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ON R^{∞}-MANIFOLDS AND Q^{∞}-MANIFOLDS, II: INFIFITE DEFICIENCY

By

Katsuro SAKAI

ABSTRACT. We introduce the notion of D-sets (D^*-sets) and establish the Unknotting Theorem for D-sets in a manifold modeled on $\boldsymbol{R}^{\infty}=\operatorname{dir}\lim \boldsymbol{R}^n$ or $Q^{\infty}=\operatorname{dir}\lim Q^n$, where Q is the Hilbert cube. This yields equality of D-sets, D^*-sets and infinite (i.e., $\boldsymbol{R}^{\infty}$- or Q^{∞}-) deficient sets. Our Theorem corresponds to a weak version of the Unknotting Theorem for infinite deficient sets proved by V.T. Liem. However our proof is elementary and short. And we give an alternative proof of the Infinite Deficient Embedding Approximation Theorem due to Liem. Using Anderson-McCharen's trick, this Approximation Theorem strengthens our Unknotting Theorem in the strong form. Moreover, we show that the union of two $\boldsymbol{R}^{\infty}$- (or Q^{∞}-) manifolds meeting in an $\boldsymbol{R}^{\infty}$- (or Q^{∞}-) manifold is also an $\boldsymbol{R}^{\infty}$- (or Q^{∞}-) manifold, and that for any space X, $X\times\boldsymbol{R}$ is an $\boldsymbol{R}^{\infty}$- (or Q^{∞}-) manifold if and only if so is $X\times I$.

0. Introduction.

Separable paracompact manifolds modeled on $\boldsymbol{R}^{\infty}=\operatorname{dir}\lim \boldsymbol{R}^n$ and $Q^{\infty}=\operatorname{dir}\lim Q^n$, where Q is the Hilbert cube, are called *$\boldsymbol{R}^{\infty}$-manifolds* and *Q^{∞}-manifolds*, respectively. These manifolds have been studied by R. E. Heisey, V.T. Liem, et al. (cf. References of [11]). In the previous paper [11], we gave a characterization of these manifolds and elementary short proofs of the Open Embedding Theorem, the Stability Theorem, the Classification Theorem, etc. This paper is a sequel of [11].

The notions of D-sets and D^*-sets are introduced in Section 1, as generalizations of closed sets contained in collared sets, and the Unknotting Theorem for D-sets in $\boldsymbol{R}^{\infty}$- (or Q^{∞}-) manifolds is established in Section 2. Our theorem yields characterizations of infinite deficiency in these manifolds, i.e., the equality of D-sets, D^*-sets and $\boldsymbol{R}^{\infty}$- or Q^{∞}-deficient sets (see Section 3), and some fundamental properties of infinite deficient sets are easily derived, e.g., (i) *a finite union of $\boldsymbol{R}^{\infty}$- (or Q^{∞}-) deficient sets is also $\boldsymbol{R}^{\infty}$- (or Q^{∞}-) deficient* [7, Proposition 1.4] (or [5, Proposition 2]); (ii) *locally $\boldsymbol{R}^{\infty}$- (or Q^{∞}-) deficient closed set is also $\boldsymbol{R}^{\infty}$- (or Q^{∞}-) deficient*

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[7, Theorem 5.1 (or 5.3)]; (iii) *a collared submanifold is* $\boldsymbol{R}^\infty$- (*or* Q^∞-) *deficient* [7, Theorem 3.3 (or 3.4)]. Using the result of Section 7 mentioned below, we have a short proof of the converse of (iii) [6, Theorem 4.2] (or [4, Theorem 2.3]) i.e., (iii)′ *an* $\boldsymbol{R}^\infty$- (*or* Q^∞-) *deficient submanifold is collared.* Thus we have an alternative proof of the Collaring Theorem. The Unknotting Theorem for infinite deficient sets was established by Liem [7], [5], in the weak form for $\boldsymbol{R}^\infty$-case (i.e., without an ambient isotopy). Our theorem corresponds to the weak version of Liem's theorem but our proof is elementary and short. In Section 4, using our theorem, we give an easy proof of the Infinite Deficient Embedding (D-Embedding) Approximation Theorem due to Liem [6], [4]. Using Anderson-McCharen's trick [1], this Approximation Theorem strengthens our Unknotting Theorem in the strong form (see Section 5).

In [2], it is shown that for any space X, $X\times\boldsymbol{R}\cong\sigma$ (resp. Σ) if and only if $X\times I\cong\sigma$ (resp. Σ), where the space σ (resp. Σ) is a metric version of $\boldsymbol{R}^\infty$ (resp. Q^∞). In Section 6, we show this valid equally to $\boldsymbol{R}^\infty$ and Q^∞. From this, we can see that a space X containing an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold M as a dense open set is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold if $X\setminus M$ is contained in a collared set in X.

Let X_1 and X_2 be closed subsets of a space X with $X=X_1\cup X_2$ and $X_0=X_1\cap X_2$. J. Mogilski [9] showed that if X_0, X_1 and X_2 are l_2-manifolds then X is also an l_2-manifold. In Section 7, we prove its $\boldsymbol{R}^\infty$- (or Q^∞) version. J.P. Henderson and J.J. Walsh [2] constructed cell-like decompositions of σ and Σ whose decomposition spaces are not homeomorphic to σ and Σ but the products with $\boldsymbol{R}$ or I are homeomorphic to σ and Σ respectively. Their examples apply equally to $\boldsymbol{R}^\infty$ and Q^∞, as mentioned in Section 7 of [2]. Then one should remark that the Mogilski's method in [9] cannot apply to the $\boldsymbol{R}^\infty$- (or Q^∞-) version.

For undefined terms and notations, refer to the previous paper [11].

1. *D*-Sets and *D**-Sets.

Let A be a closed subset of a space X. We call A a D-set in X if it satisfies the following condition:

($\mathscr{D}$) For each compact sets $C\supset C_0$ in X and each open cover $\mathcal{U}$ of X, there exists an embedding $h: C\to X$ $\mathcal{U}$-near to the inclusion $C\subset X$ with $h|C_0=\mathrm{id}$ and $h(C\setminus C_0)\subset X\setminus A$.

And A is a D^*-set in X if it satisfies the following:

($\mathscr{D}^*$) For each closed set X_0 in X and each open cover $\mathcal{U}$ of X, there exists an embedding $f: X\to X$ $\mathcal{U}$-near to id with $f|X_0=\mathrm{id}$ and $f(X\setminus X_0)\subset X\setminus A$.

Clearly *each D^*-set is a D-set.* These sets are generalizations of closed sets contained in collared subsets of X as seen below. In this section, we will observe some properties of D-sets and D^*-sets in general spaces. However those are not required in the proof of the Unknotting Theorem for D-sets in an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold (see Section 2).

