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REMARK ON SOME COMBINATORIAL CONSTRUCTION OF RELATIVE INVARIANTS

By

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It is a classical problem to determine the explicit form of relative invariants. However, if it is too complicated, it seems more important to know the mathematical structure of relative invariants than just to write down the all terms of them. Hence, in this paper, we suggest to use some principle to construct relative invariants in § 1, and as examples, we shall construct some relative invariants of $GL(n, \boldsymbol{C})$ on $\overset{3}{\wedge}\boldsymbol{C}^n$ for all $n\geqq 6$ (See Propositions 4.1, 4.3, and 4.5), including all relative invariants for $n=6, 7, 8, 9$. This work was done while the author was visiting Europe, and he would like to express his hearty thanks to Prof. H. Popp at Mannheim University in West Germany, and to Prof. D. Luna at Grenoble University in France for their mathematical stimulation and encouragement. The author also would like to express his hearty thanks to Prof. M. Sato who kindly explained his works for $n=6$.

§ 1. Let $\rho: G\rightarrow GL(V)$ be a finite-dimensional rational representation of a reductive algebraic group G, all defined over the complex number field $\boldsymbol{C}$.

A homogeneous polynomial $f(x)$ on V is called a *relative invariant* if there exists a rational character $\chi: G\rightarrow \boldsymbol{C}^{\times}$ satisfying $f(\rho(g)x)=\chi(g)f(x)$ for all $g\in G$ and $x\in V$. Now let $S^r(V)$ be the all homogeneous polynomials of degree r on V. Then the group G acts on $S^r(V)$ as $(g\phi)(x)\overset{\text{def}}{=}\phi(\rho(g)^{-1}x)$ for $\phi\in S^r(V)$, $g\in G$ and $x\in V$. We denote this representation by $\rho^{(r)}$. Since G is reductive, it is the direct sum of irreducible representations: $\rho^{(r)}=\bigoplus_i \rho_i^{(r)}$. We denote by $W_i^{(r)}$ the representation space of $\rho_i^{(r)}: S^r(V)=\bigoplus_i W_i^{(r)}$. Note that a homogeneous polynomial $f(x)$ is a relative invariant of degree r if and only if $f(x)\in W_i^{(r)}$ for some $W_i^{(r)}$ satisfying $\dim W_i^{(r)}=1$. We say that $\rho_i^{(r)}$ decomposes to $\rho_j^{(r_1)}\times\rho_k^{(r_2)}$ $(r_1+r_2=r)$ and denote this relation by $\rho_i^{(r)}\sim\rho_j^{(r_1)}\times\rho_k^{(r_2)}$ when $\rho_i^{(r)}$ is one of the irreducible components of the symmetric tensor of $\rho_j^{(r_1)}$ and $\rho_k^{(r_2)}$. This implies that the polynomials ϕ in $W_i^{(r)}$ can be obtained from those in $W_i^{(r_1)}$ and $W_k^{(r_2)}$, i.e., $\phi=\sum_t \phi_t\theta_t$ for some $\phi_t\in W_j^{(r_1)}$ and $\theta_t\in W_k^{(r_2)}$. In such a way, we can reduce the problem of

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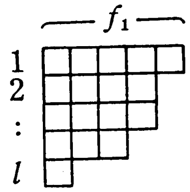
determination of a relative invariant to that of polynomials of lower degree.

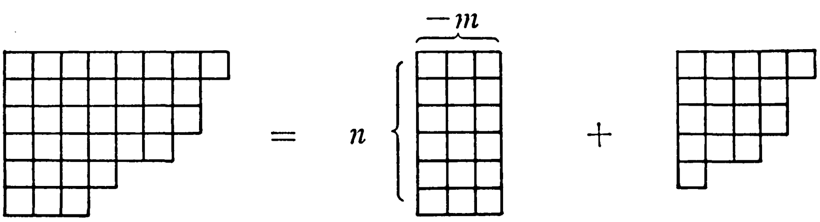
§ **2.** Since we shall deal examples for $G=GL(n, \boldsymbol{C})$ in § 3 and § 4, here, we shall review the Young diagram for $GL(n, \boldsymbol{C})$. Let $\mathcal{Y}$ be the set of the following diagrams.

$\mathcal{Y}=\Bigg\{$ 

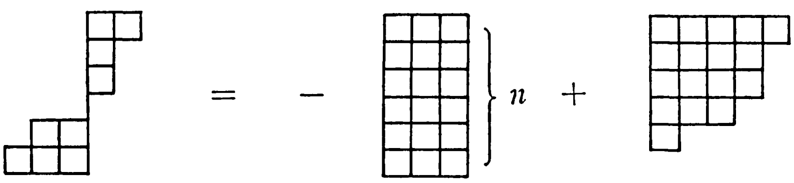
$;\quad l<n,\quad 0\leqq f_l\leqq \cdots \leqq f_1\Bigg\}.$

Then it is well-known (See [1]) that, for a given irreducible representation ρ of $GL(n, \boldsymbol{C})$, there exists uniquely an integer m such that $\tilde{\rho}(g)=(\det g)^m\cdot\rho(g)$ corresponds to one of the diagrams in $\mathcal{Y}$.

If m is negative, we correspond the following diagram to ρ.

