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GENERATORS AND RELATIONS FOR COMPACT LIE ALGEBRAS

By

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1. Introduction.

The main purpose of this paper is to provide a system of generators and relations for each of the nine types of compact simple Lie algebras. Indeed, we are able to give a presentation of each such algebra which depends only on the finite Cartan matrix (A_{ij}) which is attached to the complexification of our compact algebra. One of the main results that lies behind our work is the Theorem of Serre which gives a presentation of the simple Lie algebras over the complex field attached to (A_{ij}).

Although our main interest is with the compact Lie algebras, we work in the generality of Kac-Moody Lie algebras (see [1], [4], [7], [8]). In this setting we will be able to provide a generalization of the compact simple Lie algebras. We realize these algebras as certain forms of the Kac-Moody algebra. More specifically, if (A_{ij}) is any indecomposable Cartan matrix which is non-Euclidean we let $\mathcal{L}_C$ (resp. $\bar{\mathcal{L}}_C$) be the reduced (resp. standard) Kac-Moody Lie algebra over the complex field $\boldsymbol{C}$, (see Section 1 for more details). We define a real form $\mathcal{L}_C$ (resp. $\bar{\mathcal{L}}_C$) of $\mathcal{L}_C$ (resp. $\bar{\mathcal{L}}_C$), and show that $\mathcal{L}_C$ is the only simple homomorphic image of $\bar{\mathcal{L}}_C$. We then give generators and relations for $\bar{\mathcal{L}}_C$. The question of when $\bar{\mathcal{L}}_C=\mathcal{L}_C$ is equivalent to the question of when $\mathcal{L}_C=\bar{\mathcal{L}}_C$, and is a major unsolved question about Kac-Moody algebras. However, thanks to Serre's Theorem, we know $\mathcal{L}_C=\bar{\mathcal{L}}_C$, and hence $\mathcal{L}_C=\bar{\mathcal{L}}_C$, when (A_{ij}) is of finite type. This yields a presentation of $\mathcal{L}_C$ in this case.

The content of the paper is as follows. In Section 1 we recall the notation and a few facts about Kac-Moody algebras. In Section 2, the final section, we begin by making a study of the algebras $\mathcal{L}_C$ and $\bar{\mathcal{L}}_C$. We then go on to obtain a presentation of $\bar{\mathcal{L}}_C$, and then use this in dealing with the compact simple Lie algebras. We think it is interesting that analogues of the compact Lie algebras exist in the Kac-Moody setting. Moreover, just as the compact algebras are

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important in studying all real forms of the simple Lie algebras over $\boldsymbol{C}$ (e.g. Cartan and Iwasawa decompositions), no doubt the generalizations studied here will play a similar role. Throughout, we let $\boldsymbol{R}$ denote the real field and $\boldsymbol{C}$ the complex field.

Section 1: We use similar notation to [1] and [2] where the reader may find all the necessary facts. Also, one may consult [4], [7], [8]. Thus, (A_{ij}) will denote an $l\times l$ indecomposable Cartan matrix which is not Euclidean. $\tilde{\mathcal{L}}$ denotes the universal Kac-Moody algebra of type (A_{ij}) over $\boldsymbol{R}$, so that $\tilde{\mathcal{L}}$ is generated by $3l$ elements e_j, h_j, f_j, $1\leqq j\leqq l$, subject to the relations

(i) $[e_k, h_j]=A_{jk}e_k$,
(ii) $[f_k, h_j]=-A_{jk}f_k$,
(iii) $[h_j, h_k]=0$,
(iv) $[e_k, f_j]=\delta_{kj}h_k$, $1\leqq j$, $k\leqq l$.

