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COVERINGS OF GENERALIZED CHEVALLEY GROUPS ASSOCIATED WITH AFFINE LIE ALGEBRAS

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R. Steinberg [21] has given a presentation of a simply connected Chevalley group (=the group of k-rational points of a split, semisimple, simply connected algebraic group defined over a field k) and has constructed the (homological) universal covering of the group. In this note, we will consider an analogy for a certain family of groups associated with affine Lie algebras.

1. Chevalley groups, Steinberg groups and the functor $K_2(\Phi, \cdot)$.

Let Φ be a reduced irreducible root system in a Euclidean space $\boldsymbol{R}^n$ with an inner product $(\cdot, \cdot)$ (cf. [4], [6]). We denote by Φ^+ (resp. Φ^-) the positive (resp. negative) root system of Φ with respect to a fixed simple root system $\Pi=\{\alpha_1, \cdots, \alpha_n\}$. We suppose that α_1 is a long root (for convenience' sake). Let α_{n+1} be the negative highest root of Φ. Set $a_{ij}=2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ for each $i, j=1, 2, \cdots, n+1$. The matrices $A=(a_{ij})_{1\leq i, j\leq n}$ and $\tilde{A}=(a_{ij})_{1\leq i, j\leq n+1}$ are called a Cartan matrix of Φ and the affine Cartan matrix associated with A respectively (cf. [4], [5], [6]).

Let $G(\Phi, \cdot)$ be a Chevalley-Demazure group scheme of type Φ (cf. [1], [20]). For a commutative ring R, with 1, we call $G(\Phi, R)$ a Chevalley group over R. For each $\alpha\in\Phi$, there is a group isomorphism—"exponential map"—of the additive group of R into $G(\Phi, R): t\longmapsto x_\alpha(t)$. The elementary subgroup $E(\Phi, R)$ of $G(\Phi, R)$ is defined to be the subgroup generated by $x_\alpha(t)$ for all $\alpha\in\Phi$ and $t\in R$. We use the notation $G_1(\Phi, \cdot)$ and $E_1(\Phi, \cdot)$ (resp. $G_0(\Phi, \cdot)$ and $E_0(\Phi, \cdot)$) if $G(\Phi, \cdot)$ is simply connected (resp. of adjoint type). It is well-known that $G_1(\Phi, R)=E_1(\Phi, R)$ if R is a Euclidean domain (cf. [22, Theorem 18/Corollary 3]).

Let $St(\Phi, R)$ be the group generated by the symbols $\hat{x}_\alpha(t)$ for all $\alpha\in\Phi$ and $t\in R$ with the defining relations

(A) $\hat{x}_\alpha(s)\hat{x}_\alpha(t)=\hat{x}_\alpha(s+t)$,

(B) $[\hat{x}_\alpha(s), \hat{x}_\beta(t)]=\prod \hat{x}_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}\, s^i t^j)$,

(B)′ $\hat{w}_\alpha(u)\hat{x}_\alpha(t)\hat{w}_\alpha(-u)=\hat{x}_{-\alpha}(-u^{-2}t)$

for all $\alpha, \beta\in\Phi(\alpha+\beta\neq 0)$, $s, t\in R$ and $u\in R^*$, the units of R, where $\hat{w}_\alpha(u)=\hat{x}_\alpha(u)\hat{x}_{-\alpha}(-$

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$u^{-1})\hat{x}_\alpha(u)$ (cf. [20]). We call $St(\Phi, R)$ a Steinberg group over R.

Since the relations corresponding to (A), (B), (B)′ hold in $E_1(\Phi, R)$, there is a homomorphism θ of $St(\Phi, R)$ onto $E_1(\Phi, R)$ such that $\theta(\hat{x}_\alpha(t))=x_\alpha(t)$ for all $\alpha\in\Phi$ and $t\in R$. Put $K_2(\Phi, \cdot)=Ker[St(\Phi, \cdot)\xrightarrow{\theta} E_1(\Phi, \cdot)]$, i. e., $1\longrightarrow K_2(\Phi, \cdot)\longrightarrow St(\Phi, \cdot)\xrightarrow{\theta} E_1(\Phi, \cdot)\longrightarrow 1$ is exact. For each $\alpha\in\Phi$ and $u, v\in R^*$, we set $\{u, v\}_\alpha=\hat{h}_\alpha(uv)\hat{h}_\alpha(u)^{-1}\hat{h}_\alpha(v)^{-1}$, called a Steinberg symbol, where $\hat{h}_\alpha(u)=\hat{w}_\alpha(u)\hat{w}_\alpha(-1)$. Let $\hat{K}=\langle\{u, v\}_\alpha|\alpha\in\Phi, u, v\in R^*\rangle$. Then $\hat{K}\subseteq K_2(\Phi, R)\cap Cent(St(\Phi, R))$.