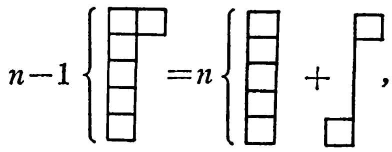


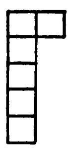
If m is positive, for example $m=3$, we correspond the following diagram to ρ.



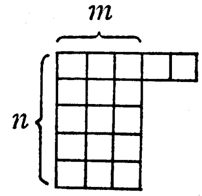
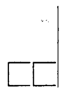
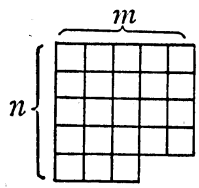
Thus any irreducible representation of $G=GL(n, \boldsymbol{C})$ corresponds uniquely to one of the Young diagrams.

EXAMPLE 2.1.  corresponds to the following representation ρ. Let V be the totality of $n\times n$ matrices X with $\operatorname{tr} X=0$. Then G acts on V by $\rho(g)X=gXg^{-1}$ for $g\in G$ and $X\in V$. Since



, this diagram  corresponds to

$\tilde{\rho}(g)=(\det g)\rho(g)$, i. e., $\tilde{\rho}(g)X=(\det g)gXg^{-1}$ for $g\in G$ and $X\in V$.

EXAMPLE 2.2. $\square\square$ corresponds to the following representation ρ. Let V be the totality of symmetric $n\times n$ matrices. Then G acts on V by $\rho(g)X=gX{}^tg$ for $g\in G$ and $X\in V$. Hence  corresponds to ρ' such that $\rho'(g)X=(\det g)^m\cdot gX{}^tg$ for $g\in G$ and $X\in V$. Similarly,  corresponds to the dual representation of ρ. Hence  corresponds to ρ'' such that $\rho''(g)X=(\det g)^m\cdot {}^tg^{-1}Xg^{-1}$ for $g\in G$ and $X\in V$.

§ **3.** Let $u_1, \cdots, u_n$ be a basis of $\boldsymbol{C}^n$. Then $G=GL(n, \boldsymbol{C})$ acts on $\boldsymbol{C}^n$ by $(gu_1, \cdots, gu_n)=(u_1, \cdots, u_n)g$ for $g\in G$. Hence G acts on $V=\overset{3}{\wedge}\boldsymbol{C}^n$ by $\rho(g)(u_i\wedge u_j\wedge u_k)=gu_i\wedge gu_j\wedge gu_k$. From now on, we shall fix this triplet (G, ρ, V) and let us consider its relative invariants.

We define the derivation $\dfrac{\partial}{\partial u_i}$ $(i=1, \cdots, n)$ by

$$\frac{\partial}{\partial u_i}(u_j\wedge u_k\wedge u_l)=\delta_{ij}\cdot u_k\wedge u_l \quad (k,\ l\neq i). \tag{3.1}$$

Then, for an element $x=\sum_{i<j<k} x_{ijk}u_i\wedge u_j\wedge u_k$ of V, we define the polarization $\hat{x}$ by

$$\hat{x}=\sum_{i=1}^{n} y_i\frac{\partial x}{\partial u_i} \text{ where } y_1, \cdots, y_n \text{ are indeterminants.} \tag{3.2}$$

For a natural number m satisfying $2m+1\leqq n$, we can define the homogeneous polynomials

$$f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x) \quad (1\leqq i_1, \cdots, i_{m-1}\leqq m-1\,;\ 1\leqq j_{2m+2}<\cdots<j_n\leqq n)$$

of degree m by

$$\underbrace{x\wedge\hat{x}\wedge\cdots\wedge\hat{x}}_{m}=\sum_{\substack{i_1,\cdots,i_{m-1}\\ j_{2m+2},\cdots,j_n}} f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x)y_{i_1}\cdots y_{i_{m-1}}v_{j_{2m+2},\cdots,j_n} \tag{3.3}$$

where $v_{j_{2m+2},\cdots,j_n}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_{j_n}=\omega$ $(=u_1\wedge\cdots\wedge u_n)$.

In other words, we have

$$f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x)\omega=x\wedge\frac{\partial x}{\partial u_{i_1}}\wedge\cdots\wedge\frac{\partial x}{\partial u_{i_{m-1}}}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_n\,. \tag{3.4}$$

For $n=2m+1$, we put $f^{i_1,\cdots,i_{m-1}}(x)=f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x)$, i. e.,

$$\underbrace{x\wedge\hat{x}\wedge\cdots\wedge\hat{x}}_{m}=\Big(\sum_{i_1,\cdots,i_{m-1}} f^{i_1,\cdots,i_{m-1}}(x)y_{i_1}\cdots y_{i_{m-1}}\Big)\omega\,.$$

PROPOSITION 3.1. (1) $f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(\rho(tI_n)x)=t^{3m}f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x)$, i. e., $f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(tx)=t^m\cdot f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x)$ for $t\in\boldsymbol{C}^\times$.

(2) $f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(\rho(c)x)=\dfrac{c_{i_1}\cdots c_{i_{m-1}}}{c_{j_{2m+2}}\cdots c_{j_n}}f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x)$

for $c=\begin{pmatrix} c_1 & & 0\\ & \ddots & \\ 0 & & c_n\end{pmatrix}\in SL(n,\boldsymbol{C})$.

