\ ,$$

the first normal space N_p^1 at p is contained in the subspace $F'N_p=\phi\mathcal{D}(p)$. Thus the reduction theorem due to Dajczer [2] and Magid [7] in the anti-de Sitter space can be applied to this situation, and hence there exists a $(2n-1)$-dimensional complete and totally geodesic submanifold N_0 of H_1^{2m+1} such that $N\subset N_0$ and $T_p(N_0)=N_p\oplus F'N_p$ for any point p of N. Fix a point p in N and put $\boldsymbol{R}_2^{2n+2}=T_p(N_0)\oplus\boldsymbol{R}_p$. Then $\boldsymbol{R}_2^{2n+2}$ is a complex linear subspace of $\boldsymbol{R}_2^{2m+2}=\boldsymbol{C}_1^{m+1}$ and H_1^{2n+1}

which is defined by the intersection of $\boldsymbol{R}_2^{2n+2}$ and H_1^{2m+1} is the $(2n+1)$-dimensional anti-de Sitter space with the metric induced from that of H_1^{2m+1}. Since the geodesics of H_1^{2m+1} are just the intersections of P and H_1^{2m+1}, where P is a plane through the origin O in $\boldsymbol{C}_1^{m+1}$ which meets H_1^{2m+1}, the anti-de Sitter space H_1^{2n+1} is a complete and totally geodesic submanifold of H_1^{2m+1}. Obvious it follows $T_p H_1^{2n+1}=T_pN_0$. Apart from this viewpoint, it is seen in [13] that for complete and totally geodesic semi-Riemannian submanifolds N_1 and N_2 of a semi-Riemannian manifold if there is a point p in N_1 and N_2 at which the tangent spaces coincide, then N_1 coincides with N_2. This implies that $N_0=H_1^{2n+1}$. Moreover, since H_1^{2n+1} is invariant under the multiplication by $e^{i\theta}$, N is totally real in H_1^{2n+1}.

This concludes the proof.

The argument developed in this section can be applied to the case where the submanifold in the unit sphere S^{2m+1} is totally real. Accordingly the following property is verified. The proof is omitted.

PROPOSITION 3.3. *Let N be an $(n+1)$-dimensional complete and totally real submanifold tangent to the structure vector field E of S^{2m+1} with the Sasakian structure (ϕ, E, ω, G). If the f-structure in the normal bundle is parallel, there exists a totally geodesic unit sphere S^{2n+1} of S^{2m+1}, in which N is totally real.*

REMARK. When the proof of the above Proposition is checked carefully, it is seen that the condition that the f-structure in the normal bundle is parallel can be replaced by the apparently weaker one that $f'\alpha$ vanishes identically.

REMARK. Under the additional condition that the mean curvature vector field is not trivial and parallel in the normal bundle, the compact N is minimally contained in a hypersurface of positive curvature in S^{2n+1} [5].

4. Lorentzian circle bundles over a submanifold of H_mC.

Let H_1^{2m+1} be a $(2m+1)$-dimensional anti-de Sitter space of $\boldsymbol{C}_1^{m+1}$ equipped with the Hermitian form F on $\boldsymbol{C}_1^{m+1}$. Let $U(1,m)$ be the set of matrices A in $GL(m+1,C)$ such that $F(Az,Aw)=F(z,w)$ for each z and w in $\boldsymbol{C}_1^{m+1}$. Then the group $U(1,m)$ acts transitively on H_1^{2m+1} and the group $S^1=\{e^{i\theta}\}$ acts freely on H_1^{2m+1} by $z\to e^{i\theta}z$. The orbit $\{e^{i\theta} z\}$ lies in negative definite plane spanned by z and iz. Let $\bar{M}$ be the base manifold of the principal fiber bundle H_1^{2m+1} with the structure group S^1. For any point z in H_1^{2m+1} let T_z' be the subspace of $T_z H_1^{2m+1}$ defined by $T_z'=\{w\in T_zH_1^{2m+1}: \mathrm{Re}F(iz,w)=0\}$. Then the restriction of F to T_z' is positive definite and the orthogonal sum decomposition $T_zH_1^{2m+1}=T_z'\oplus$