Definition. R is called universal for Φ if $K_2(\Phi, R)=\hat{K}$.

Let $E_u(\Phi, R)=St(\Phi, R)/\hat{K}$. Then the homomorphism θ induces a homomorphism $\bar{\theta}$ of $E_u(\Phi, R)$ onto $E_1(\Phi, R)$. We see:

"R is universal for Φ"

$\Leftrightarrow$ "$\bar{\theta}$ is an isomorphism"

$\Rightarrow$ "θ is a central extension."

EXAMPLE 1 (cf. [20], [21], [22]). Let k be a field.

(1) $St(\Phi, k)$ is connected if $(\Phi, |k|)\neq(A_1, 2)$, $(B_2, 2)$, $(G_2, 2)$ and $(A_1, 3)$.

(2) k is universal for each Φ.

(3) $St(\Phi, k)$ is a universal covering of $E_1(\Phi, k)$ with a few exceptions.

2. The case of Laurent polynomial rings.

Let $k[T]$ be the ring of polynomials in T with coefficients in a field k, and $\mathfrak{M}$ the maximal ideal of $k[T]$ generated by T. Let $k[T, T^{-1}]$ be the ring of Laurent polynomials in T and T^{-1} with coefficients in k. We identify $k[T]$ with a subring of $k[T, T^{-1}]$ naturally. Set

$$U=\langle x_\alpha(f), x_\beta(g)|\alpha\in\Phi^+,\ \beta\in\Phi^-,\ f\in k[T],\ g\in\mathfrak{M}\rangle,$$
$$N=\langle w_\alpha(tT^m)|\alpha\in\Phi,\ t\in k^*,\ m\in\boldsymbol{Z}\rangle,$$
$$H=\langle h_\alpha(t)|\alpha\in\Phi,\ t\in k^*\rangle, \text{ and}$$
$$B=\langle U, H\rangle$$

as subgroups of $E(\Phi, k[T, T^{-1}])$, where $w_\alpha(u)=x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)$ and $h_\alpha(u)=w_\alpha(u)w_\alpha(-1)$.

THEOREM 2 ([17]).

(1) $B\cap N=H$.

(2) $(E(\Phi, k[T, T^{-1}]), B, N)$ is a Tits system.

COROLLARY 3.
(1) The canonical homomorphism $\psi: E_1(\Phi, k[T, T^{-1}]) \longrightarrow E_0(\Phi, k[T, T^{-1}])$ is a central extension.
(2) $Ker\,\psi=\{\prod_{i=1}^{n} h_{\alpha_i}(t_i) \mid \prod_{i=1}^{n} t_i^{\langle\beta,\alpha_i\rangle}=1$ for all $\beta\in\Phi\}$, where $\langle\beta, \alpha_i\rangle=2(\beta, \alpha_i)/(\alpha_i, \alpha_i)$, and $t_i\in k^*$.

We define the subgroups $\hat{U}, \hat{N}, \hat{H}, \hat{B}$ of $St(\Phi, k[T, T^{-1}])$:

$$\begin{aligned}
\hat{U}&=\langle \hat{x}_\alpha(f),\ \hat{x}_\beta(g) \mid \alpha\in\Phi^+,\ \beta\in\Phi^-,\ f\in k[T],\ g\in\mathfrak{M}\rangle,\\
\hat{N}&=\langle \hat{w}_\alpha(tT^m) \mid \alpha\in\Phi,\ t\in k^*,\ m\in\boldsymbol{Z}\rangle,\\
\hat{H}&=\langle \hat{h}_\alpha(t) \mid \alpha\in\Phi,\ t\in k^*\rangle\hat{K},\\
\hat{B}&=\langle\hat{U}, \hat{H}\rangle
\end{aligned}$$

We denote by U_u, N_u, H_u and B_u the canonical images of $\hat{U}, \hat{N}, \hat{H}$ and $\hat{B}$ in $E_u(\Phi, k[T, T^{-1}])$ respectively. Then $(E_u(\Phi, k[T, T^{-1}]), B_u, N_u)$ and $(St(\Phi, k[T, T^{-1}]), \hat{B}, \hat{N})$ are Tits systems, which is established by using the same technique as in [17].