(3) For $r\neq s$, we have

$$\begin{aligned} & f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(d\rho(E_{rs})x)\\ & =\sum_{l=1}^{m-1}\delta_{ri_l}\cdot f^{i_1,\cdots,\overset{l}{\overset{\vee}{s}},\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x)-\sum_{k=2m+2}^{n}\delta_{sj_k}f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,\underset{\overset{\wedge}{j_k}}{r},\cdots,i_n}(x)\,,\end{aligned}$$

where E_{rs} denotes the matrix unit.

PROOF. (1) is obvious. (2): By the action of $\rho(c)$, any term of the right-hand side of (3.4) becomes

$$\begin{aligned} & c_{k_1}c_{k_2}c_{k_3}x_{k_1k_2k_3}u_{k_1}\wedge u_{k_2}\wedge u_{k_3}\wedge c_{i_1}c_{i_1'}c_{i_1''}x_{i_1i_1'i_1''}u_{i_1'}\wedge u_{i_1''}\wedge\cdots\\ & \cdots\wedge c_{i_{m-1}i'_{m-1}i''_{m-1}}x_{i_{m-1}i'_{m-1}i''_{m-1}}u_{i'_{m-1}}\wedge u_{i''_{m-1}}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_{j_n}\\ & =\frac{c_{i_1}\cdots c_{i_{m-1}}}{c_{j_{2m+2}}\cdots c_{j_n}}(c_{k_1}c_{k_2}c_{k_3}c_{i_1'}c_{i_1''}\cdots c_{i'_{m-1}}c_{i''_{m-1}}c_{j_{2m+2}}\cdots c_{j_n})(x_{k_1k_2k_3}\cdots)\omega\\ & =\frac{c_{i_1}\cdots c_{i_{m-1}}}{c_{j_{2m+2}}\cdots c_{j_n}}\cdot x_{k_1k_2k_3}u_{k_1}\wedge u_{k_2}\wedge u_{k_3}\wedge x_{i_1i_1'i_1''}u_{i_1'}\wedge u_{i_1''}\wedge\cdots\wedge u_{j_n}\end{aligned}$$

since $c_{k_1}c_{k_2}c_{k_3}c_{i_1'}c_{i_1''}\cdots c_{i'_{m-1}}c_{i''_{m-1}}c_{j_{2m+2}}\cdots c_{j_n}=c_1\cdots c_n=1$.

(3): We put $x=x(u_1, \cdots, u_n)=\sum x_{ijk}u_i\wedge u_j\wedge u_k$.
Then we have

$$\frac{\partial}{\partial u_i}(\rho(g)x)=\frac{\partial}{\partial u_i}x(gu_1, \cdots, gu_n)=\sum_k\Big(\frac{\partial x}{\partial u_k}\Big)(gu_1, \cdots, gu_n)\frac{\partial(gu_k)}{\partial u_i} \tag{3.5}$$

$$=\sum_k g_{ik}\frac{\partial x}{\partial u_k}(gu_1, \cdots, gu_n) \quad \text{for} \quad g=(g_{ij})\in G.$$

For $g=\exp t\, E_{rs}$, we have

$$\frac{d}{dt}\rho(g)x\,|_{t=0}=\sum_{k=1}^{n}\frac{d(gu_k)}{dt}\wedge\frac{\partial x}{\partial u_k}(gu_1, \cdots, gu_n)\,|_{t=0}=u_r\wedge\frac{\partial x}{\partial u_s}. \tag{3.6}$$

Similarly, by using (3.5), we have

$$\frac{d}{dt}\Big(\frac{\partial}{\partial u_i}\rho(g)x\Big)\Big|_{t=0}=u_r\wedge\frac{\partial^2 x}{\partial u_s\partial u_i}+\delta_{ir}\frac{\partial x}{\partial u_s} \quad \text{for} \quad g=\exp t\, E_{rs}. \tag{3.7}$$

Using (3.4)$\sim$(3.7), we have

$$f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(d\rho(E_{rs})x)\omega=\frac{d}{dt}f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(\rho(g)x)\omega\,|_{t=0} \tag{3.8}$$

$$=\sum_{l=1}^{m-1}\delta_{ri_l}\cdot f^{i_1,\cdots,\overset{l}{\vee}{s},\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x)+A, \quad \text{where}$$

$$A=\Big(u_r\wedge\frac{\partial x}{\partial u_s}\Big)\wedge\frac{\partial x}{\partial u_{i_1}}\wedge\cdots\wedge\frac{\partial x}{\partial u_{i_{m-1}}}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_{j_n} \tag{3.9}$$

$$+\sum_{l=1}^{m-1}x\wedge\frac{\partial x}{\partial u_{i_1}}\wedge\cdots\wedge\Big(u_r\wedge\frac{\partial^2 x}{\partial u_s\partial u_{i_l}}\Big)\wedge\cdots\wedge\frac{\partial x}{\partial u_{i_{m-1}}}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_{j_n}.$$