We will start to prove the following lemma:

1-1 LEMMA: *Let $\alpha\colon X\to[0,\infty)$ be a map of a paracompact space X and $\mathcal{U}$ a collection of open sets in $X\times\boldsymbol{R}$ such that for each $x\in X$ there is a $U\in\mathcal{U}$ containing $\{x\}\times[0,\alpha(x)]$. Then there exists a map $\beta\colon X\to(0,\infty)$ such that for each $x\in X$ there is a $U\in\mathcal{U}$ containing $\{x\}\times[-\beta(x),\alpha(x)+\beta(x)]$.*

PROOF: For each $x\in X$, choose an open neighborhood $U(x)$ of x in X and an $\varepsilon(x)>0$ so that $U(x)\times[-\varepsilon(x),\alpha(x)+\varepsilon(x)]$ is contained in some $U\in\mathcal{U}$. The open cover $\{U(x)|x\in X\}$ has a locally finite open refinement $\{V_\lambda|\lambda\in\Lambda\}$. From normality of X, there is an open cover $\{W_\lambda|\lambda\in\Lambda\}$ such that $\mathrm{cl}\,W_\lambda\subset V_\lambda$ for each $\lambda\in\Lambda$. For each $\lambda\in\Lambda$, choose $x_\lambda\in X$ so that $V_\lambda\subset U(x_\lambda)$ and take a Urysohn map $u_\lambda\colon X\to I$ with $u_\lambda(X\setminus V_\lambda)=0$ and $u_\lambda(\mathrm{cl}\,W_\lambda)=1$. Then we define $\beta\colon X\to(0,\infty)$ by

$$\beta(x)=\sup\{\varepsilon(x_\lambda)u_\lambda(x)|\lambda\in\Lambda\}$$

The continuity of β follows from local finiteness of $\{V_\lambda|\lambda\in\Lambda\}$. It is obvious that β has the required property. □

1-2 PROPOSITION. *Let A be a closed subset of a paracompact perfectly normal space X. If A is contained in some collared set in X, then A is a D^*-set in X, hence a D-set in X.*

PROOF: Let B be a collared set in X with $A\subset B$. Then we have an open embedding $k\colon B\times[0,1)\to X$ such that $k(x,0)=x$ for each $x\in B$. Let X_0 be a closed set in X and $\mathcal{U}$ an open cover of X. Now we will construct an embedding $f\colon X\to X$ $\mathcal{U}$-near to id with $f|X_0=\mathrm{id}$ and $f(X\setminus X_0)\subset X\setminus A$. Let W be an open set in X with $A\subset W\subset\mathrm{cl}\,W\subset k(B\times[0,1))$. Each $x\in(B\cap W)\setminus X_0$ has an open neighborhood V_x in X which is contained in $W\setminus X_0$ and some $U\in\mathcal{U}$. From Lemma 1-1, we have a map $\beta\colon(B\cap W)\setminus X_0\to(0,1)$ such that each $\{x\}\times[0,\beta(x)]$ is contained in some $k^{-1}(V_{x'})$. Take a map $\gamma\colon B\to I$ with $\gamma^{-1}(0)=(B\setminus W)\cup(B\cap X_0)$ and define a map $\alpha\colon B\to[0,1)$ by

$$\alpha(x)=\begin{cases}\beta(x)\gamma(x) & \text{for } x\in(B\cap W)\setminus X_0,\\ 0 & \text{for } x\in(B\setminus W)\cup(B\cap X_0).\end{cases}$$

We define an embedding $h\colon B\times[0,1)\to B\times[0,1)$ by

$$h(x,t)=\begin{cases}\left(x,\dfrac{1}{2}t+\dfrac{1}{2}\alpha(x)\right) & \text{if } 0\le t\le\alpha(x),\\ (x,t) & \text{otherwise.}\end{cases}$$

Observe that h is $k^{-1}(\mathcal{U})$-near to id and $h|(B\times[0,1)\setminus k^{-1}(W))\cup k^{-1}(X_0)=\mathrm{id}$. Then khk^{-1} extends to an embedding $f\colon X\to X$ with $f|X\setminus k(B\times[0,1))=\mathrm{id}$. Clearly f is the desired embedding. □

The following is trivial:

1-3 PROPOSITION: (1) *Any closed subsed of a D-set (resp. D^*-set) in an arbitrary space X is also a D-set (resp. D^*-set) in X.*

(2) *A finite union of D^*-sets of an arbitrary space X is also a D^*-set in X.*

(3) *A discrete union of D-sets (resp. D^*-sets) of a Hausdorff (resp. arbitrary) space X is also a D-set (resp. D^*-set) in X.*

(4) *If A is a D-set in a Hausdorff space X, then for any open subset U of X, $A\cap U$ is a D-set in U.*

(5) *If A is a closed subset of a Hausdorff (resp. normal) space X which is a D-set (resp. D^*-set) in an open subset of X, then A is a D-set (resp. D^*-set) in X.*

A closed subset A of a space X is a *local D-set* (resp. a *local D^*-set*) in X if each $x\in A$ has an open neighborhood U in X such that $A\cap U$ is a D-set (resp. D^*-set) in U. Then using Michael's theorem for local properties [8], we easily obtain

1-4 PROPOSITION: *Any local D^*-set in a paracompact space X is a D^*-set in X.*

PROOF: Let A be a local D^*-set in X. By (1) and (5) in 1-3, each $x\in A$ has a closed neighborhood in A which is a D^*-set in X. Using [8, Theorem 5-5], the result follows from (1), (2) and (3) in 1-3. □

In the above proof, we only use the fact that each $x\in A$ has a neighborhood A_x in A which is contained in some open subset U of X as a D^*-set in U. We will call such a closed set A a *weakly local* D^*-set in X. Similarly a *weakly local D-set* in X is defined.

1-5 COROLLARY: *Any locally compact set A in an $\boldsymbol{R}^\infty$ (or Q^∞-) manifold M is a D^*-set in M, hence a D-set in M.*

PROOF: Because of similarity, we show only the $\boldsymbol{R}^\infty$-case. Each $x\in A$ has a compact neighborhood A_x in A which is contained in an open subset U of M

homeomorphic to an open set on $\boldsymbol{R}^\infty$. Since $\boldsymbol{R}^\infty \cong I^\infty = \operatorname{dir}\lim I^n$, there is an open embedding $g: U \to I^\infty$. From compactness, $g(A_x) \subset I^n$ for some n. Hence $g(A_x)$ is contained in a collared set in I^∞, so A_x is also contained in a collared set in U. By Proposition 1-2, A_x is a D^*-set in U. Thus A is a weakly local D^*-set in M. The above remark of Proposition 1-4 assure that A is a D^*-set in X. □

2. Unknotting Theorem for D-sets.

An embetting $f: X \to Y$ of a space X into a space Y is called a *D-embetting* if $f(X)$ is a D-set in Y. We prove the following Unknotting Theorem for D-sets (D-embeddings) in an $\boldsymbol{R}^\infty$- or Q^∞-manifold.

2-1 UNKNOTTING THEOREM for D-sets: *Let M be an $\boldsymbol{R}^\infty$- or Q^∞-manifold, $f: A \to M$ a D-embedding of a D-set A in M and $\mathcal{U}$, $\mathcal{V}$ open covers of M. If f is $\mathcal{U}$-homotopic to the inclusion $A \subset M$, then f extends to a homeomorphism $\tilde{f}: M \to M$ which is* $\operatorname{st}(\mathcal{U}, \mathcal{V})$*-near to* id.

The main lemma for our Unknotting Theorem is the following which is a direct consequence of [11, Lemma 1-5] and the definition of D-sets.

2-2 LEMMA: *Let C be a D-set in an $\boldsymbol{R}^\infty$-manifold (resp. a Q^∞-manifold) M and $f: B \to M$ a map from a finite dimensional compact metric space (resp. a compact metric space) B to M that restricts to an embedding $f|A: A \to M$ on a closed subset A of B. Then for each open cover $\mathcal{U}$ of M, there exists an embedding $g: B \to M$ such that $g|A = f|A$, $g(B \setminus A) \subset M \setminus C$ and g is $\mathcal{U}$-homotopic to f stationarily on A.*

It is easy to see that each $\boldsymbol{R}^\infty$- or Q^∞-manifold is an ANE for compact metric spaces, hence for countable direct limits of compact metric spaces. If X is a countable direct limit of compact metric spaces, then so are a closed subspace of X and the product space $X \times I$. We use the next Homotopy Entension Theorem (cf. Proof of [3, Ch. IV, Theorem 2.2]).