THEOREM 4.
(1) $G_1(\Phi, k[T])$ is presented by the generators $\tilde{x}_\gamma(f)$ and $\tilde{w}_\alpha(t)$ for all $\alpha\in\Pi$, $\gamma\in\Phi^+$, $f\in k[T]$ and $t\in k^*$, and the defining relations (R1)—(R9):

(R1) $\tilde{x}_\gamma(f)\tilde{x}_\gamma(g)=\tilde{x}_\gamma(f+g)$,
(R2) $\tilde{w}_\alpha(t)^{-1}=\tilde{w}_\alpha(-t)$,
(R3) $\tilde{w}_\alpha(t)\tilde{x}_\alpha(u)\tilde{w}_\alpha(-t)=\tilde{x}_\alpha(-t^2u^{-1})\tilde{w}_\alpha(t^2u^{-1})\tilde{x}_\alpha(-t^2u^{-1})$,
(R4) $[\tilde{x}_\gamma(f), \tilde{x}_\delta(g)]=\prod \tilde{x}_{i\gamma+j\delta}(N_{\gamma,\delta,i,j} f^i g^j)$,
(R5) $\tilde{h}_\alpha(t)\tilde{h}_\alpha(u)=\tilde{h}_\alpha(tu)$,
(R6) $\underbrace{\tilde{w}_\alpha(t)\tilde{w}_\beta(u)\tilde{w}_\alpha(t)\cdots}_{q}=\underbrace{\tilde{w}_\beta(u)\tilde{w}_\alpha(t)\tilde{w}_\beta(u)\cdots}_{q}$,
(R7) $\tilde{w}_\alpha(t)\tilde{x}_\rho(f)\tilde{w}_\alpha(-t)=\tilde{x}_{\rho'}(ct^{-\langle\rho,\alpha\rangle}f)$,
(R8) $\tilde{h}_\alpha(t)\tilde{x}_\alpha(f)\tilde{h}_\alpha(t^{-1})=\tilde{x}_\alpha(t^2f)$,
(R9) $\tilde{w}_\alpha(t)\tilde{h}_\beta(u)\tilde{w}_\alpha(-t)=\tilde{h}_\beta(u)\tilde{h}_\alpha(u^{-\langle\alpha,\beta\rangle})$

for all $\alpha, \beta\in\Pi(\alpha\neq\beta)$, $\gamma, \delta\in\Phi^+$, $\rho\in\Phi^+-\{\alpha\}$, $f, g\in k[T]$ and $t, u\in k^*$, where $\tilde{h}_\alpha(t)=\tilde{w}_\alpha(t)\tilde{w}_\alpha(-1)$, and $N_{\gamma,\delta,i,j}$ and c are as in [20] or [22], and each side of the equation in (R6) is the product of q symbols, and $q=2, 3, 4$ or 6 if $(\boldsymbol{R}\alpha+\boldsymbol{R}\beta)\cap\Phi$ is of type $A_1\times A_1$, A_2, B_2 or G_2 respectively, and $\langle\gamma, \alpha\rangle=2(\gamma, \alpha)/(\alpha, \alpha)$ and $\rho'=\rho-\langle\rho, \alpha\rangle\alpha$.
(2) $k[T]$ is universal for each root system Φ.

PROOF. (1) One can get this presentation of $G_1(\Phi, k[T])$ by using the same argument as in [23], [24] and [25]. (2) It follows from (1) that $k[T]$ is universal. (By using an amalgamated free product decomposition of $G_1(\Phi, k[T])$ which is described in [26], Rehmann [19] has given a different proof of the statement (2) from ours.) q. e. d.

THEOREM 5. $k[T, T^{-1}]$ is universal for each root system Φ.