We shall show that $A=0$ if $s\neq j_{2m+2}, \cdots, j_n$. Since A is a multiple of $\omega=u_1\wedge\cdots\wedge u_n$, any non-zero term A contains u_s, and hence we have

$$A=\sum_{l=1}^{m-1}\Big(u_r\wedge\frac{\partial x}{\partial u_s}\Big)\wedge\frac{\partial x}{\partial u_{i_1}}\wedge\cdots\wedge\Big(u_s\wedge\frac{\partial^2 x}{\partial u_s\partial u_{i_l}}\Big)\wedge\cdots\wedge\frac{\partial x}{\partial u_{i_{m-1}}}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_{j_n}$$

$$+\sum_{l=1}^{m-1}\Big(u_s\wedge\frac{\partial x}{\partial u_s}\Big)\wedge\frac{\partial x}{\partial u_{i_1}}\wedge\cdots\wedge\Big(u_r\wedge\frac{\partial^2 x}{\partial u_s\partial u_{i_l}}\Big)\wedge\cdots\wedge\frac{\partial x}{\partial u_{i_{m-1}}}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_{j_n}$$

$$+\sum_{l=1}^{m-1}\sum_{k\neq l}x\wedge\frac{\partial x}{\partial u_{i_1}}\wedge\cdots\wedge\Big(u_s\wedge\frac{\partial^2 x}{\partial u_s\partial u_{i_k}}\Big)\wedge\cdots\wedge\Big(u_r\wedge\frac{\partial^2 x}{\partial u_s\partial u_{i_l}}\Big)\wedge\cdots$$

$$\cdots\wedge\frac{\partial x}{\partial u_{i_{m-1}}}\wedge\cdots\wedge u_{j_n}=0$$

because the first and the second terms cancel each other and the last term is clearly zero. Finally we shall show that

$$A=-f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,\underset{\wedge}{r},\cdots,j_n}(x)\omega \qquad (\text{with } r \text{ in place of } \hat{j}_k)$$

if $s=j_k$ for some $k=2m+2,\cdots,n$. Note that if

$$x=x_s+u_s\wedge\frac{\partial x}{\partial u_s}\quad\left(\text{resp. } \frac{\partial x}{\partial u_i}=\left(\frac{\partial x}{\partial u_i}\right)_s+u_s\wedge\frac{\partial^2 x}{\partial u_s\partial u_i}\right),$$

then

$$u_s\wedge x_s=u_s\wedge x\quad\left(\text{resp. } u_s\wedge\left(\frac{\partial x}{\partial u_i}\right)_s=u_s\wedge\frac{\partial x}{\partial u_i}\right).$$

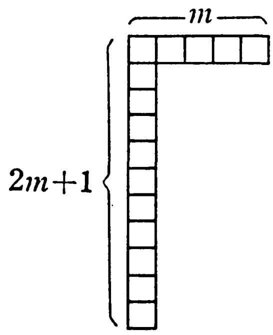
Then

$$\begin{aligned}
A=&-\left(u_s\wedge\frac{\partial x}{\partial u_s}\right)\wedge\frac{\partial x}{\partial u_{i_1}}\wedge\cdots\wedge\frac{\partial x}{\partial u_{i_{m-1}}}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_r\wedge\cdots\wedge u_{j_n}\\
&-\sum_{l=1}^{m-1}x\wedge\frac{\partial x}{\partial u_{i_1}}\wedge\cdots\wedge\left(u_s\wedge\frac{\partial^2 x}{\partial u_s\partial u_{i_l}}\right)\wedge\cdots\wedge\frac{\partial x}{\partial u_{i_{m-1}}}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_r\wedge\cdots\wedge u_{j_n}\\
=&-\left[x_s+u_s\wedge\frac{\partial x}{\partial u_s}\right]\wedge\left[\left(\frac{\partial x}{\partial u_{i_1}}\right)_s+u_s\wedge\frac{\partial^2 x}{\partial u_s\partial u_{i_1}}\right]\wedge\cdots\wedge\left[\left(\frac{\partial x}{\partial u_{i_{m-1}}}\right)_s+u_s\wedge\frac{\partial^2 x}{\partial u_s\partial u_{i_{m-1}}}\right]\\
&\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_r\wedge\cdots\wedge u_{j_n}\\
=&-x\wedge\frac{\partial x}{\partial u_{i_1}}\wedge\cdots\wedge\frac{\partial x}{\partial u_{i_{m-1}}}\wedge u_{j_{2m+2}}\wedge\cdots\wedge u_r\wedge\cdots\wedge u_{j_n}\\
=&-f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,r,\cdots,j_n}(x)\omega.
\end{aligned}$$

Q. E. D.

REMARK 3.2. This proposition implies that $S^m(V)=S^m\left(\begin{smallmatrix}\square\\ \square\end{smallmatrix}\right)$ contains the following Young diagram.

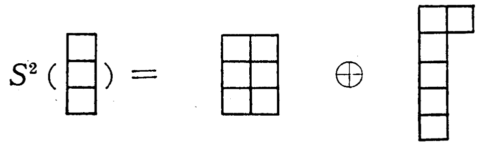
(3.10)



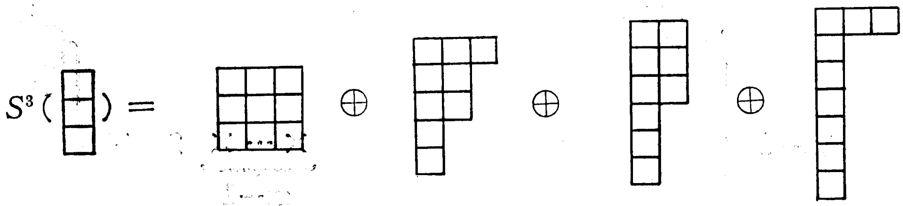
This corresponds to the polynomials $f^{i_1,\cdots,i_{m-1}}_{j_{2m+2},\cdots,j_n}(x)$.