The reduced Kac-Moody algebra is defined to be $\tilde{\mathcal{L}}/\mathcal{R}$ where $\mathcal{R}$ is the radical of $\tilde{\mathcal{L}}$. We let this be denoted by $\mathcal{L}$ and recall that $\mathcal{R}$ is the unique maximal ideal of $\tilde{\mathcal{L}}$. The standard Kac-Moody algebra is denoted $\bar{\mathcal{L}}$ and is $\tilde{\mathcal{L}}$ factored by the ideal Ξ, where we recall that Ξ is generated by the elements $e_j(ad\ e_k)^{-A_{kj}+1}$, $f_j(ad\ f_k)^{-A_{kj}+1}$, $1\leqq j\neq k\leqq l$, and the elements $\{h\in\tilde{H}\,|\,\alpha_j(h)=0, 1\leqq j\leqq l\}$. Here, as usual, $\tilde{H}$ is the linear span of $h_1, \cdots h_l$ in $\tilde{\mathcal{L}}$. V_Z denotes the free abelian group with basis $\alpha_1, \cdots, \alpha_l$ and V_Z acts on $\tilde{H}$ by $\alpha_j(h_k)=A_{kj}$ for $1\leqq j$, $k\leqq l$. Γ_1 denotes the roots of $\tilde{\mathcal{L}}$ and Γ_2 will denote the roots of $\mathcal{L}$ and $\bar{\mathcal{L}}$, (they have the same roots). Thus, we have the decomposition $\mathcal{L}=H\oplus\sum_{\alpha\in\Gamma_2}\mathcal{L}_\alpha$; similarly for $\tilde{\mathcal{L}}$ and $\bar{\mathcal{L}}$. As usual, η denotes the automorphism of $\mathcal{L}$ (or $\bar{\mathcal{L}}$ or $\tilde{\mathcal{L}}$) which satisfies $\eta(e_j)=f_j$, $\eta(f_j)=e_j$, $1\leqq j\leqq l$. Here, we are of course using the convention of letting e_j, h_j, f_j, $1\leqq j\leqq l$, denote the image in $\mathcal{L}$ or $\bar{\mathcal{L}}$ of the corresponding elements of $\tilde{\mathcal{L}}$. Note that $\bar{\mathcal{L}}$ has a unique maximal ideal which is the kernel of the natural map of $\bar{\mathcal{L}}$ onto $\mathcal{L}$.

If S denotes any one of the algebras $\mathcal{L}$, $\bar{\mathcal{L}}$, $\tilde{\mathcal{L}}$ we let S_C denote the complexification of S. Thus, $S_C=\boldsymbol{C}\otimes_{\boldsymbol{R}}S$, and we let τ be the map $\bar{\ }\otimes\eta$ of S_C, where the "over bar" denotes complex conjugation. Thus, τ is a semi-linear automorphism of S_C of period 2. We define $\mathcal{S}_C$ to be the fixed points of S_C under τ. Clearly $\mathcal{S}_C$ is real form of S_C. We let $S_+=\{s\in S\,|\,\eta(s)=s\}$ and $S_-=\{s\in S\,|\,\eta(s)=-s\}$. Then we have the decompositions

$$S=S_+\oplus S_-,\quad \mathcal{S}_C=S_+\oplus iS_-. \tag{1.1}$$

Of course, $\mathcal{L}_C$ is our generalization of the simple compact Lie algebras. Indeed, we will see in the next section that $\mathcal{L}_C$ is a finite dimensional simple compact Lie algebra when (A_{ij}) is of finite type. For the present we remark, since $\mathcal{L}_C$ is simple, and $\mathcal{L}_C$ is a real form of $\mathcal{L}_C$, that $\mathcal{L}_C$ is simple. Moreover, it is clear, and follows from the corresponding fact for $\tilde{\mathcal{I}}_C$ (or $\tilde{\mathcal{I}}_C$), that $\tilde{\mathcal{I}}_C$ (or $\tilde{\mathcal{I}}_C$) has a unique maximal ideal which is the kernel of the obvious natural homomorphism onto $\mathcal{L}_C$.

Section 2: As in the previous section we let S denote any one of the algebras $\mathcal{L}$, $\bar{\mathcal{I}}$, or $\tilde{\mathcal{I}}$. Letting H_S denote the image of $\tilde{H}$ in S we have the decomposition $S=H_S\oplus\sum_{\alpha\in\Gamma}S_\alpha$, where Γ is the root system of S. Let $n_\alpha=\dim S_\alpha$, and for $\alpha\in\Gamma^+$ we let $x_{\alpha,1},\cdots,x_{\alpha,n_\alpha}$ be a basis of the space S_α chosen from among the elements $[e_{j_1},\cdots,e_{j_t}]$ where $\alpha_{j_1}+\cdots+\alpha_{j_t}=\alpha$. It's worth noting that the n_α's are known when $S=\mathcal{L}$ or $S=\bar{\mathcal{I}}$, (see [3]). Thus, for example, $x_{\alpha_j,1}=e_j$ for $1\leqq j\leqq l$. Let $x_{-\alpha,j}=\eta(x_{\alpha,j})$ for $\alpha\in\Gamma^+$, $1\leqq j\leqq n_\alpha$, so that $x_{-\alpha,1},\cdots,x_{-\alpha,n_\alpha}$ is a basis of $S_{-\alpha}$. Then we have