PROOF. In the following commutative diagram:

$$\begin{array}{ccc} B_uN_uB_u=E_u(\Phi, k[T, T^{-1}]) & \xrightarrow{\bar{\theta}} & E_1(\Phi, k[T, T^{-1}])=B_1N_1B_1 \\ \uparrow & & \uparrow \\ E_u(\Phi, k[T]) & \xrightarrow{\sim} & E_1(\Phi, k[T])\supseteq B_1 \end{array}$$

we have $Ker\,\bar{\theta}\subseteq B_u$. On the other hand, $B_u\simeq B_1$ by the universality of $k[T]$. Therefore $\bar{\theta}$ is an isomorphism. q. e. d.

By taking $T=1$, the sequence $0\to k\to k[T, T^{-1}]$ splits, so $K_2(\Phi, k)$ is a direct summand of $K_2(\Phi, k[T, T^{-1}])$. Then:

THEOREM 6 ([2]).
(1) $K_2(A_1, k[T, T^{-1}])=K_2(A_1, k)\oplus S$, where $S=\langle\{T, t\}_\alpha | t\in k^*\rangle$ and α is a fixed root.
(2) $S\simeq k^*$ if $k^2=k$ (i. e. k is a square root closed field).

COROLLARY 7 (cf. [2], [12], [13]).
(1) $K_2(\Phi, k[T, T^{-1}])=K_2(\Phi, k)\oplus S$, where $S=\langle\{T, t\}_\alpha | t\in k^*\rangle$ and α is a fixed long root.
(2) S is isomorphic to a factor group of k^* if $\Phi\neq C_n$ $(n\geq 1)$.
(3) S is isomorphic to a factor group of k^* if $k^2=k$.

PROOF. (1) and (3) follow from Theorem 6. If $\Phi\neq C_n$ $(n\geq 1)$, then A_2 can be embedded in the long roots of Φ. By Matsumoto's theorem, one sees (2). q. e. d.

REMARK 8. The statements of Theorem 5, Theorem 6 and Corollary 7 have been confirmed by Hurrelbrink [7] in the case when $\Phi\neq G_2$. He has directly calculated the relations of $G_1(\Phi, k[T, T^{-1}])$ of type $\Phi=A_1, A_2$, and B_2, and by using this has proved Theorem 5 for $\Phi\neq G_2$. Our proof of Theorem 5 is different from his, and contains the case of type G_2.

As an application of [20, (5.3) Theorem/Remarks] and Theorem 5, we can establish the following theorem.

THEOREM 9. If char $k=0$, then $St(\Phi, k[T, T^{-1}])$ is a universal covering of $E_0(\Phi, k[T, T^{-1}])$.

3. Kac-Moody Lie algebras and generalized Chevalley groups.

An $l\times l$ integral matrix $C=(c_{ij})$ is called a generalized Cartan matrix if (i) $c_{ii}=2$, (ii) $i\neq j\Rightarrow c_{ij}\leq 0$, and (iii) $c_{ij}=0\Longleftrightarrow c_{ji}=0$. From now on, we suppose char $k=0$. We denote by $L_1=L_1(C)$ the Lie algebra over k generated by the $3l$ generators $e_1,\cdots,e_l,h_1,\cdots,h_l,f_1,\cdots,f_l$ with the defining relations $[h_i,h_j]=0$, $[e_i,f_j]=\delta_{ij}h_i$, $[h_i,e_j]=c_{ji}e_j$, $[h_i,f_j]=-c_{ji}f_j$ for all $1\leq i,j\leq l$, and $(\mathrm{ad}\,e_i)^{-c_{ji}+1}e_j=0$, $(\mathrm{ad}\,f_i)^{-c_{ji}+1}f_j=0$ for all $1\leq i\neq j\leq l$. Then the generators $e_1,\cdots,e_l,h_1,\cdots,h_l,f_1,\cdots,f_l$ are linearly independent in L_1. We view L_1 as a $\boldsymbol{Z}^l$-graded Lie algebra defined by $\deg(e_i)=(0,\cdots,0,1,0,\cdots,0)$, $\deg(h_i)=(0,\cdots,0)$ and $\deg(f_i)=(0,\cdots,0,-1,0,\cdots,0)$, where ± 1 are in the i-th position. Then there is the maximal homogeneous ideal $R_1=R_1(C)$ of L_1 such that $R_1\cap(\sum_{i=1}^{l}kh_1+\cdots+kh_l)=0$. Set $L=L(C)=L_1/R_1$, called the Kac-Moody Lie algebra over k associated with a generalized Cartan matrix C (cf. [3], [5], [8], [10], [14]). The algebra L is also $\boldsymbol{Z}^l$-graded. For each l-tuple $(n_1,\cdots,n_l)\in\boldsymbol{Z}^l$, we let $L(n_1,\cdots,n_l)$ denote the homogeneous subspace of degree $(n_1,\cdots,n_l)$ in L. We identify e_i,h_i,f_i with their images in L. Then:

PROPOSITION 10.
(1) $L(n_1,\cdots,n_l)$ is the subspace of L spanned by the elements $[e_{i_1},[e_{i_2},\cdots,[e_{i_{r-1}},e_{i_r}]\cdots]]$ (resp. $[f_{i_1},[f_{i_2},\cdots,[f_{i_{r-1}},f_{i_r}]\cdots]]$), where e_j (resp. f_j) occurs $|n_j|$ times, if $(n_1,\cdots,n_l)$ belongs to $(\boldsymbol{Z}_+)^l-\{0\}$ (resp. $(\boldsymbol{Z}_-)^l-\{0\}$).
(2) $L(0,\cdots,0)=kh_1+\cdots+kh_l$.
(3) $L(n_1,\cdots,n_l)=0$ otherwise.

Put $L_0=L_0(C)=kh_1+\cdots+kh_l$. For each $i=1,\cdots,l$, we define a degree derivation D_i on L such that $D_i(x)=n_ix$ for all $x\in L(n_1,\cdots,n_l)$. Set $D_0=kD_1+\cdots+kD_l$, viewed as an abelian Lie algebra of dimension l. For a subspace $D\subseteq D_0$, let $L^e=L(C)^e=D\times L$ (semidirect product) and $(L_0)^e=D\times L_0$ (direct product). For each $j=1,\cdots,l$, let γ_j be an element of $((L_0)^e)^*$, the dual of $(L_0)^e$, such that $[h,e_j]=\gamma_j(h)e_j$ for all $h\in(L_0)^e$. We note that $\gamma_j(h_i)=c_{ji}$ for all $i,j=1,\cdots,l$. We will choose and fix a subspace D of D_0 such that $\gamma_1,\cdots\gamma_l$ are linearly independent in $((L_0)^e)^*$. This is possible, since $\gamma_i(D_j)=\delta_{ij}$. Set $L^\gamma=\{x\in L\,|\,[h,x]=\gamma(h)x$ for all $h\in(L_0)^e\}$ for each $\gamma\in((L_0)^e)^*$. It is easily seen that $L^{n_1\gamma_1+\cdots+n_l\gamma_l}=L(n_1,\cdots,n_l)$ for all $(n_1,\cdots,n_l)\in\boldsymbol{Z}^l$. In particular, $L^{\gamma_i}=ke_i$, $L^0=L_0$ and $L^{-\gamma_i}=kf_i$.

Let $\Delta=\Delta(C)=\{\gamma\in((L_0)^e)^*\,|\,L^\gamma\neq 0\}$, called the root system of L. Set $\Gamma=\sum_{i=1}^{l}\boldsymbol{Z}\gamma_i$, a free $\boldsymbol{Z}$-submodule of $((L_0)^e)^*$. The Weyl group $W=W(C)$ is defined to be the subgroup of $GL(((L_0)^e)^*)$ generated by w_i for all $i=1,\cdots,l$, where w_i is an endomorphism of $((L_0)^e)^*$ such that $w_i(\gamma)=\gamma-\gamma(h_i)\gamma_i$. Then Δ and Γ are W-stable. Also W acts on L_0 naturally: $w_i(h_j)=h_j-c_{ij}h_i$. Hence we see $(w\gamma)(wh)=\gamma(h)$ for all $w \in W$, $\gamma \in ((L_0)^e)^*$ and $h \in L_0$.