span $\{iz\}$ is given. Moreover we have a connection in H_1^{2m+1} such that T_z' s, z in H_1^{2m+1}, are the horizontal subspaces. The natural projection $\pi: H_1^{2m+1} \to M$ induces a linear onto isomorphism $d\pi: T_z' \to T_{\pi(z)}M$. The complex structure $w \to iw$ in T_z' is compatible with the action of S^1 and induces the almost complex structure J on $\bar{M}$ such that $d\pi \circ i = J \circ d\pi$. A scalar product g on each tangent space $\bar{M}_x$ at each point x in $\bar{M}$ is defined by $g(X,Y) = -\mathrm{Re}F(U,V)$, where U and V are elements in T_z' such that $\pi(z)=x$, $d\pi(U)=X$ and $d\pi(V)=Y$. Then $(\bar{M}, g, J)$ is the m-dimensional complex hyperbolic space $\boldsymbol{H_m C}$ with constant holomorphic sectional curvature -4 ([3] and [17]). It is well known that H_1^{2m+1} is the principal S^1-bundle over $\boldsymbol{H_m C}$ with the projection $\pi: H_1^{2m+1} \to \boldsymbol{H_m C}$, which is the Riemannian submersion in the sence of O'Neill [12] with fundamental tensor J and totally geodesic time-like fibers. For any point z in H_1^{2m+1} we put $E_z = -Jz$ in $T_z H_1^{2m+1}$, and then the orthogonal sum decomposition $T_z H_1^{2m+1} = T_{\pi(z)} \boldsymbol{H_m C} \oplus \mathrm{span}\,\{E_z\}$ is given up to identification. Let (ϕ, E, ω, G) be the Sasakian structure equipped in H_1^{2m+1} and (J, g) the Kaehlerian structure in $\boldsymbol{H_m C}$. Let $*$ be the horizontal lift of the Riemannian submersion $\pi: H_1^{2m+1} \to \boldsymbol{H_m C}$. By the construction.

$$(4.1) \qquad (JX)^* = \phi X^*, \quad G(X^*, Y^*) = g(X,Y)$$

for any vector fields X and Y on $\mathfrak{X}(\boldsymbol{H_m C})$. The above decomposition gives the following relationships between the semi-Riemannian conncetion $\bar{D}$ of H_1^{2m+1} and the Riemannian connection ∇ of $\boldsymbol{H_m C}$:

$$(4.2) \qquad \bar{D}_{x^*} Y^* = (\nabla_X Y)^* - G(\phi X^*, Y^*)E, \quad \bar{D}_{x^*} E = -\phi X^*.$$

For an n-dimensional submanifold M of $\boldsymbol{H_m C}$, one can construct the Lorentzian submanifold $N = \pi^{-1}(M)$ which is the principal S^1-bundle over M with time-like totally geodesic fibers and the projection π [6]. Moreover, π is compatible with the fibration $\pi: H_1^{2m+1} \to \boldsymbol{H_m C}$, that is, the diagram

$$\begin{array}{ccc} N & \xrightarrow{i'} & H_1^{2m+1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{i} & \boldsymbol{H_m C} \end{array}$$

is commutative, where i and i' are the respective immersions. This shows that $N_z = (M_{\pi(z)}) \oplus \mathrm{span}\{E_z\}$. The Gauss formulas for the immersions i' and i and the equation (4.2) yield

$$(4.3) \qquad \begin{cases} D_{x^*} Y^* = (\nabla_X Y)^* - G(\phi X^* Y^*)E, \\ \alpha(X^*, Y^*) = \beta(X, Y)^*, \end{cases}$$

while the Weingarten formulas for the immersions i' and i give rise to the following relationships between the shape operators A and B:

(4.4) $$\begin{cases} A_\xi{}^*Y^* = (B_\xi Y)^* + G(A_\xi{}^*Y^*, E)E, \\ D_x{}^{*\perp}\xi^* = (\nabla_x{}^{\perp}\xi)^*. \end{cases}$$

On the other hand, for orthogonal operators (P,F), (p,f) and (P',F'), (p',f') of the immersions i' and i respectively, (4.1) means that

(4.5) $$\begin{cases} (PX)^* = P'X^*, \ (FX)^* = F'X^*, \\ (p\xi)^* = p'\xi^*, \ (f\xi)^* = f'\xi^*, \end{cases}$$

and using (3.3) and (4.4), we have

(4.6) $$(B_\xi X)^* = A_\xi{}^*X^* + G(F'X^*, \xi^*)E, \ A_\xi{}^*E = p'\xi^*.$$

From (4.5) it follows easily that N is totally real in H_1^{2m+1} if and only if M is totally real in $\boldsymbol{H}_m\boldsymbol{C}$ [18].

Now, for the diagram mentioned above, the relation between the parallelism of the operators f and f' is investigated.

LEMMA 4.1. *If the f-structure f in the normal bundle of M is parallel, then so does the operator f'.*

PROOF. Under the assumption that the operator f is parallel, for any vector field X in $\mathcal{X}(M)$ and any normal field ξ in $\mathcal{X}^{\perp}(M)$, it follows from (4.4), (4.5) and the above equation that $D_x{}^*f'(\xi^*)=0$. Since $d\pi : \mathcal{X}^{\perp}(N)\to\mathcal{X}^{\perp}(M)$ is an isometric isomorphism, it means $D_x{}^*f'=0$.

On the other hand, for any normal field τ in $\mathcal{X}^{\perp}(N)$, the last equation of (3.4) is reduced to $D_E f'(\tau) = -F'(A_\tau E - p'\tau)$, which together with (3.3) gives $D_E f' = 0$. This means that f' is parallel.