LEMMA 2.1. *$\{x_{\alpha,j}+x_{-\alpha,j}\mid\alpha\in\Gamma^+,\ 1\leqq j\leqq n_\alpha\}$ is a basis of S_+.*

PROOF. Let $x=h+\sum_{\alpha\in\Gamma^+}\left(\sum_{j=1}^{n_\alpha}a_{\alpha,j}x_{\alpha,j}+b_{\alpha,j}x_{-\alpha,j}\right)$ be an element in S_+ where $h\in H_S$. Then $\eta(x)=x$ implies that $h=0$ and $a_{\alpha,j}=b_{\alpha,j}$ for all $\alpha\in\Gamma^+$, $1\leqq j\leqq n_\alpha$. Thus, x is in the linear span of the set $\{x_{\alpha,j}+x_{-\alpha,j}\mid\alpha\in\Gamma^+,\ 1\leqq j\leqq n_\alpha\}$. The rest is clear. □

We now define some elements of interest to us. Let $x_j=e_j+f_j$, $y_j=i(e_j-f_j)$, and $z_j=ih_j$ for $1\leqq j\leqq l$. Clearly x_j, y_j, $z_j\in S_C$ for $1\leqq j\leqq l$.

LEMMA 2.2. *S_+ is generated by the elements $x_1,\cdots,x_l$.*

PROOF. Let M be the subalgebra of S_+ generated by the elements $x_1,\cdots,x_l$. By Lemma 2.1 it is enough to show that for all $\alpha\in\Gamma^+$ and $j\in\{1,\cdots,n_\alpha\}$ that $x_{\alpha,j}+x_{-\alpha,i}\in M$. We do this by induction on $l(\alpha)$, where $l(\alpha)$ is defined to be $\sum C_j$, when $\alpha=\sum C_j\alpha_j$; the case when $l(\alpha)=1$ being clear because $\dim S_{\alpha_k}=1$ for $1\leqq k\leqq l$, so that $x_{\alpha_k,1}+x_{-\alpha_k,1}=x_k$ for $1\leqq k\leqq l$. Assume $\alpha\in\Gamma^+$ and $l(\alpha)=n+1$ where $n\geqq1$ and that if $\beta\in\Gamma^+$ and $l(\beta)\leqq n$ that $x_{\alpha,j}+x_{-\beta,j}\in M$ for $1\leqq j\leqq n_\beta$. Now, since $l(\alpha)\geqq2$ then for any $j\in\{1,\cdots,n_\alpha\}$ we may assume that there exists some $\beta\in\Gamma^+$, $k\in\{1,\cdots,l\}$ such that $l(\beta)=n$, and $x_{\alpha,j}=[x_{\beta,t},e_k]$, for some $t\in\{1,\cdots,n_\alpha\}$. (This is because $x_{\alpha,j}=[e_{j_1},\cdots,e_{j_n},e_k]$ for some k and $x_{\alpha_k,1}=e_k$). Thus, $x_{\alpha,j}+x_{-\alpha,j}=x_{\alpha,j}+\eta(x_{\alpha,j})=[x_{\beta,t},e_k]+[x_{-\beta,t},f_k]$.