2-3 HOMOTOPY EXTENSION THEOREM: *Let Y be an ANE for $\mathcal{C}$ and $\mathcal{U}$ an open cover of Y, where $\mathcal{C}$ is a closed hereditary (=weakly hereditary) class of normal spaces such that $X \times I \in \mathcal{C}$ for all $X \in \mathcal{C}$. If $h: A \times I \to Y$ is a $\mathcal{U}$-homotopy of a closed set A in $X \in \mathcal{C}$ such that h_0 extends to a map $f: X \to Y$, then h extends to a $\mathcal{U}$-homotopy $\tilde{h}: X \times I \to Y$ with $\tilde{h}_0 = f$.*

PROOF of THEOREM 2-1: Write $M = \operatorname{dir}\lim X_n$ where each X_n is a finite dimensional compact metric subspace (or compact metric subspace) of X_{n+1}. From paracompactness, $\mathcal{V}$ admits a sequence of open star-refinements

$$\mathcal{V} >^* \mathcal{V}_1 >^* \mathcal{V}_2 >^* \mathcal{V}_3 >^* \cdots .$$

Inductively, we define open covers $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \cdots$ as follows:

$$\mathcal{W}_0 = \mathcal{U} \text{ and } \mathcal{W}_n = \mathrm{st}(\mathcal{W}_{n-1}, \mathcal{V}_n), \qquad n = 1, 2, \cdots .$$

Then each $\mathcal{W}_n$ refines $\mathrm{st}(\mathcal{U}, \mathcal{V})$.

Put $n_1 = 1$. From the Homotopy Extension Theorem, $f: A \to M$ extends to a map $f_1': M \to M$ $\mathcal{W}_0$-homotopic to id. Using Lemma 2-2, we have an embedding $f_1: X_{n_1} \to M$ such that

$$\begin{aligned} &f_1 | X_{n_1} \cap A = f | X_{n_1} \cap A, \\ &f_1(X_{n_1} \setminus A) \subset M \setminus f(A) \quad \text{and} \\ &f_1 \overset{\mathcal{V}_1}{\simeq} f_1' | X_{n_1} \quad \text{stationarily on } X_{n_1} \cap A. \end{aligned}$$

Then f_1 extends to an embedding $\tilde{f}_1: X_{n_1} \cup A \to M$ with $\tilde{f}_1 | A = f$. Clearly $\tilde{f}_1$ is $\mathcal{V}_1$-homotopic to $f_1' | X_{n_1} \cup A$ stationarily on A, hence $\mathcal{W}_1$-homotopic to the inclusion $X_{n_1} \cup A \subset M$.

Choose an $m_1 \geq 1$ so that $f_1(X_{n_1}) \subset X_{m_1}$. Since $\tilde{f}_1^{-1}$ is $\mathcal{W}_1$-homotopic to the inclusion $f_1(X_{n_1}) \cup f(A) \subset M$, $\tilde{f}_1^{-1}$ extends to a map $g_1': M \to M$ $\mathcal{W}_1$-homotopic to id by the Homotopy Extension Theorem. From Lemma 2-2, we have an embedding $g_1: X_{m_1} \to M$ such that

$$\begin{aligned} &g_1 | f_1(X_{n_1}) \cup (X_{m_1} \cap f(A)) = \tilde{f}_1^{-1} | f_1(X_{n_1}) \cup (X_{m_1} \cap f(A)), \\ &g_1(X_{m_1} \setminus f(A)) \subset M \setminus A \quad \text{and} \\ &g_1 \overset{\mathcal{V}_2}{\simeq} g_1' | X_{m_1} \quad \text{stationarily on } f_1(X_{n_1}) \cup (X_{m_1} \cap f(A)). \end{aligned}$$

Then $g_1 | f_1(X_{n_1}) = f_1^{-1}$ and g_1 extends to an embedding $\tilde{g}_1: X_{m_1} \cup f(A) \to M$ with $\tilde{g}_1 | f(A) = f^{-1}$ which is $\mathcal{V}_2$-homotopic to $g_1' | X_{m_1} \cup f(A)$ stationarily on $f(A)$, hence $\mathcal{W}_2$-homotopic to the inclusion $X_{m_1} \cup f(A) \subset M$.

Choose an $n_2 > n_1$ so that $g_1(X_{m_1}) \subset X_{n_2}$. Similarly as above, using the Homotopy Extension Theorem and Lemma 2-2, we have an embedding $f_2: X_{n_2} \to X_{m_2}$, $m_2 > m_1$, such that $f_2 | g_1(X_{m_1}) = g_1^{-1}$ and f_2 extends to an embedding $\tilde{f}_2: X_{n_2} \cup A \to M$ with $\tilde{f}_2 | A = f$ which is $\mathcal{W}_3$-homotopic to the inclusion $X_{n_2} \cup A \subset M$.

Thus by induction, we have the following commutative diagram of embeddings:

$$\begin{tikzcd}[column sep=small]
X_{n_1} \cup A & & X_{n_2} \cup A & & X_{n_3} \cup A & & \cdots \\
\cup & & \cup & & \cup & & \\
X_{n_1} \arrow[d, "f_1"'] & \subset & X_{n_2} \arrow[d, "f_2"'] & \subset & X_{n_3} \arrow[d, "f_3"'] & \subset & \cdots \\
X_{m_1} \arrow[urr, "g_1"] & \subset & X_{m_2} \arrow[urr, "g_2"] & \subset & X_{m_3} \arrow[urr, "g_3"] & \subset & \\
\cap & & \cap & & \cap & & \\
X_{m_1} \cup f(A) & & X_{m_2} \cup f(A) & & X_{m_3} \cup f(A) & & \cdots ,
\end{tikzcd}$$

where each f_i extends to an embedding $\tilde{f}_i: X_{n_i}\cup A \to X_{m_i}\cup f(A)\subset M$ with $\tilde{f}_i|A=f$ which is $\mathcal{W}_{2i-1}$-homotopic to the inclusion $X_{n_i}\cup A\subset M$ and each g_i extends to an embedding $\tilde{g}_i: X_{m_i}\cup f(A)\to X_{n_{i+1}}\cup A\subset M$ with $\tilde{g}_i|f(A)=f^{-1}$ which is $\mathcal{W}_{2i}$-homotopic to the inclusion $X_{m_i}\cup f(A)\subset M$. Then $f_1, f_2, \cdots$ induce a homeomorphism $f_\infty: M \to M$ which is $\mathrm{st}(\mathcal{U}, \mathcal{V})$-near to id and extends f. $\square$

3. $\boldsymbol{R}^\infty$- or $\boldsymbol{Q}^\infty$-deficient Sets.

Let E be a pointed space with the base point $0\in E$. A closed subset A of a space X is said to be *E-deficient* in X if there exists a homeomorphism $f: X\to X \times E$ with $f(A)\subset X\times\{0\}$. And A is *locally E-deficient* in X if each $x\in A$ admits an open neighborhood U in X such that $A\cap U$ is E-deficient in U. Taking $(\boldsymbol{R}^\infty, 0)$, $(Q^\infty, 0)$ or $(I, 0)$ as $(E, 0)$, we obtain the notions of (local) $\boldsymbol{R}^\infty$-deficiency, (local) Q^∞-deficiency of (local) I-deficiency, respectively. For example, as easily seen, compact sets in an $\boldsymbol{R}^\infty$- or Q^∞-manifold are $\boldsymbol{R}^\infty$- or Q^∞-deficient.

In the case that $(E\times E, (0, 0))\cong(E, 0)$, e.g., $(E, 0)=(\boldsymbol{R}^\infty, 0)$ or $(Q^\infty, 0)$, *for each E-deficient set A in a space X, there exists a homeomorphism $g: X\to X\times E$ such that $g(x)=(x, 0)$ for each $x\in A$.* In fact, let $h: E\to E\times E$ and $f: X\to X\times E$ be homeomorphisms such that $h(0)=(0, 0)$ and $f(A)\subset X\times\{0\}$, then $g=(f^{-1}\times \mathrm{id}_E)\circ(\mathrm{id}_X\times h)\circ f: X\to X\times E$ is the desired homeomorphism.