For example, it is known that

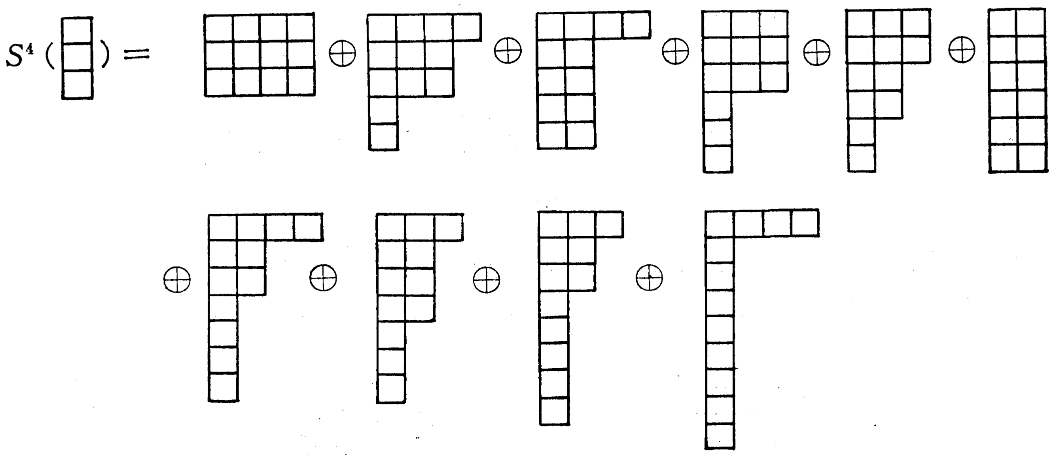
(3.11)



(3.12)



(3.13)



§ 4. Example (I) First let us consider the following polynomials of degree $m(m-1)$ for $n=3m$ $(m\geqq 2)$.

(4.1)
$$F^{i_1,\cdots,i_{m-1}}_{j_1,\cdots,j_{m-1}}(x)$$
$$=\sum_{\substack{i^\mu_\nu \\ (\mu\neq\nu)}} f^{i^1_1,\cdots,i^1_{m-1}}_{i^2_1,\cdots,i^{m-1}_1,j_1}(x)\cdot f^{i^2_1,\cdots,i^2_{m-1}}_{i^1_2,i^3_2,\cdots,i^{m-1}_2,j_2}(x)\cdots f^{i^{m-1}_1,\cdots,i^{m-1}_{m-1}}_{i^1_{m-1},\cdots,i^{m-2}_{m-1},j_{m-1}}(x)$$

where $i_k=i^k_k$ for $k=1,\cdots,m-1$.
By Proposition 3.1, we have

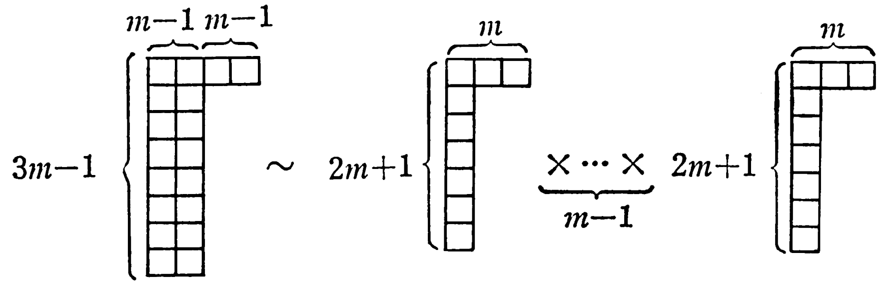
(4.2)
$$F^{i_1,\cdots,i_{m-1}}_{j_1,\cdots,j_{m-1}}(\rho(c)x)=\frac{c_{i_1}\cdots c_{i_{m-1}}}{c_{j_1}\cdots c_{j_{m-1}}}F^{i_1,\cdots,i_{m-1}}_{j_1,\cdots,j_{m-1}}(x)$$

$$\text{for}\quad c=\begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_{3m}\end{pmatrix}\in SL(3m,\boldsymbol{C}).$$

(4.3)
$$F^{i_1,\cdots,i_{m-1}}_{j_1,\cdots,j_{m-1}}(d\rho(E_{rs})x)$$

$$= \sum_{l=1}^{m-1} \delta_{r i_l} F^{i_1,\cdots,\overset{l}{\underset{\vee}{}}s,\cdots,i_{m-1}}_{j_1,\cdots,j_{m-1}}(x) - \sum_{l=1}^{m-1} \delta_{s i_l} F^{i_1,\cdots,i_{m-1}}_{j_1,\cdots,\underset{l}{\hat{r}},\cdots,j_{m-1}}(x) \quad \text{for} \quad r \neq s.$$

Therefore, (4.1) corresponds to the following decomposition.