Next, note that $x_{\beta,t}+x_{-\beta,t}\in M$ and $e_k+f_k\in M$, so that $[x_{\beta,t}+x_{-\beta,t},e_k+f_k]=x_{\alpha,j}+x_{-\alpha,j}+[x_{\beta,t},f_k]+[x_{-\beta,t},e_k]\in M$. But the element $[x_{\beta,t},f_k]+[x_{-\beta,t},e_k]$ is in $S_+\cap(S_{\beta-\alpha_k}\oplus S_{-(\beta-\alpha_k)})$, and hence by Lemma 2.1 is an $\boldsymbol{R}$-linear combination of the elements $x_{\gamma,s}+x_{-\gamma,s}$ for $s\in\{1, \cdots, n_\gamma\}$ and $\gamma=\beta-\alpha_k$. Thus, by induction $[x_{\beta,t}, f_k]+[x_{-\beta,t}, e_k]$ is in M. It now follows that $x_{\alpha,j}+x_{-\alpha,j}\in M$, as desired. □

LEMMA 2.3. *The elements ih_j, $1\leqq j\leqq l$, together with the elements $i(x_{\alpha,k}-x_{-\alpha,k})$ for $\alpha\in\Gamma^+$, $k\in\{1, \cdots, n_\alpha\}$ from a basis of iS.*

PROOF. It is enough to show that the elements h_j, $1\leqq j\leqq l$, together with the elements $x_{\alpha,k}-x_{-\alpha,k}$, for $\alpha\in\Gamma^+$, $k\in\{1, \cdots, n_\alpha\}$, span S_-. Let M be the subspace spanned by these elements and note that h_j, $1\leqq j\leqq l$, $x_{\alpha,k}+x_{-\alpha,k}$, $x_{\alpha,k}-x_{-\alpha,k}$, for $\alpha\in\Gamma^+$, $k\in\{1, \cdots, n_\alpha\}$ form a basis of S, so that $S=S_+\oplus M$. Since we also have $S=S_+\oplus S_-$ and $M\subseteqq S_-$, it follows that $M=S_-$. □

PROPOSITION 2.4. *S_C is generated by the $3l$ elements x_j, y_j, z_j, $1\leqq j\leqq l$.*

PROOF. $1/2[x_j, z_j]=1/2[e_j+f_j, ih_j]=(i/2)(2e_j-2f_j)=y_j$, $1\leqq j\leqq l$. Thus, letting M be the subalgebra of S_C generated by x_j, z_j, $1\leqq j\leqq l$, we have that $y_j\in M$, $1\leqq j\leqq l$. By Lemma 2.1 and Lemma 2.3 we know that S_C has basis ih_j, $1\leqq j\leqq l$, $(x_{\alpha,k}+x_{-\alpha,k})$, $i(x_{\alpha,k}-x_{-\alpha,k})$, for $\alpha\in\Gamma^+$, $k\in\{1, \cdots, n_\alpha\}$; hence it is enough to show that these elements are in M. By Lemma 2.2 we have that $(x_{\alpha,k}+x_{-\alpha,k})\in M$ for $\alpha\in\Gamma^+$, $k\in\{1, \cdots, n_\alpha\}$. It is clear that $iH_S\subseteqq M$. Next, let $\alpha\in I^+$, $k\in\{1, \cdots, n_\alpha\}$ and choose $h\in H_S$ such that $\alpha(h)\neq 0$, (this is possible since (A_{ij}) is not Euclidean). Then we get $ih\in M$ and $x_{\alpha,k}+x_{-\alpha,k}\in M$ so that $[x_{\alpha,k}+x_{-\alpha,k}, ih]=\alpha(h)(i(x_{\alpha,k}-x_{-\alpha,k}))\in M$. It follows that $i(x_{\alpha,k}-x_{-\alpha,k})\in M$. □

DEFINITION 2.5. Let j, $k\in\{1, \cdots, l\}$, $j\neq k$ and let s, $t\in\boldsymbol{Z}$.

We define the integer $C_{s,t}^{(j,k)}$ as follows:

$C_{0,0}^{(j,k)}=1$, $C_{s,t}^{(j,k)}=0$ if either $s<0$, $t<0$, or if $t>s$. Otherwise $C_{s,t}^{(j,k)}$ is defined inductively by $C_{s,t}^{(j,k)}=C_{s-1,t-1}^{(j,k)}+(s-1)[A_{kj}+(s-2)]C_{s-2,t}^{(j,k)}$. Note that $C_{s,s}^{(j,k)}=1$ for all $s\geqq 0$, and that $C_{s,t}^{(j,k)}=0$ if $(-1)^s\neq(-1)^t$.