Let $F_0(C, k)$ be the subgroup of $Aut(L)$ generated by $\exp \operatorname{ad} te_i$ and $\exp \operatorname{ad} tf_i$ for all $t \in k$ and $i=1, \cdots, l$. Let V be a standard L^e-module with a highest weight $\lambda, \neq 0$ (cf. [5], [10]). We let $F_V(C, k)$ denote the subgroup of $GL(V)$ generated by $\exp te_i$ and $\exp tf_i$ for all $t \in k$ and $i=1, \cdots, l$. These groups $F_0(C, k)$ and $F_V(C, k)$ have Tits systems respectively (cf. [11], [16]). Then there is a homomorphisms ν of $F_V(C, k)$ onto $F_0(C, k)$ such that $\nu(\exp te_i) = \exp \operatorname{ad} te_i$ and $\nu(\exp tf_i) = \exp \operatorname{ad} tf_i$ for all $t \in k$ and $i=1, \cdots, l$ (cf. [11]), and ν is central (cf. [18]).

4. The affine case.

Let Φ, A and $\tilde{A}$ be as in §1. Then we can regard $L(A)$ as a subalgebra of $L(\tilde{A})$ naturally. We note that $R_1(A) = R_1(\tilde{A}) = 0$, and that $\Delta(A) \approx \Phi \cup \{0\}$ and $\Delta(\tilde{A}) \approx \Delta(A) \times \boldsymbol{Z}$ (cf. [5], [9], [15]). Also we identify $W(A)$ with a subgroup of $W(\tilde{A})$. Therefore we have the following commutative diagram.

$$\begin{array}{ccc} W(A) \times L_0(A) & \longrightarrow & L_0(A) \\ \downarrow & & \downarrow \\ W(\tilde{A}) \times L_0(\tilde{A}) & \longrightarrow & L_0(\tilde{A}) \end{array}$$

We take an element σ of $W(A)$ such that $\sigma(\alpha_1) = \alpha_{n+1}$. Put $h_0 = \sigma(h_1)$ and $h_\xi = h_{n+1} - h_0$. Then $\gamma_i(h_0) = \gamma_i(\sigma h_1) = (\sigma^{-1}\gamma_i)(h_1) = \langle \sigma^{-1}\alpha_i, \alpha_1 \rangle = \langle \alpha_i, \alpha_{n+1} \rangle = a_{i, n+1}$ and $\gamma_i(h_\xi) = 0$. Therefore $\mathcal{Z} = kh_\xi$ is the center of $L(\tilde{A})$, and we have an exact sequence of Lie algebras over k (cf. [5], [8], [15]):

$$0 \longrightarrow \mathcal{Z} \longrightarrow L(\tilde{A}) \xrightarrow{\pi} k[T, T^{-1}] \underset{k}{\otimes} L(A) \longrightarrow 0.$$

Hence the map π induces an isomorphism $\tilde{\pi}$ of $F_0(\tilde{A}, k)$ onto $E_0(\Phi, k[T, T^{-1}])$ such that

$$\begin{aligned} &\tilde{\pi}(\exp \operatorname{ad} te_i) = x_{\alpha_i}(t) && \text{for all } 1 \leq i \leq n, \\ &\tilde{\pi}(\exp \operatorname{ad} te_{n+1}) = x_{\alpha_{n+1}}(tT), \\ &\tilde{\pi}(\exp \operatorname{ad} tf_i) = x_{-\alpha_i}(t) && \text{for all } 1 \leq i \leq n, \\ &\tilde{\pi}(\exp \operatorname{ad} tf_{n+1}) = x_{-\alpha_{n+1}}(tT^{-1}). \end{aligned}$$

Since $St(\Phi, k[T, T^{-1}])$ is a universal covering of $E_0(\Phi, k[T, T^{-1}])$ (cf. Theorem 9), there is a unique homomorphism, denoted by ϕ, of $St(\Phi, k[T, T^{-1}])$ into $F_V(\tilde{A}, k)$ such that the following diagram is commutative.

$$\begin{tikzcd} & F_V(\tilde{A}, k) \arrow[rr, "\nu"] & & F_0(\tilde{A}, k) \arrow[d, "\tilde{\pi}", "\wr"'] \\ St(\Phi, k[T, T^{-1}]) \arrow[ur, "\phi"] \arrow[r, "\theta"] & E_1(\Phi, k[T, T^{-1}]) \arrow[rr, "\psi"] & & E_0(\Phi, k[T, T^{-1}]) \end{tikzcd}$$