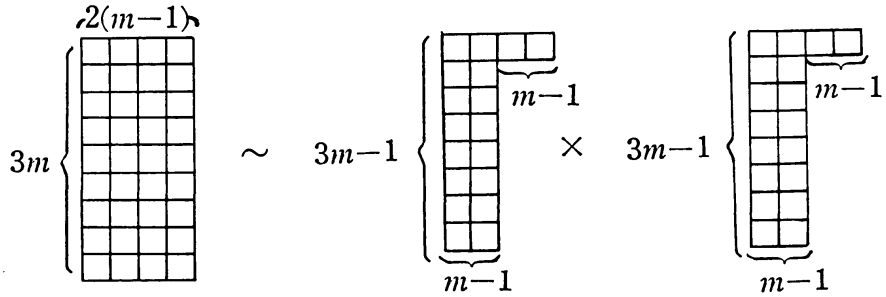
(4.4) 

Then, by (4.2) and (4.3), we see that the polynomial

$$f(x) = \sum_{\substack{i_1,\cdots,i_{m-1} \\ j_1,\cdots,j_{m-1}}} F^{i_1,\cdots,i_{m-1}}_{j_1,\cdots,j_{m-1}}(x) \cdot F^{j_1,\cdots,j_{m-1}}_{i_1,\cdots,i_{m-1}}(x) \tag{4.5}$$

is a relative invariant of degree $2m(m-1)$ on $\overset{3}{\wedge} \boldsymbol{C}^{3m}$.

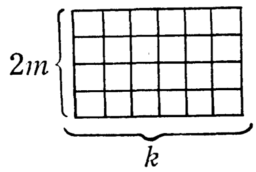
This (4.5) corresponds to the following decomposition.

(4.6) 

PROPOSITION 4.1. There exists a relative invariant $f(x)$ given by (4.5) of degree $2m(m-1)$ on $\overset{3}{\wedge} \boldsymbol{C}^{3m}$.

REMARK 4.2. For $m=2$, there exists a relative invariant $f(x)$ of degree 4 on $\overset{3}{\wedge} \boldsymbol{C}^{6}$. In this case, we have 6×6 matrix $\phi(x) = (F_{ij}(x))$ where $F_{ij}(x) = f^i_j(x)$ in (4.1), satisfying $\phi(\rho(g)x) = \det g \cdot g\phi(x)g^{-1}$ for $g \in GL(6, \boldsymbol{C})$ (See Example 2.1). Then we have $\operatorname{tr}\phi(x) = 0$ and $f(x) = \operatorname{tr}\phi(x)^2$. In fact, $\phi(x)^2$ is a non-zero scalar matrix, i. e., $\phi(x)^2 = \frac{1}{6} f(x) I_6$ (See [3]). This case has been investigated by M. Sato (See [2]).

For $m=3$, there exists a relative invariant of degree 12 on $\overset{3}{\wedge} \boldsymbol{C}^{9}$.

EXAMPLE (II) Assume that there exists a relative invariant of degree r for $n=2m$. This implies that $S^r\left(\begin{smallmatrix}\Box\\ \Box\\ \Box\end{smallmatrix}\right)$ contains  such that $2mk=3r$.

In particular, we have $k=\dfrac{3r}{2m}\in\boldsymbol{Z}$. For $n=2m+1$, this implies that there exist homogeneous polynomials $F_{i_1,\dots,i_k}(x)$ of degree r, symmetric with respect to indices $i_1,\cdots,i_k$, satisfying

(4.7) $$F_{i_1,\dots,i_k}(\rho(c)x)=\frac{1}{c_{i_1}\cdots c_{i_k}}F_{i_1,\dots,i_k}(x)$$

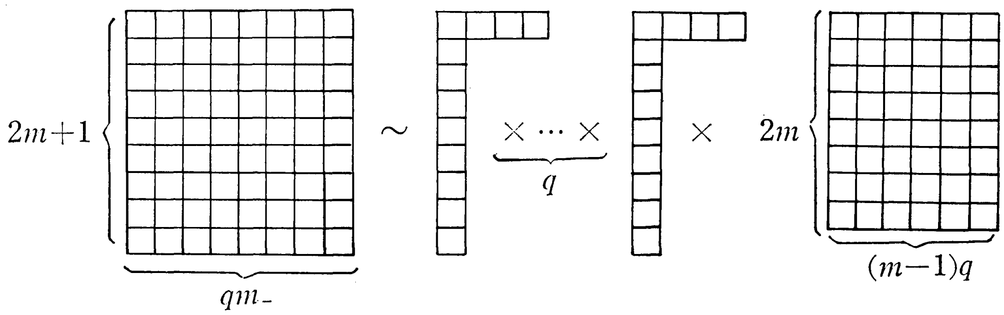
$$\text{for}\quad c=\begin{pmatrix}c_1 & & \\ & \ddots & \\ & & c_n\end{pmatrix}\in SL(n,\boldsymbol{C}),$$

(4.8) $$F_{i_1,\dots,i_k}(d\rho(E_{rs})x)=-\sum_{l=1}^{k}\delta_{si_l}\cdot F_{i_1,\dots,\underset{\hat{l}}{r},\dots,i_k}(x)\qquad\text{for}\quad r\neq s.$$

Now assume that $k\equiv 0 \bmod(m-1)$, i.e., $k=q(m-1)$ for some q. Then, by Proposition 3.1, (4.7) and (4.8), we can see that the homogeneous polynomial

(4.9) $$f(x)=\sum_{i_1^1,\dots,i_{m-1}^q} f^{i_1^1,\dots,i_{m-1}^1}(x)\cdots f^{i_1^q,\dots,i_{m-1}^q}(x)F_{i_1^1,\dots,i_{m-1}^q}(x)$$

is a relative invariant of degree $r'=r+mq\left(=\dfrac{2m+1}{2m-2}\cdot r\right)$ for $n=2m+1$. This (4.9) corresponds to the following decomposition.