PROPOSITION 2.6. *The elements x_j, y_i, z_j for $1\leqq j\leqq l$ satisfy the following relations:*

F_1 $y_j=1/2[x_j, z_j]$, $1\leqq j\leqq l$,

F_2 $[x_j, z_k]=A_{kj}y_j$, $1\leqq j, k\leqq l$,

F_3 $[y_j, z_k]=-A_{kj}x_j$, $1\leqq j, k\leqq l$,

F_4 $[z_j, z_k]=0$, $1\leqq j, k\leqq l$,

F_5 $[x_j, x_k]+[y_j, y_k]=0$, $1\leqq j, k\leqq l$,

F_6 $[x_j, y_k]+[x_k, y_j]=-4\delta_{jk}z_j, \quad 1\leqq j, k\leqq l,$

F_7 $e_j(ad\ e_k)^{2n}+f_j(ad\ f_k)^{2n}=\sum_{t=0}^{n}(-1)^{n-t}C_{2n,2t}^{(j,k)}x_j(ad\ x_k)^{2t},$

for $n\geqq 0,\ 1\leqq j,\ k\leqq l,\ j\neq k,$

F_8 $e_j(ad\ e_k)^{2n+1}+f_j(ad\ f_k)^{2n+1}=\sum_{t=0}^{n}(-1)^{n-t}C_{2n+1,2t+1}^{(j,k)}x_j(ad\ x_k)^{2t+1},$

for $n\geqq 0,\ 1\leqq j,\ k\leqq l,\ j\neq k,$

F_9 $i(e_j(ad\ e_k)^{2n}-f_j(ad\ f_k)^{2n})=\sum_{t=0}^{n}(-1)^{n-t}C_{2n,2t}^{(j,k)}y_j(ad\ x_k)^{2t},$

for $n\geqq 0,\ 1\leqq j,\ k\leqq l,\ j\neq k,$

F_{10} $i(e_j(ad\ e_k)^{2n+1}-f_j(ad\ f_k)^{2n+1})=\sum_{t=0}^{n}(-1)^{n-t}C_{2n+1,2t+1}^{(j,k)}y_j(ad\ x_k)^{2t+1},$

for $n\geqq 0,\ 1\leqq j,\ k\leqq l,\ j\neq k.$

In particular, these relations are a consequence of the definitions and the relations (i)-(iv) *of Section 1.*

PROOF. The relations F_1 through F_6 are easy to establish so we do F_7 and F_8 together, by induction on n. In doing this we write $C_{s,t}$ for $C_{s,t}^{(j,k)}$. Also, we will use the following well known formulas.

$$e_j(ad\ e_k)^t(ad\ f_k)=(tA_{kj}+t(t-1))e_j(ad\ e_k)^{t-1} \quad \text{for } t\geqq 1,$$

$$f_j(ad\ f_k)^t(ad\ e_k)=(tA_{kj}+t(t-1))f_j(ad\ f_k)^{t-1} \quad \text{for } t\geqq 1.$$

When $n=0$, $e_j(ad\ e_k)^{2n}+f_j(ad\ f_k)^{2n}=e_j+f_j=x_j$ while

$$\sum_{t=0}^{n}(-1)^{n-t}C_{2n,2t}x_j(ad\ x_k)^{2t}=C_{0,0}x_j=x_j,$$

so F_7 holds when $n=0$. By definition, $[e_j, e_k]+[f_j, f_k]=[x_j, x_k]$ for $j\neq k$, so F_8 holds when $n=0$. Also F_7 holds when $n=1$, since we have

$$\begin{aligned}
&e_j(ad\ e_k)^2+f_j(ad\ f_k)^2\\
&=[e_j(ad\ e_k)+f_j(ad\ f_k), e_k+f_k]-[f_j(ad\ f_k), e_k]-[e_j(ad\ e_k), f_k]\\
&=x_j(ad\ x_k)^2-A_{kj}f_j-A_{kj}e_j\\
&=C_{2,2}x_j(ad\ x_k)^2-C_{2,0}x_j.
\end{aligned}$$