Using Theorem 2-1, we can obtain the following characterization of infinite deficiency in an $\boldsymbol{R}^\infty$- or Q^∞-manifold.

3-1 THEOREM: *Let A be a closed subset of an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold M. The followings are equivalent:*

(i) *A is $\boldsymbol{R}^\infty$- (or Q^∞-) deficient in M.*

(ii) *A is I-deficient in M.*

(iii) *A is contained in a collared closed submanifold of M.*

(iv) *A is contained in a collared set in M.*

(v) *A is a D^*-set in M.*

(vi) *A is a D-set in M.*

PROOF: (i)$\to$(ii) is derived from $(\boldsymbol{R}^\infty\times I, (0, 0))\cong(\boldsymbol{R}^\infty, 0)$ or $(Q^\infty\times I, (0, 0))\cong(Q^\infty, 0)$. (ii)$\to$(iii)$\to$(iv) are trivial. (iv)$\to$(v) is Proposition 1-2. (v)$\to$(vi) is obvious. We prove (vi)$\to$(i). By the Stability Theorem (e.g., see [11]), there is a homeomorphism $h: M\times\boldsymbol{R}^\infty\to M$ (or $h: M\times Q^\infty\to M$) homotopic to the projection. Let $i: M\to M\times \{0\}\subset M\times\boldsymbol{R}^\infty$ (or $\subset M\times Q^\infty$) be the natural injection. Using (i)$\to$(vi), $hi(A)$ is a D-set in M, hence $hi|A$ is a D-embedding homotopic to the inclusion $A\subset M$. Then $hi|A$ extends to a homeomorphism $g: M\to M$. Since $g^{-1}hi|A=\mathrm{id}$, that is, $h^{-1}g|A=$

$i|A$, $h^{-1}g: M \to M\times \boldsymbol{R}^\infty$ (or $h^{-1}g: M\to M\times Q^\infty$) is a homeomorphism with $h^{-1}g(A)=A\times\{0\}$. The proof of theorem is complete. □

In the above proof of (iv)→(i), a homeomorphism $g^{-1}h: M\times\boldsymbol{R}^\infty\to M$ (or $g^{-1}h: M\times Q^\infty\to M$) can be chosen arbitrarily close to the projection because h can be so. Thus we have

3-2 COROLLARY: *Let M be an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold and A a D-set in M. Then for each open cover $\mathcal{U}$ of M, the projection $p: M\times\boldsymbol{R}^\infty\to M$ (or $p: M\times Q^\infty\to M$) is $\mathcal{U}$-homotopic to a homeomorphism stationarily on $A\times\{0\}$.*

In the above corollary, we can replace $\boldsymbol{R}^\infty$ or Q^∞ by I and $\boldsymbol{R}$ because $I\times\boldsymbol{R}^\infty\cong\boldsymbol{R}\times\boldsymbol{R}^\infty\cong\boldsymbol{R}^\infty$ and $I\times Q^\infty\cong\boldsymbol{R}\times Q^\infty\cong Q^\infty$. This is used in Sections 4 and 5.

Using our characterization of infinite deficiency, one can easily obtain the fundamental properties of infinite deficient sets in $\boldsymbol{R}^\infty$- or Q^∞-manifolds. For example, the properties mentioned Introduction have been seen in Section 1 and those proofs are fairly easy.

4. Approximation Theorems.

First, we prove the following Closed Embedding Approximation Theorem:

4-1 CLOSED EMBEDDING APPROXIMATION THEOREM: *Let M be an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold, X a countable direct limit of finite dimensional compact metric spaces (or compact metric spaces) and $f: X\to M$ a map that restricts to a D-embedding on a closed subset A of X. Then for each open cover $\mathcal{U}$ of M, there exists a closed embedding $g: X\to M$ such that $g|A=f|A$ and g is $\mathcal{U}$-near to f (moreover g is $\mathcal{U}$-homotopic to f stationarily on A).*

We use the next easily observed lemma:

4-2 LEMMA: *Let $f: X=\operatorname{dir}\lim X_n\to Y=\operatorname{dir}\lim Y_n$ be a map between countable direct limits of compact metric spaces. If f is injective and $f(X)\cap Y_n=f(X_n)$ for each $n\in\boldsymbol{N}$, then f is a closed embedding.*

PROOF OF THEOREM 4-1: Write $X=\operatorname{dir}\lim X_n$ and $M=\operatorname{dir}\lim Y_n$, where each X_n and Y_n are finite dimensional compact metric (or compact metric) subspaces of X_{n+1} and Y_{n+1} respectively. From paracompactness, $\mathcal{U}$ admits a sequence of open star-refinements

$$\mathcal{U} >^* \mathcal{U}_1 >^* \mathcal{U}_2 >^* \mathcal{U}_3 >^* \cdots.$$

Inductively, we define open covers $\mathcal{V}_1$, $\mathcal{V}_2$, $\cdots$ as follows:

$$\mathcal{V}_1=\mathcal{U}_1 \text{ and } \mathcal{V}_{n+1}=\mathrm{st}(\mathcal{V}_n, \mathcal{U}_{n+1}), \qquad n=1,2,\cdots.$$

Then each $\mathcal{V}_n$ refines $\mathcal{U}$.

Put $n_1=1$. From Lemma 2-2, we have an embedding $g_1: X_{n_1}\to M$ such that

$$g_1|A\cap X_{n_1}=f|A\cap X_{n_1},$$
$$g_1(X_{n_1}\setminus A)\subset M\setminus f(A) \text{ and}$$
$$g_1 \overset{\mathcal{V}_1}{\simeq} f|X_{n_1} \text{ stationarily on } A\cap X_{n_1}.$$

Then g_1 extends to an embedding $\tilde{g}_1: X_{n_1}\cup A\to M$ with $\tilde{g}_1|A=f|A$ which is $\mathcal{V}_1$-homotopic to $f|X_{n_1}\cup A$ stationarily on A. By the Homotopy Extension Theorem, $\tilde{g}_1$ extends to a map $g_1': X\to M$ $\mathcal{V}_1$-homotopic to f stationarily on A. Choose $m_1 \geq 1$ so that $g_1(X_{n_1})\subset Y_{m_1}$ and put

$$X_1^*=X_{n_1}\cup(A\cap f^{-1}(Y_{m_1})) \text{ and } g_1^*=\tilde{g}_1|X_1^*: X_1^*\to Y_{m_1}.$$

Note that X_1^* is compact and g_1^* is an embedding such that

$$g_1^*|A\cap X_1^*=f|A\cap X_1^* \text{ and}$$
$$g_1^* \overset{\mathcal{V}_1}{\simeq} f|X_1^* \text{ stationarily on } A\cap X_1^*.$$

Choose an $n_2>n_1$ so that $X_1^*\subset X_{n_2}$. Since Y_{m_1} is compact, Y_{m_1} is a D-set in M from deficiency. Hence $f(A)\cup Y_{m_1}$ is also a D-set in M by 1-3 with 3-1. From Lemma 2-2, we have an embedding $g_2: X_{n_2}\to M$ such that

$$g_2|X_{n_1}\cup(A\cap X_{n_2})=g_1'|X_{n_1}\cup(A\cap X_{n_2}),$$
$$g_2(X_{n_2}\setminus(X_{n_1}\cup A))\subset M\setminus(f(A)\cup Y_{m_1}) \text{ and}$$
$$g_2 \overset{\mathcal{U}_2}{\simeq} g_1'|X_{n_2} \text{ stationarily on } X_{n_1}\cup(A\cap X_{n_2}).$$