(4.10) 

PROPOSITION 4.3. Assume that there exists a relative invariant of degree r of $(GL(2m,\boldsymbol{C}),\ \begin{smallmatrix}\Box\\ \Box\\ \Box\end{smallmatrix},\ \overset{3}{\wedge}\boldsymbol{C}^{2m})$. Then we have $k=\dfrac{3r}{2m}\in\boldsymbol{Z}$. If $k\equiv 0 \bmod(m-1)$, then there exists a relative invariant $f(x)$ given by (4.9) of degree $r'=\dfrac{2m+1}{2m-2}r$ of $(GL(2m+1,\boldsymbol{C}),\ \begin{smallmatrix}\Box\\ \Box\\ \Box\end{smallmatrix},\ \overset{3}{\wedge}\boldsymbol{C}^{2m+1})$.

REMARK 4.4. For any relative invariant $f(x)$ of $\left(GL(2m, \boldsymbol{C}), \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \stackrel{3}{\wedge}\boldsymbol{C}^{2m}\right)$, $f(x)^{m-1}$ satisfies the condition in Proposition 4.3. For $m=3$ and $r=4$ (See Remark 4.2), we have $k=2\equiv 0 \bmod(m-1)$, and hence, there exists a relative invariant $f(x)$ of degree 7 on $\stackrel{3}{\wedge}\boldsymbol{C}^7$. It is known that any relative invariant on $\stackrel{3}{\wedge}\boldsymbol{C}^7$ is of the form $cf(x)^m$ ($c\in\boldsymbol{C}^{\times}$, $m\in\boldsymbol{Z}$) (See [2]). In this case, (3.10) corresponds to 7×7 symmetric matrices $\phi(x)=(f^{ij}(x))$ with $f^{ij}(x)$ in (3.4), satisfying $\phi(\rho(g)x)=(\det g)g\cdot\phi(x){}^tg$ for $g\in G$, and $\begin{array}{|c|c|}\hline \ & \ \\\hline \ & \ \\\hline \ & \ \\\hline \ & \ \\\hline \ & \ \\\hline \ & \ \\\hline\end{array} \in S^4\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)$ (See also (3.13)) corresponds to 7×7 symmetric matrices $\phi^*(x)=(F_{ij}(x))$ with $F_{ij}(x)=\sum_{k,l} f^k_{il}(x)f^l_{kj}(x)$ (i. e., (4.6) for $m=2$ and $n=7$), satisfying $\phi^*(\rho(g)x)=(\det g)^2\cdot{}^tg^{-1}\phi^*(x)g^{-1}$ for $g\in G$ (See Example 2.2). Put $\Phi(x)=\phi(x)\cdot\phi^*(x)$. Then we have $\Phi(\rho(g)x)=(\det g)^3\cdot g\Phi(x)g^{-1}$ for $g\in G$, and $f(x)=\operatorname{tr}\Phi(x)$. In fact, $\Phi(x)$ is a non-zero scalar matrix: $\Phi(x)=\frac{1}{7}f(x)I_7$ (See [3]).

EXAMPLE (III) Assume that there exists a relative invariant of degree r for $n=2m+1$. This implies that $S^r\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)$ contains $2m+1\left\{\underbrace{\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\\hline \ & \ & \ & \ & \ \\\hline \ & \ & \ & \ & \ \\\hline \ & \ & \ & \ & \ \\\hline \ & \ & \ & \ & \ \\\hline\end{array}}_{k}\right.$ such that $3r=(2m+1)k$. In particular, we have $k=\dfrac{3r}{2m+1}\in\boldsymbol{Z}$. For $n=2m+2$, this implies that there exist homogeneous polynomials $F_{i_1,\dots,i_k}(x)$ of degree r, symmetric with respect to indices $i_1,\cdots,i_k$, satisfying (4.7) and (4.8) (for $n=2m+2$). Now assume that $k\equiv 0 \bmod(m-2)$, i. e., $k=q(m-2)$ for some q. Then we can define polynomials

$$F^{i_1^1,\dots,i_{m-2}^1,\dots,i_1^q,\dots,i_{m-2}^q}_{j_{2m+3}^1,\dots,j_n^1,\dots,j_{2m+3}^q,\dots,j_n^q}(x) \tag{4.11}$$

$$=\sum_{k_1,\dots,k_q=1}^{n} f^{i_1^1,\dots,i_{m-2}^1 k_1}_{k_q j_{2m+3}^1,\dots,j_n^1}(x)\cdot f^{i_1^2,\dots,i_{m-2}^2 k_2}_{k_1 j_{2m+3}^2,\dots,j_n^2}(x)\cdots f^{i_1^q,\dots,i_{m-2}^q k_q}_{k_{q-1} j_{2m+3}^q,\dots,j_n^q}(x)$$

of degree mq for $n\geqq 2m+2$. For $n=2m+2$, we denote it simply by $F^{i_1^1,\dots,i_{m-2}^q}(x)$. Then, by Proposition 3.1, we have

$$F^{i_1^1,\dots,i_{m-2}^q}_{j_{2m+3}^1,\dots,j_n^q}(\rho(c)x)=\frac{c_{i_1^1}\cdots c_{i_{m-2}^q}}{c_{j_{2m+3}^1}\cdots c_{j_n^q}}F^{i_1^1,\dots,i_{m-2}^q}_{j_{2m+3}^1,\dots,j_n^q}(x) \tag{4.12}$$