Next, assume that $m\geqq 1$ and that F_7 holds when $n\leqq m$, and that F_8 holds when $n\leqq m-1$. We show F_8 holds when $n=m$. We have that

$$
\begin{aligned}
& e_j(ad\ e_k)^{2m+1}+f_j(ad\ f_k)^{2m+1} \\
&=[e_j(ad\ e_k)^{2m}+f_j(ad\ f_k)^{2m},\ e_k+f_k] \\
&\quad -[e_j(ad\ e_k)^{2m},\ f_k]-[f_j(ad\ f_k)^{2m},\ e_k] \\
&=[\sum_{t=0}^{m}(-1)^{m-t}C_{2m,2t}x_j(ad\ x_k)^{2t},\ x_k] \\
&\quad -(2mA_{kj}+2m(2m-1))e_j(ad\ e_k)^{2m-1}-(2mA_{kj}+2m(2m-1))f_j(ad\ f_k)^{2m-1} \\
&=\sum_{t=0}^{m}(-1)^{m-t}C_{2m,2t}x_j(ad\ x_k)^{2t+1} \\
&\quad -(2mA_{kj}+2m(2m-1))\sum_{t=0}^{m-1}(-1)^{m-1-t}C_{2m-1,2t+1}x_j(ad\ x_k)^{2t+1}.
\end{aligned}
$$

This equals

$$
\begin{aligned}
&\sum_{t=0}^{m-1}\{(-1)^{m-t}C_{2m,2t}-(2mA_{kj}+2m(2m-1))(-1)^{m-1-t}C_{2m-1,2t+1}\}\,x_j(ad\ x_k)^{2t+1} \\
&\quad +C_{2m,2m}x_j(ad\ x_k)^{2m+1}.
\end{aligned}
$$

Now

$$
C_{2m,2t}+2m(A_{kj}+(2m-1))C_{2m-1,2t-1}=C_{2m+1,2t+1}
$$

and

$$
\begin{aligned}
&(-1)^{m-t}C_{2m,2t}-(2mA_{kj}+2m(2m-1))(-1)^{m-1-t}C_{2m-1\ 2t+1} \\
&=(-1)^{m-t}(C_{2m,2t}+(2mA_{kj}+2m(2m-1))C_{2m-1,2t+1}) \\
&=(-1)^{m-t}(C_{2m,2t}+2m(A_{kj}+(2m-1))C_{2m-1\ 2t+1}) \\
&=(-1)^{m-t}C_{2m+1,2t+1}\,.
\end{aligned}
$$

Thus, we have that

$$
e_j(ad\ e_k)^{2m+1}+f_j(ad\ f_k)^{2m+1}=\sum_{t=0}^{m}(-1)^{m-t}C_{2m+1,2t+1}x_j(ad\ x_k)^{2t+1},
$$

as desired.

Next, one lets $m\geqq 1$ and assumes F_8 holds when $n\leqq m$ and that F_7 holds when $n\leqq m$ and shows that F_7 holds when $n=m+1$. This is similar to the above and so is ommitted. In the same way F_9 and F_{10} can be shown to hold. □

DEFINITION 2.7. Let $F\mathcal{L}=F\mathcal{L}(X_j, Y_j, Z_j \mid 1\leqq j\leqq l)$ be the free Lie algebra over $\boldsymbol{R}$ generated by the $3l$-symbols X_j, Y_j, Z_j, $1\leqq j\leqq l$. Recall that (A_{ij}) is a fixed $l\times l$ indecomposable Cartan matrix which is not of Euclidean type. Let J denote the ideal of $F\mathcal{L}$ generated by the following elements: (R_1—R_{10})

R_1: $Y_j - 1/2[X_j, Z_j]$, $1 \leqq j \leqq l$,

R_2: $[X_j, Z_k] - A_{kj} Y_j$, $1 \leqq j, k \leqq l$,

R_3: $[Y_j, Z_k] + A_{kj} X_j$, $1 \leqq j, k \leqq l$,

R_4: $[Z_j, Z_k]$, $1 \leqq j, k \leqq l$,

R_5: $[X_j, X_k] + [Y_j, Y_k]$, $1 \leqq j, k \leqq l$,

R_6: $[X_j, X_k] + [X_k, Y_j] + 4\delta_{jk} Z_j$, $1 \leqq j, k \leqq l$.