Since $g_2|A\cap X_{n_2}=f|A\cap X_{n_2}$ and $g_2(X_{n_2}\setminus A)\subset M\setminus f(A)$, g_2 extends to an embedding $\tilde{g}_2: X_{n_2}\cup A\to M$ with $\tilde{g}_2|A=f|A$. Then $\tilde{g}_2$ is clearly $\mathcal{U}_2$-homotopic to $g_1'|X_{n_2}\cup A$ stationarily on $X_{n_1}\cup A$. By the Homotopy Extension Theorem, $\tilde{g}_2$ extends to a map $g_2': X\to M$ which is $\mathcal{U}_2$-homotopic to g_1' stationarily on $X_{n_1}\cup A$, hence $\mathcal{V}_2$-homotopic to f stationarily on A. Choose an $m_2>m_1$ so that $g_2(X_{n_2})\subset Y_{m_2}$ and put

$$X_2^*=X_{n_2}\cup(A\cap f^{-1}(Y_{m_2})) \text{ and } g_2^*=\tilde{g}_2|X_2^*: X_2^*\to Y_{m_2}.$$

Then X_2^* is compact and g_2^* is an embedding such that

$$g_2^*|X_1^*=g_1^*, \qquad g_2^*|A\cap X_2^*=f|A\cap X_2^*,$$
$$g_2^*(X_2^*\setminus X_1^*)\subset M\setminus Y_{m_1} \text{ and}$$
$$g_2^* \overset{\mathcal{V}_2}{\simeq} f|X_2^* \text{ stationarily on } A\cap X_2^*.$$

Thus inductively, we have integers $1=n_1<n_2<\cdots$, $1\leq m_1<m_2<\cdots$ and embeddings $g_i^*: X_i^* \to Y_{m_i}$ of compact sets X_i^* in X, $i=1,2,\cdots$, such that

$$X_{n_i} \subset X_i^* \subset X_{n_{i+1}}, \quad g_{i+1}^*|X_i^* = g_i^*, \quad g_i^*|A\cap X_i^* = f|A\cap X_i^*,$$
$$g_i^*(X_i^* \setminus X_j^*) \subset M \setminus Y_{m_j} \text{ if } j<i, \text{ and}$$
$$g_i^* \overset{\mathcal{V}_i}{\simeq} f|X_i^* \text{ stationarily on } A\cap X_i^*.$$

Since $X=\operatorname{dir}\lim X_i^*$ and $M=\operatorname{dir}\lim Y_{m_i}$, embeddings $g_1^*, g_2^*, \cdots$ induce a map $g: X\to M$ extending $f|A$ which is clearly injective and $\mathcal{U}$-near to f. By Lemma 4-2, g is a closed embedding.

For the additional statement, we can construct a $\mathcal{U}$-homotopy between f and g since each g_i' is $\mathcal{U}_i$-homotopic to g_{i-1}' stationarily on $X_{n_i}\cup A$ (where $g_0'=f$ and $X_{n_0}=\emptyset$). Otherwise, if we assume by the Open Embedding Theorem that M is an open set in $\boldsymbol{R}^\infty$ (or Q^∞) and each element of $\mathcal{U}$ is convex, then the additional statement is immediate. □

The following Approximation Theorem has been proved by V.T. Liem. Using Theorem 4-1 and Corollary 3-2, we give an easy alternative proof.

4-3 D-EMBEDDING APPROXIMATION THEOREM [4], [6]: *Let M be an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold, X a countable direct limit of finite dimenisional compact metric (or compact metric) spaces and $f: X\to M$ a map that restricts to a D-embedding on a closed subsed A of X. Then for each open cover $\mathcal{U}$ of M, there exists a D-embedding $g: X\to M$ such that $g|A=f|A$ and g is $\mathcal{U}$-homotopic to f stationarily on A.*

PROOF: By theorem 4-1, we may assume without loss of generality that f is a closed embedding. From Corollary 3-2 (cf. its remark), the projection $p: M\times I\to M$ is $\mathcal{U}$-homotopic to a homeomorphism $h: M\times I\to M$ stationarily on $f(A)\times\{0\}$. Let $i: M\to M\times\{0\}\subset M\times I$ be the natural injection. The embedding $g=hif: X\to M$ is the desired one. □

For open embeddings, we can strengthen the Open Embedding Approximation Theorem [11]:

4-4 OPEN EMBEDDING APPROXIMATION THEOREM (strong version): *Let M and N be $\boldsymbol{R}^\infty$- (or Q^∞-) manifolds and A an R^∞- (or Q^∞-) deficient set in M. Then for any open cover $\mathcal{U}$ of N, any map $f: M\to N$ is $\mathcal{U}$-homotopic to an open embedding $g: M\to N$ such that $g(A)$ is an $\boldsymbol{R}^\infty$- (or Q^∞-) deficient set in N. If $f|A: A\to N$ is an $\boldsymbol{R}^\infty$- (or Q^∞-) deficient embedding, then f and g are $\mathcal{U}$-homotopic stationarily on A (of course $f|A=g|A$).*

PROOF: From Theorem 3-3, we may assume that $f|A: A\to N$ is an $\boldsymbol{R}^\infty$- (or Q^∞-) deficient embedding. Apply the proof of [11, Theorem 2-2] in which [11, Lemma 1-5] is replaced with Lemma 2-2 in this paper (cf. Proof of Theorem 4-1). □

As an immediate consequence, we have a strong version of the Open Embedding Theorem:

4-5 OPEN EMBEDDING THEOREM (strong version): *Let M be an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold and A an $\boldsymbol{R}^\infty$- (or Q^∞-) deficient set in M. Then M can be embedded in $\boldsymbol{R}^\infty$ (or Q^∞) so that M is open and A is closed and $\boldsymbol{R}^\infty$- (or Q^∞-) deficient in $\boldsymbol{R}^\infty$ (or Q^∞).*

5. **Unknotting Theorem** (strong version).

Using Theorems 2-1, 4-3, Proposition 1-3 (with 3-1) and Corollary 3-2, we can prove the following strong version of Theorem 2-1 by Anderson-McCharen's trick in [1].

5-1 UNKNOTTING THEOREM (strong version): *Let M be an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold, A a D-set in M and $\mathcal{U}$, $\mathcal{V}$ open covers of M. If a D-embedding $f: A\to M$ is $\mathcal{U}$-homotopic to the inclusion $i: A\subset M$, then f extends to a homeomorphism $\tilde{f}: M\to M$ which is ambiently invertibly* $\mathrm{st}(\mathcal{U}, \mathcal{V})$*-isotopic to* id. *Moreover if the homotopy $\Phi: i\simeq f$ is stationary on a closed subset A_0 of A and* $\mathrm{cl}\,\Phi((A\setminus A_0)\times I)$ *is contained in an open subset W of M, then the isotopy $\Psi: \mathrm{id}\simeq\tilde{f}$ can be chosen to be stationary on $A_0\cup(M\setminus W)$.*

For the sake of completeness, we include the details. First we prove the below:

5-2 LEMMA: *Let $\gamma: Y\to[0,\infty)$ be a map of a paracompact space Y, W an open set in $Y\times\boldsymbol{R}$ and $\mathcal{U}$ an open cover of $Y\times\boldsymbol{R}$ such that if $y\in\mathrm{cl}(Y\setminus\gamma^{-1}(0))$ then $\{y\}\times[0,\gamma(y)]\subset W\cap U$ for some $U\in\mathcal{U}$. Then there exists an ambient invertible $\mathcal{U}$-isotopy $\theta: Y\times\boldsymbol{R}\times I\to Y\times\boldsymbol{R}$ stationary on $\gamma^{-1}(0)\times\boldsymbol{R}\cup(Y\times\boldsymbol{R}\setminus W)$ such that $\theta_0=$* id *and $\theta_1(y,0)=(y,\gamma(y))$ for each $y\in Y$.*