Then, by the relation $\hat{h}_\alpha(t)\hat{x}_\alpha(a)\hat{h}_\alpha(t)^{-1}=\hat{x}_\alpha(t^2a)$, we see

$$\begin{aligned}
&\phi(\hat{x}_{\alpha_i}(a))=\exp ae_i && \text{for all } 1\leq i\leq n,\\
&\phi(\hat{x}_{\alpha_{n+1}}(aT))=\exp ae_{n+1}, &&\\
&\phi(x_{-\alpha_i}(a))=\exp af_i && \text{for all } 1\leq i\leq n,\\
&\phi(x_{-\alpha_{n+1}}(aT^{-1}))=\exp af_{n+1}, &&\\
&\phi(\hat{w}_{\alpha_i}(t))=w_i(t) && \text{for all } 1\leq i\leq n,\\
&\phi(\hat{w}_{\alpha_{n+1}}(tT))=w_{n+1}(t), &&\\
&\phi(\hat{h}_{\alpha_i}(t))=h_i(t) && \text{for all } 1\leq i\leq n,\\
&\phi(\{T, t\}_{\alpha_{n+1}}\hat{h}_{\alpha_{n+1}}(t))=h_{n+1}(t), &&
\end{aligned}$$

where $w_i(t)=(\exp te_i)\,(\exp -t^{-1}f_i)\,(\exp te_i)$ and $h_i(t)=w_i(t)w_i(-1)$ for each $i=1, 2, \cdots, n+1$, and $a\in k$ and $t\in k^*$. In particular, ϕ is an epimorphism. Thus:

THEOREM 11. *$St(\Phi, k[T, T^{-1}])$ is a universal covering of $F_V(\tilde{A}, k)$.*

Finally in this note, we will discuss the kernel of ϕ. Since $Ker\,\phi\subseteq Ker\,(\theta\phi)$, an element x of $Ker\,\phi$ can be written as $\prod_{i=1}^n \hat{h}_{\alpha_i}(t_i)\prod_p\{a_p, b_p\}_{\alpha_1}^{r_p}\prod_{j=1}^q\{T, c_j\}_{\alpha_{n+1}}^{s_j}$, where $t_i, a_p, b_p, c_j\in k^*$ and $r_p, s_j\in \boldsymbol{Z}_+$. Then $\phi(\{T, c_j\}_{\alpha_{n+1}})=h_{n+1}(c_j)\sigma h_1(c_j)^{-1}\sigma^{-1}$. On each weight space V_μ of V (cf. [5], [10]), $\phi(x)=\prod_{i=1}^n t_i^{\mu(h_i)}\prod_{j=1}^q c_j^{\mu(h_{n+1})s_j}c^{-(\sigma^{-1}\mu)(h_1)s_j}=\prod_{i=1}^n t_i^{\mu(h_i)}\prod_{j=1}^q c_j^{\mu(h_{n+1})s_j}c_j^{-\mu(h_0)s_j}=\prod_{i=1}^n t_i^{\mu(h_i)}\prod_{j=1}^q c_j^{\mu(h_\xi)s_j}$. Therefore:

$$\begin{aligned}
&\phi(x)=1\\
&\Leftrightarrow \prod_{i=1}^n t_i^{\mu(h_i)}\prod_{j=1}^q c_j^{\mu(h_\xi)s_j}=1 \qquad \text{for all weight } \mu.
\end{aligned}$$

Put $P=\langle \prod_{i=1}^n \hat{h}_{\alpha_i}(t_i)\prod_{j=1}^q\{T, c_j\}_{\alpha_{n+1}}^{s_j} \mid \prod_{i=1}^n t_i^{\mu(h_i)}\prod_{j=1}^q c_j^{\mu(h_\xi)s_j}=1$ for all weight μ of $V\rangle$.

THEOREM 12. *$Ker\,\phi=K_2(\Phi, k)\oplus P$.*

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Note added in proof. Recently H. Garland [Publ. IHES 52 (1980), 181-312] has constructed a subgroup F_1 of $Aut(V)$ containing $F_V(\tilde{A}, k)$, and has shown that $St(\Phi, k((T)))$ is a universal covering of F_1, where $k((T))$ is the T-adic completion of $k[T, T^{-1}]$. Then the composite map $St(\Phi, k[T, T^{-1}]) \to St(\Phi, k((T))) \to F_1$ coinsides with the covering map of $F_V(\tilde{A}, k)$