Next, let $j, k \in \{1, \cdots, l\}$, $j \neq k$. Let $m = -A_{kj} + 1$. If m is even we put $n = m/2$. Then

R_7: $\sum_{t=0}^{n} (-1)^{n-t} C_{2n, 2t}^{(j,k)} X_j (ad\ X_k)^{2t}$, and

R_9: $\sum_{t=0}^{n} (-1)^{n-t} C_{2n, 2t}^{(j,k)} Y_j (ad\ X_k)^{2t}$.

If m is odd we put $n = \dfrac{m-1}{2}$. Then

R_8: $\sum_{t=0}^{n} (-1)^{n-t} C_{2n+1, 2t+1}^{(j,k)} X_j (ad\ X_k)^{2t+1}$, and

R_{10}: $\sum_{t=0}^{n} (-1)^{n-t} C_{2n+1, 2t+1}^{(j,k)} Y_j (ad\ X_k)^{2t+1}$.

Finally, we let $\bar{\mathcal{L}}_C = \dfrac{F\mathcal{L}}{J}$ and let $\bar{L}_C$ be the complexification of $\bar{\mathcal{L}}_C$. Let $E_j, F_j, H_j \in \bar{L}_C$ be defined by

$$E_j = 1/2(X_j - iY_j), \quad F_j = 1/2(X_j + iY_j),$$

and

$$H_j = -iZ_j, \quad 1 \leqq j \leqq l.$$

PROPOSITION 2.8. *The algebras $\bar{\mathcal{L}}_C$ and $\bar{L}_C$ are isomorphic. In particular,* R_1-R_{10} *provides a presentation of $\bar{\mathcal{L}}_C$. Moreover, $\bar{L}_C$ has a unique maximal ideal and the corresponding simple factor is isomorphic to $\mathcal{L}_C$.*

PROOF. We first note that formulas (i)-(iv) of section 1 hold in $\bar{L}_C$. Indeed, R_4 implies that $[H_j, H_k] = 0$ for $1 \leqq j, k \leqq l$. Now

$$\begin{aligned}[E_k, H_j] &= 1/2[X_k - iY_k, -iZ_j] \\ &= -i/2[X_k, Z_j] - 1/2[Y_k, Z_j] \\ &= -i/2\, A_{jk} Y_k + 1/2\, A_{jk} X_k \quad (\text{by } \mathrm{R}_2 \text{ and } \mathrm{R}_3) \\ &= A_{jk}(1/2(X_k - \mathrm{i} Y_k)) = A_{jk} E_k\,,\end{aligned}$$

as desired. Similarly, one finds that $[F_k, H_j]=-A_{jk}F_k$ and that $[E_j, F_k]=\delta_{jk}H_j$.

By Proposition 2.6 we obtain a Lie algebra homomorphism ϕ from $\bar{L}_C$ onto the subalgebra of $\bar{\mathcal{L}}_C$ generated by the elements $x_j, y_j, z_j, 1\leqq j\leqq l$; and by Proposition 2.4 this is the algebra $\bar{\mathcal{L}}_C$. Thus, ϕ is a surjective homomorphism of $\bar{L}_C$ onto $\bar{\mathcal{L}}_C$.

Since the relations (i)-(iv) of Section 1 hold in $\bar{L}_C$ we get a Lie algebra homomorphism $\tilde{\Psi}$ from the universal Kac-Moody algebra $\tilde{\mathcal{L}}_C$ to $\bar{L}_C$ such that $\tilde{\Psi}(e_j)=E_j$, $\tilde{\Psi}(f_j)=F_j$, and $\tilde{\Psi}(h_j)=H_j$, $1\leqq j\leqq l$. Now formulas F_1-F_{10} hold in $\bar{L}_C$ thanks to Proposition 2.6. Thus, since R_7-R_{10} hold in $\bar{L}_C$ we see that $E_j(ad\ E_k)^{-A_{kj}+1}=0=F_j(ad\ F_k)^{-A_{kj}+1}$. It follows that $\tilde{\Psi}$ induces a homomorphism Ψ of $\bar{\mathcal{L}}_C$ to $\bar{L}_C$. Clearly, $\Psi(x_j)=X_j$, $\Psi(y_j)=Y_j$, and $\Psi(z_j)=Z_j$, $1\leqq j\leqq l$, so that $\Psi(\bar{\mathcal{L}}_C)=\bar{L}_C$. Finally, it is clear that $\phi\circ\Psi=id_{\bar{\mathcal{L}}_C}$ and $\Psi\circ\phi=id_{\bar{L}_C}$, so that $\bar{\mathcal{L}}_C$ and $\bar{L}_C$ are isomorphic. As in Section 1 it is clear that $\bar{L}_C$ has a unique maximal ideal with the corresponding simple factor being isomorphic to $\mathcal{L}_C$. $\square$