PROOF: From Lemma 1-1, we have maps $\alpha,\beta: Y\to\boldsymbol{R}$ such that for each $y\in Y$, $\alpha(y)<0\le\gamma(y)<\beta(y)$ and $\{y\}\times[\alpha(y),\beta(y)]$ is contained in some $U\in\mathcal{U}$, moreover if $y\in\mathrm{cl}(Y\setminus\gamma^{-1}(0))$ then $\{y\}\times[\alpha(y),\beta(y)]\subset W$. Then the desired isotopy $\theta: Y\times\boldsymbol{R}\times I\to Y\times\boldsymbol{R}$ is defined by

$$\theta(y,s,t)=\begin{cases}\left(y,\dfrac{\alpha(y)-t\gamma(y)}{\alpha(y)}s+t\gamma(y)\right) & \text{if } \gamma(y)\neq 0 \text{ and } \alpha(y)\leq s\leq 0,\\ \left(y,\dfrac{\beta(y)-t\gamma(y)}{\beta(y)}s+t\gamma(y)\right) & \text{if } \gamma(y)\neq 0 \text{ and } 0\leq s\leq \beta(y),\\ (y,s) & \text{otherwise.}\end{cases} \quad \square$$

PROOF of THEOREM 5-1: Let $\mathcal{V}'$ be an open star-refinement of $\mathcal{V}$ and W' an open set in M such that $\mathrm{cl}\,\Phi((A\setminus A_0)\times I)\subset W'\subset \mathrm{cl}\,W'\subset W$.

First, we will construct an ambient invertible $\mathcal{V}'$-isotopy $\Psi': M\times I\to M$ stationary on $A_0\cup(M\setminus W')$ such that $\Psi'_0=\mathrm{id}$ and $\Psi'_1(f(A\setminus A_0))\cap A=\emptyset$. From Proposition 1-3 with Theorem 3-1, $A\cup f(A)$ is a D-set in M. Using Corollary 3-2 (cf. its remark), we have a homeomorphism $h: M\times \boldsymbol{R}\to M$ such that $h|(A\cup f(A))\times\{0\}=p|(A\cup f(A))\times\{0\}$, where $p: M\times\boldsymbol{R}\to M$ is the projection. Put $W'_0=p(M\times\{0\}\cap h^{-1}(W'))$. From Lemma 1-1, we have a map $\beta': W'\to(0,1]$ such that for each $x\in W'_0$ there is a $V'\in\mathcal{V}'$ with $\{x\}\times[0,\beta'(x)]\subset h^{-1}(V')\cap h^{-1}(W')$. Choose an open set G in M so that $\mathrm{cl}\,f(A\setminus A_0)=\mathrm{cl}\,(f(A)\setminus A_0)\subset G\subset \mathrm{cl}\,G\subset W'_0$. Take a map $\beta'': M\to I$ with $\beta''^{-1}(0)=A_0\cup(M\setminus G)$ and define a map $\beta: M\to I$ by

$$\beta(x)=\begin{cases}\beta'(x)\beta''(x) & \text{if } x\in W'_0,\\ 0 & \text{if } x\notin W'_0.\end{cases}$$

Then $A_0\subset\beta^{-1}(0)$, $f(A\setminus A_0)\subset M\setminus\beta^{-1}(0)\subset G$ and for each $x\in W'_0$ there is $V'\in\mathcal{V}'$ with $\{x\}\times[0,\beta(x)]\subset h^{-1}(V')\cap h^{-1}(W')$. Hence by Lemma 5-2, we have an ambient invertible $h^{-1}(\mathcal{V}')$-isotopy $\theta': M\times\boldsymbol{R}\times I\to M\times\boldsymbol{R}$ stationary on $A_0\times\boldsymbol{R}\cup(M\times\boldsymbol{R}\setminus h^{-1}(W'))$ such that $\theta'_0=\mathrm{id}$ and $\theta'_0(x,0)=(x,\beta(x))$ for each $x\in M$. The desired isotopy $\Psi': M\times I\to M$ is defined by $\Psi'(x,t)=h\theta'(h^{-1}(x),t)$.

Next, we will construct an ambient invertible $\mathrm{st}^2(\mathcal{U},\mathcal{V}')$-isotopy $\Psi'': M\times I\to M$ stationary on $A_0\cup(M\setminus W)$ such that $\Psi''_0=\mathrm{id}$ and $\Psi''_1|A=\Psi'_1 f$. Using $\Phi: i\simeq f$ and $\Psi': \mathrm{id}\simeq\Psi'_1$, we can obtain a $\mathrm{st}(\mathcal{U},\mathcal{V}')$-homotopy $\Phi': A\times I\to M$ stationary on A_0 such that $\Phi'_0=i$, $\Phi'_1=\Psi'_1 f$ and $\Phi'((A\setminus A_0)\times I)\subset W'$. Let $\alpha: M\to I$ be a map with $\alpha^{-1}(0)=A_0$. Denote

$$K=\{(x,0,t)\,|\,x\in A,\ 0\leq t\leq\alpha(x)\}\subset M\times I\times\boldsymbol{R}\quad\text{and}$$
$$L=\{(x,0,t)\in K\,|\,t=0\ \text{ or }\ t=\alpha(x)\}.$$

Then K is a D-set in $M\times I\times\boldsymbol{R}$ because it is contained in a collared set (1-2). Define a map $\Phi'': K\to M$ by

$$\Phi''(x,0,t)=\begin{cases}\Phi'\left(x,\dfrac{t}{\alpha(x)}\right) & \text{if } x\notin A_0,\\ x & \text{if } x\in A_0.\end{cases}$$

Observe $\Phi''(x,0,0)=x$, $\Phi''(x,0,\alpha(x))=\Psi_1'(x)$ for each $x\in A$ and $\Phi''(L)=A\cup\Psi_1'f(A)$ is a D-set in M (1-3 with 3-1). Since $\Psi_1'f(A\setminus A_0)\cap A=\emptyset$, $\Phi''|L$ is a closed embedding, so a D-embedding. Note that $\Phi''(K\setminus A_0\times\{0\}\times\{0\})\subset W'$. Let W'' be an open set in M with $\mathrm{cl}\,W'\subset W''\subset\mathrm{cl}\,W''\subset W$ and $\mathcal{V}''$ an open cover of M which refines both covers $\mathcal{V}'$ and $\{W'', M\setminus\mathrm{cl}\,W'\}$. By Theorem 4-3, Φ'' is $\mathcal{V}''$-homotopic to a D-embedding $\Phi''':K\to M$ stationary on L. Then Φ''' is homotopic to $q|K$ because so is Φ''', where $q:M\times I\times\boldsymbol{R}\to M$ is the projection. Since q is a near homeomorphism by the Stability Theorem, Φ''' is homotopic to the restriction of a homeomorphism from $M\times I\times\boldsymbol{R}$ onto M. Using Theorem 2-1, Φ''' extends to a homeomorphism $g:M\times I\times\boldsymbol{R}\to M$. For each $x\in A\setminus A_0$, choose a $U\in\mathcal{U}$ so that

$$\Phi''(\{x\}\times\{0\}\times[0,\alpha(x)])=\Phi'(\{x\}\times I)\subset\mathrm{st}\,(U,\mathcal{V}')\cap W'\,.$$

Since Φ'' and Φ''' are $\mathcal{V}''$-near,

$$\begin{aligned}\Phi'''(\{x\}\times\{0\}\times[0,\alpha(x)])&\subset\mathrm{st}\,(\mathrm{st}\,(U,\mathcal{V}'),\mathcal{V}')\cap W''\\&=\mathrm{st}^2\,(U,\mathcal{V}')\cap W''\,.\end{aligned}$$

Hence

$$\{x\}\times\{0\}\times[0,\alpha(x)]\subset g^{-1}(\mathrm{st}^2\,(U,\mathcal{V}')\cap W'')\,.$$