THEOREM 4.2. *Let $\boldsymbol{H}_m\boldsymbol{C}$ be a complex m-dimentional complex hyperbolic space of constant holomorphic sectional curvature -4, and M an n-dimensional complete and totally real submanifold of $\boldsymbol{H}_m\boldsymbol{C}$. If the f-structure in the normal bundle of M is parallel, then there exists a complex n-dimensional totally geodesic submanifold $M_0=\boldsymbol{H}_n\boldsymbol{C}$ of $\boldsymbol{H}_m\boldsymbol{C}$ in which M is totally real.*

PROOF. By means of Proposition 3.2 and Lemma 4.1, there exists a $(2n+1)$-dimensional totally geodesic anti-de Sitter space $H_1^{2n+1}=N_0$ of H_1^{2m+1}, in which N is totally real. The restriction of the projection $\pi : H_1^{2m+1}\to\boldsymbol{H}_m\boldsymbol{C}$ to H_1^{2n+1} is denoted by the same π, and let M_0 be the image of N_0 under π. Then, since N_0 is invariant under the multiplication by $e^{i\theta}$, the image M_0 is a submanifold of $\boldsymbol{H}_m\boldsymbol{C}$. Induce a metric on M_0 so that the projection $\pi : N_0\to M_0$ is a Riemannian submersion. For the isometric immersion $i'_0 : N\to N_0$ and the totally geodesic immersion $i'_1 : N_0\to H_1^{2m+1}$, smooth mappings i_0 (resp. i_0) of M into M_0 (resp. M_0 into

$\boldsymbol{H}_m\boldsymbol{C}$) can be chosen in such a way that the diagrams are commutative. Then i_0 and i_1 make both isometric immersions, because $\pi : N_0 \to M_0$ is also the Riemannian submersion. On the right diagram, let α_1 and β_1 be the second fundamental forms for i'_1 and i_1. By the similar way to (4.3) we get $\alpha_1(X^*, Y^*) = \beta_1(X, Y)^*$ for any X and Y in $\mathcal{X}(M_0)$, which implies that β_1 vanishes identically, i.e., M_0 is also totally geodesic. Since M_0 is a Kaehlerian submanifold of $\boldsymbol{H}_m\boldsymbol{C}$, it is the complex hyperbolic space $\boldsymbol{H}_n\boldsymbol{C}$ and M is totally real in $\boldsymbol{H}_n\boldsymbol{C}$.

This concludes the proof.

5. Principal circle bundles over a submanifold of $\boldsymbol{P}_m\boldsymbol{C}$.

Let M be an n-dimensional totally real submanifold of a complex m-dimensional complex projective space $\boldsymbol{P}_m\boldsymbol{C}$. Then one can construct a principal circle bundle over the submanifold M with the projection π in such a way that π is compatible with the Hopt fibration $\pi : S^{2m+1} \to \boldsymbol{P}_m\boldsymbol{C}$. Namely, the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(M) & \longrightarrow & S^{2m+1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & \boldsymbol{P}_m\boldsymbol{C} \end{array}$$

By the similar verification to that stated in the previous section, the following theorem is proved.

THEOREM 5.1. *Let M be an n-dimensional complete and totally real submanifold of $\boldsymbol{P}_m\boldsymbol{C}$. If the f-structure in the normal bundle on M is parallel, then there exists a totally geodesic submanifold $M_0 = \boldsymbol{P}_n\boldsymbol{C}$ of $\boldsymbol{P}_m\boldsymbol{C}$ in which M is totally real.*

The proof will be sketched. Let (ϕ, E, ω, G) be the Sasakian structure admitted in S^{2m+1} and (J, g) the Kaehlerian structure in $\boldsymbol{P}_m\boldsymbol{C}$. By these structures, the set (P, F), (p, f) and (P', F'), (p', f') of orthogonal operators are defined for the isometric immersions $M \to \boldsymbol{P}_m\boldsymbol{C}$ and $N = \pi^{-1}(M) \to S^{2m+1}$. As is well known, M is totally real in $\boldsymbol{P}_m\boldsymbol{C}$ if and only if N is totally real in S^{2m+1}, the parallelism of the f-strucure f derives that F and p are both parallel by the same discussion as that developed in the previous sections, and moreover the operators F', p' and f' are also so. These properties imply that the distribution $F'\mathcal{D}$ defined by $p \to F'N_p$ is parallel with respect to the normal connection and dim $F'\mathcal{D} = n$, and furthermore the first normal space N_p^1 is the subspace of $F'N_p$. By means of the reduction theorem of Erbacher [4], it yields that there exists a totally geodesic S^{2n+1} of S^{2m+1} in which N is totally real (Proposition 3.3.). The restriction of the Hopf fibration to S^{2n+1} gives a complete totally geodesic Kaehlerian submanifold $M_0=\pi(S^{2n+1})$ of $\boldsymbol{P}_m\boldsymbol{C}$. Hence M_0 is the complex projective space $\boldsymbol{P}_n\boldsymbol{C}$ and M is totally real in M_0

Thus the proof is complete.

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Kyungpook University
Taegu, 635
Korea

University of Tsukuba
Ibaraki, 305
Japan