Assume now that (A_{ij}) is one of the 9 types of $l\times l$ indecomposable finite Cartan matrices. Then by Serre's Theorem $\mathcal{L}_C=\bar{\mathcal{L}}_C$ is the split simple Lie algebra of type (A_{ij}) over $\boldsymbol{C}$. Let $(\cdot, \cdot)$ denote the Killing form $\mathcal{L}_C$ and let κ be as in Section 1. As in [6 pg. 147-149] a compact subalgebra C of $\mathcal{L}_C$ has basis ih_j, $1\leqq j\leqq l$, $e_\alpha+e_{-\alpha}$, $i(e_\alpha-e_{-\alpha})$; for $\alpha\in\Delta$, (the root system of $\mathcal{L}_C$) and $e_\alpha\in\mathcal{L}_\alpha$ is chosen such that $\kappa(e_\alpha)=e_{-\alpha}$ and $(e_\alpha, e_{-\alpha})=-1$, for all $\alpha\in\Delta$.

We are going to show that $x_j, y_j\in C$, $1\leqq j\leqq l$. As usual, h_α denotes the element in H satisfying $\alpha(h)=(h_\alpha, h)$ for all $h\in H$, $\alpha\in\Delta$. Then $[e_j, f_j]=h_j$ and $\alpha_j(h_j)=2$ imply that

$$(e_j, f_j)=\frac{-2}{(\alpha_j, \alpha_j)}, \quad 1\leqq j\leqq l,$$

since

$$\begin{aligned}2=(h_j, h_{\alpha_j})&=([e_j, f_j], h_{\alpha_j})=(e_j, [f_j, h_{\alpha_j}])\\ &=-\alpha_j(h_{\alpha_j})(e_j, f_j)=(-\alpha_j, \alpha_j)(e_j, f_j).\end{aligned}$$

Let

$$\lambda_j=\left(\frac{(\alpha_j, \alpha_j)}{2}\right)^{1/2}\in\boldsymbol{R}, \quad 1\leqq j\leqq l.$$

Then $(\lambda_j e_j, \lambda_j f_j)=-1$ so that, as part of our basis of C, we can take $e_{\alpha_j}=\lambda_j e_j$, $e_{-\alpha_j}=\lambda_j f_j$. Then $e_{\alpha_j}+e_{-\alpha_j}=\lambda_j(e_j+f_j)\in C$, hence $x_j=\lambda_j^{-1}(e_{\alpha_j}+e_{-\alpha_j})\in C$, and similarly $y_j\in C$ for $1\leqq j\leqq l$. It follows that $\mathcal{L}_C\subseteqq C$. But $(\mathcal{L}_C)_C=\mathcal{L}_C=C_C$ so that $\mathcal{L}_C=C$. This completes the proof of the following result.

THEOREM 2.9. *Let* (A_{ij}) *be an* $l\times l$ *indecomposable Cartan matrix of finite type. Then the Lie algebra generated by the* $3l$ *elements* X_j, Y_j, Z_j, $1\leqq j\leqq l$, *satisfying the relations* R_1-R_{10} *is the compact simple Lie algebra of type* (A_{ij}).

One consequence of this result is the following Corollary.

COROLLARY 2.10. *Let* (A_{ij}) *be an* $l\times l$ *indecomposable Cartan matrix of finite type. Then there is one and only one simple Lie algebra generated by* $3l$ *elements* X_j, Y_j, Z_j, $1\leqq j\leqq l$, *satisfying the relations* R_1-R_6. *Moreover, this algebra is compact.*

PROOF. The algebra $\tilde{\mathcal{L}}_C$ satisfies R_1-R_6, and has a unique simple factor. This factor is compact. □

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