Let W''' be an open set in M with $\mathrm{cl}\,W''\subset W'''\subset\mathrm{cl}\,W'''\subset W$. For each $x\in\mathrm{cl}\,(A\setminus A_0)$ there is some $U\in\mathcal{U}$ such that

$$\{x\}\times\{0\}\times[0,\alpha(x)]\subset g^{-1}(\mathrm{st}^2\,(U,\mathcal{V}'))\cap g^{-1}(W''')\,.$$

Let N be an open neighborhood of $\mathrm{cl}\,(A\setminus A_0)\times\{0\}$ in $M\times I$ such that if $y\in N$ then

$$\{y\}\times[0,\alpha r(y)]\subset g^{-1}(\mathrm{st}^2\,(U,\mathcal{V}'))\cap g^{-1}(W''')$$

for some $U\in\mathcal{U}$, where $r:M\times I\to M$ is the projection. Take a Urysohn map k: $M\times I\to I$ with $k(M\times I\setminus N)=0$ and $k(\mathrm{cl}\,(A\setminus A_0)\times\{0\})=1$ and define a map $\gamma:M\times I\to I$ by $\gamma(y)=k(y)\cdot\alpha r(y)$. Then observe that $\gamma|A\times\{0\}=\alpha r|A\times\{0\}$ and for each $y\in\mathrm{cl}\,(M\times I\setminus\gamma^{-1}(0))$, there is a $U\in\mathcal{U}$ such that

$$\{y\}\times[0,\gamma(y)]\subset g^{-1}(\mathrm{st}^2\,(U,\mathcal{V}'))\cap g^{-1}(W)\,.$$

Hence by Lemma 5-2, we have an ambient invertible $g^{-1}(\mathrm{st}^2\,(\mathcal{U},\mathcal{V}'))$-isotopy θ'': $M\times I\times\boldsymbol{R}\times I\to M\times I\times\boldsymbol{R}$ stationary on $A_0\times I\times\boldsymbol{R}\cup(M\times I\times\boldsymbol{R}\setminus g^{-1}(W))$ such that $\theta_0''=\mathrm{id}$ and $\theta_1''(x,0,0)=(x,0,\gamma(x,0))=(x,0,\alpha(x))$ for each $x\in M$. Recall that for each $x\in A$, $g(x,0,0)=x$ and $g(x,0,\alpha(x))=\Psi_1'f(x)$. Then the desired isotopy $\Psi'':M\times I\to M$ is defined by $\Psi''(x,t)=g\theta''(g^{-1}(x),t)$.

Finally, we define an ambient invertible isotopy $\Psi:M\times I\to M$ by $\Psi_t=\Psi_t'^{-1}\Psi_t''$, $t\in I$. This isotopy is stationarily on $A_0\cup(M\setminus W)$. And it is a $\mathrm{st}\,(\mathcal{U},\mathcal{V})$-isotopy because

$$\begin{aligned}\mathrm{st}\,(\mathrm{st}^2\,(\mathcal{U},\mathcal{V}'),\mathcal{V}') &= \mathrm{st}\,(\mathrm{st}\,(\mathrm{st}\,(\mathcal{U},\mathcal{V}'),\mathcal{V}'),\mathcal{V}') \\ &= \mathrm{st}\,(\mathcal{U},\mathrm{st}\,\mathcal{V}') \\ &< \mathrm{st}\,(\mathcal{U},\mathcal{V}). \quad \square\end{aligned}$$

6. Enlargement of Manifolds.

In previous paper [11], we gave a characterization of $\boldsymbol{R}^\infty$- or Q^∞-manifolds. The following is its variation as mentioned after Lemma 1–5 in [11].

6–1 THEOREM: (a) *A countable direct limit X of finite dimentional compact metric spaces is an $\boldsymbol{R}^\infty$-manifold if and only if X is an ANE for (finite dimentional) compact metric spaces and it has the following property:*

($\mathcal{A}'_f$) *Let $f: B\to X$ be a map from a finite dimensional compact metric space B into X that restricts to an embedding on a closed subset A of B. Then there exists an embedding $g: B\to X$ such that $f|A=g|A$.*

(b) *A countable direct limit X of compact metric spaces is a Q^∞-manifold if and only if X is an ANE for compact metric spaces and it has the property ($\mathcal{A}'$) that is the above property ($\mathcal{A}'_f$) with the phase "finite dimensional" deleted.*

Using the above characterization, we prove the following $\boldsymbol{R}^\infty$- (or Q^∞-) version of [2, Theorem 3].

6–2 THEOREM: *For any space X, $X\times\boldsymbol{R}$ is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold if and only if so is $X\times[0,\infty]$, hence if and only if so is $X\times I$.*

PROOF. The "if" part is trivial since $X\times\boldsymbol{R}$ can be embedded in $X\times[0,\infty)$ as an open set. We must prove the "only if" part. Because of similarity, we show only $\boldsymbol{R}^\infty$-case.

First, we note that $X\times[0,\infty)$ is an ANE for compact metric spaces which is a countable direct limit of finite dimensional compact metric spaces, since so is $X\times\boldsymbol{R}$. Then we may prove that $X\times[0,\infty)$ has property ($\mathcal{A}'_f$). Let $f: B\to X\times[0,\infty)$ be a map from a finite dimensional compact metric space B into $X\times[0,\infty)$ that restricts to an embedding on a closed subset A of B. From Corollary 1–5, $pf(A)\times\boldsymbol{R}$ is a D-set in an $\boldsymbol{R}^\infty$-manifold $X\times\boldsymbol{R}$, where $p: X\times\boldsymbol{R}\to X$ is the projection. Using Lemma 2–2, we have an embedding $g: B\to X\times\boldsymbol{R}$ such that $g|A=f|A$ and $g(B\setminus A)\subset X\times\boldsymbol{R}\setminus pf(A)\times\boldsymbol{R}$. If we can construct a homeomorphism $h: X\times\boldsymbol{R}\to X\times\boldsymbol{R}$ so that $h|pf(A)\times\boldsymbol{R}=\mathrm{id}$ and $h(g(B))\subset X\times[0,\infty)$, then $hg: B\to X\times[0,\infty)$ is an embedding with $hg|A=f|A$, so $X\times[0,\infty)$ has property ($\mathcal{A}'_f$).

Now, we will construct such a homeomorphism. From compactness of $g(B)$, we may assume that $g(B)\subset X\times(-1,\infty)$. For each $n\in\boldsymbol{N}$, put

$$D_n = p(g(B) \cap X \times (-\infty, -2^{-n}]) .$$

Then each D_n is a closed set in X missing $pf(A)$. Let $k_n: X \to I$ be a map with $k_n(pf(A))=0$ and $k_n(D_n)=1$. Define a map $k: X \to I$ by

$$k(x) = \sum_{n=1}^{\infty} 2^{-n} k_n(x) \qquad \text{for each } x \in X.$$

Then clearly $k(pf(A))=0$ and $x \in D_n$ implies $k(x) \geq 2^{-(n-1)}$. It follows that

$$g(B) \subset \{(x, t) \in X \times \boldsymbol{R} \mid t \geq -k(x)\}$$

because if $g(y)=(x, t) \in X \times (-2^{-(n-1)}, -2^{-n}]$ then $x \in D_n$ so $t > -2^{-(n-1)} \geq -k(x)$. The desired homeomorphism $h: X \times \boldsymbol{R} \to X \times \boldsymbol{R}$ is defined by

$$h(x, t) = (x, t + k(x)) \qquad \text{for each } (x, t) \in X \times \boldsymbol{R}.$$

For the additional statement, the "if" part is trivial and the "only if" part follows from $X \times I = X \times [0, 1) \cup X \times (0, 1]$. □

H. Toruńczyk [12] showed that if a complete ANR X contains an l_2-manifold whose complement is a Z-set in X then X is necessarily an l_2-manifold. For σ- or Σ-manifolds, the similar statement holds (see [12, Theorem 5.2]). For $\boldsymbol{R}^\infty$- or Q^∞-manifolds, we have the following:

6–3 PROPOSITION: *Let M be an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold which is embedded in a space X as a dense set. If $X \setminus M$ is contained in a union $\bigcup_{\lambda \in \Lambda} A_\lambda$ of collared sets A_λ, $\lambda \in \Lambda$, in X and $X \setminus M$ or $\bigcup_{\lambda \in \Lambda} A_\lambda$ is closed in X, then X is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold.*

PROOF: For each $\lambda \in \Lambda$, let $k_\lambda: A_\lambda \times [0, 1) \to X$ be an open embedding such that $k_\lambda(x, 0) = x$ for each $x \in A_\lambda$. Since $k_\lambda(A_\lambda \times (0, 1))$ is an open subset of M, $A_\lambda \times (0, 1)$ is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold, hence so is $A_\lambda \times [0, 1)$. Note if $\bigcup_{\lambda \in \Lambda} A_\lambda$ is closed in X then $X \setminus \bigcup_{\lambda \in \Lambda} A_\lambda$ is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold because it is an open subset of M. Thus

$$\{k_\lambda(A_\lambda \times [0, 1)) \mid \lambda \in \Lambda\} \cup \{M\} \quad \text{or} \quad \{k_\lambda(A_\lambda \times [0, 1)) \mid \lambda \in \Lambda\} \cup \{X \setminus \textstyle\bigcup_{\lambda \in \Lambda} A_\lambda\}$$

is an open cover of X all whose member is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold. Hence X is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold. □

In Section 1, we introduced D-sets and D^*-sets as generalizations of closed sets contained in collared sets. One should notice that Z-sets in an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold are not necessarily infinite deficient, hence not contained in collared sets [5], whereas Z-sets in an l_2- (or σ- or Σ-) manifold are infinite deficient, hence contained in collared sets.

6-4 PROBLEM: Let X be an ANE for compact metric spaces which is a coutable direct limit of finite dimensional compact metric (or compact metric) spaces. *If X contains an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold whose complement is a D-set in X, then is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold?*

7. Union of Two $\boldsymbol{R}^\infty$- (or $\boldsymbol{Q}^\infty$-) Manifolds.

In this section, we prove the following theorem:

7-1 THEOREM: *Let X_1 and X_2 be closed subsets of a space X with $X=X_1\cup X_2$ and $X_0=X_1\cap X_2$. If X_0, X_1 and X_2 are $\boldsymbol{R}^\infty$- (or Q^∞-) manifolds then so is X.*

Although this is the $\boldsymbol{R}^\infty$- (or Q^∞-) version of the Mogilski's result [9], his method cannot apply as mentioned in Introduction. We use the characterization of $\boldsymbol{R}^\infty$- and Q^∞-manifolds, i. e., Theorem 6-1. To prove the theorem, we first show the following lemma:

7-2 LEMMA: *Let Y and Z be closed subspaces of a space X with $X=Y\cup Z$. If $Y=\operatorname{dir}\lim Y_n$ and $Z=\operatorname{dir}\lim Z_n$ where each Y_n and Z_n are closed in Y_{n+1} and Z_{n+1} respectively, then $X=\operatorname{dir}\lim(Y_n\cup Z_n)$.*

PROOF: Let $A\subset X$. Assume that $A\cap(Y_n\cup Z_n)$ is closed in $Y_n\cup Z_n$ for each $n\in\boldsymbol{N}$. Since $A\cap Y_n$ is closed in Y_n for each $n\in\boldsymbol{N}$, $A\cap Y$ is closed in Y, hence it is closed in X. Similarly $A\cap Z$ is closed in X. Therefore A is closed in X. Since $X=\bigcup_{n\in\boldsymbol{N}}(Y_n\cup Z_n)$, this implies $X=\operatorname{dir}\lim(Y_n\cup Z_n)$. □

PROOF of THEOREM 7-1: Because of similarity, we proove only $\boldsymbol{R}^\infty$-case. From the above lemma, X is a countable direct limit of finite dimensional compact metric spaces. Note that X is an ANE for compact metric spaces. Therefore we may show that X has property $(\mathcal{A}'_f)$ in Theorem 6-1. Let $f: B\to X$ be a map from a finite dimensional compact metric space B into X that restricts to an embedding on a closed subset A of B. Put $B_i=f^{-1}(X_i)$ and $A_i=A\cap B_i$ for $i=0,1,2$. First using [11, Lemma 1-5], we replace $f|B_0$ with an embedding $g_0: B_0\to X_0$ such that $g_0|A_0=f|A_0$ and g_0 is homotopic to $f|B_0$ stationarily on A_0. Then g_0 extends to an embedding $g_0': B_0\cup A_1\to X_1$ which is homotopic to $f|B_0\cup A_1$ stationarily on A_1. By the Homotopy Extension Theorem, g_0' extends to a map $g_0'': B_1\to X_1$ which is homotopic to $f|B_1$ stationarily on A_1. Using again [11, Lemma 1-5], we have an embedding $g_1: B_1\to X_1$ such that $g_1|B_0\cup A_1=g_0'$, hence $g_1|B_0=g_0$ and $g_1|A_1=f|A_1$. From compactness, $g_1(B_1)\cap X_2$ is a D-set in X_2 (1-5). Similarly as above, but using Lemma 2-2, we have an embedding $g_2: B_2\to X_2$ such that $g_2|B_0=g_0$, $g_2|A_2=f|A_2$ and moreover $g_2(B_2\setminus B_0)\cap g_1(B_1)=\emptyset$. Then we can define an embedding $g: B\to X$ by $g|B_1=$

g_1 and $g|B_2=g_2$. Clearly $g|A=f|A$. □

Since examples of Henderson-Walsh [2] apply equally to $\boldsymbol{R}^\infty$ and Q^∞, as mentioned in Section 7 of [2], we have spaces Y and Z such that $Y\not\cong\boldsymbol{R}^\infty$ and $Z\not\cong Q^\infty$ but $Y\times I\cong Y\times\boldsymbol{R}\cong\boldsymbol{R}^\infty$ and $Z\times I\cong Z\times\boldsymbol{R}\cong Q^\infty$ (cf. Theorem 6-2). Let $X_1=Y\times[0,1]$ (or $Z\times[0,1]$) and $X_2=Y\times[1,2]$ (or $Z\times[1,2]$). Then $X=X_1\cup X_2\cong X_1\cong X_2\cong\boldsymbol{R}^\infty$ (or Q^∞) but $X_0=X_1\cap X_2\not\cong\boldsymbol{R}^\infty$ (or Q^∞), so the assumption in Theorem 7-1 that X_0 is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold is not essential.

Because of examples of Henderson-Walsh [2], Mogilski's method in [9] cannot apply to the σ- (or Σ-) version of Theorem 7-1. However, using Mogilski's characterization of σ- and Σ-manifolds [10], this can be proved similarly as Theorem 7-1.

As an application, we prove the following Collaring Theorem due to Liem:

7-3 Collaring Theorem [7, Theorems 3-3 and 3-4]: *Let N be a closed $\boldsymbol{R}^\infty$- (or Q^∞-) submanifold of an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold M. Then N is $\boldsymbol{R}^\infty$- (or Q^∞-) deficient in M if and only if N is collared in M.*

Proof: The "if" part is follows from Theorem 3-1. To prove the "only if" part, put $L=M\times\{0\}\cup N\times I$. From Theorem 7-1, L is an $\boldsymbol{R}^\infty$- (or Q^∞-) manifold. The projection $p:L\to M$ is a fine homotopy equivalence, so a near homeomorphism (e.g., see [11, Theorem 2-3]). Using Theorem 2-1, we have a homeomorphism $h:L\to M$ such that $h(x,1)=x$ for each $x\in L$. Then N is collared in M. □

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Institute of Mathematics
University of Tsukuba
Sakura-mura, Ibaraki, 305 JAPAN