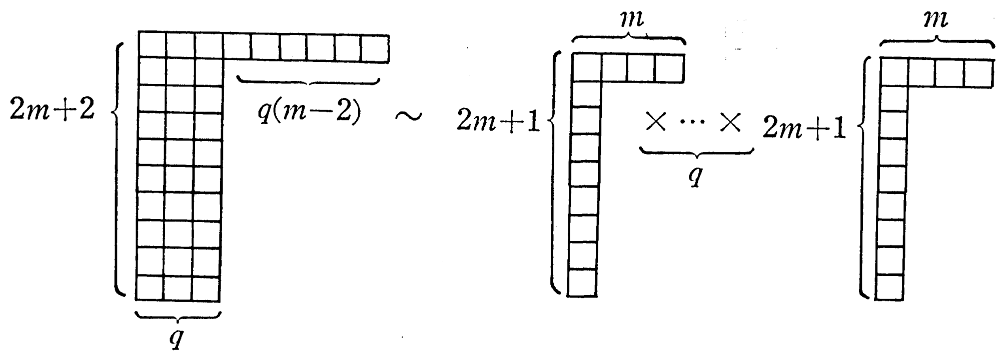
$$\text{for}\quad c=\begin{pmatrix} c_1 & & 0\\ & \ddots & \\ 0 & & c_n\end{pmatrix}\in SL(n, \boldsymbol{C}),$$

$$(4.13)\qquad F^{i_1^1,\dots,i_{m-2}^q}_{j_{2m+3}^1,\dots,j_n^q}(d\rho(E_{rs})x)$$

$$=\sum \delta_{ri_\nu^\mu} F^{i_1^1,\dots,s,\dots,i_{m-2}^q}_{j_{2m+3}^1,\dots,j_n^q}(x)-\sum \delta_{sj_\nu^\mu} F^{i_1^1,\dots,i_{m-2}^q}_{j_{2m+3}^1,\dots,r,\dots,j_n^q}(x)\qquad \text{for}\quad r\neq s.$$

This implies that (4.11) corresponds to the following diagrams.

(4.14)



Then, by (4.7), (4.8), (4.12) and (4.13), one can see that

$$(4.15)\qquad f(x)=\sum_{i_1^1,\dots,i_{m-2}^q=1}^{n} F^{i_1^1,\dots,i_{m-2}^q}(x)\cdot F_{i_1^1,\dots,i_{m-2}^q}(x)$$

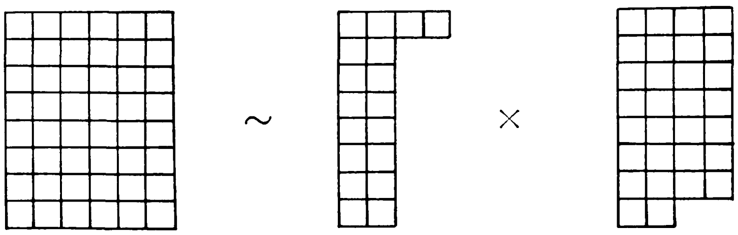
is a relative invariant of degree $r'=r+mq\ \left(=\frac{2(m^2-1)}{(2m+1)(m-2)}r\right)$ on $\overset{3}{\wedge}\boldsymbol{C}^{2m+2}$.

PROPOSITION 4.5. Assume that there exists a relative invariant of degree r of $(GL(2m+1, \boldsymbol{C}), \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \overset{3}{\wedge}\boldsymbol{C}^{2m+1})$. Then we have $k=\frac{3r}{2m+1}\in\boldsymbol{Z}$. If $k\equiv 0 \bmod (m-2)$, then there exists a relative invariant $f(x)$ given by (4.15) of degree $r'=\frac{2(m^2-1)}{(2m+1)(m-2)}\cdot r$ of $(GL(2m+2, \boldsymbol{C}), \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \overset{3}{\wedge}\boldsymbol{C}^{2m+2})$.

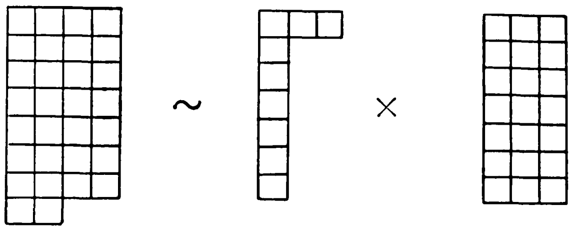
REMARK 4.6. For $m=3$ and $r=7$ (See Remark 4.4), we have $k=3\equiv 0 \bmod (m-2)$, and hence there exists a relative invariant $f(x)$ of degree $r'=16$ on $\overset{3}{\wedge}\boldsymbol{C}^8$. It is known that any relative invariant of $(GL(8, \boldsymbol{C}), \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \overset{3}{\wedge}\boldsymbol{C}^8)$ is of the form $cf(x)^m$ with $c\in\boldsymbol{C}^\times$ and $m\in\boldsymbol{Z}$.

In this case, we can also use (4.4) for $m=3$ and the following decompositions (4.16) and (4.17).

(4.16)



(4.67)



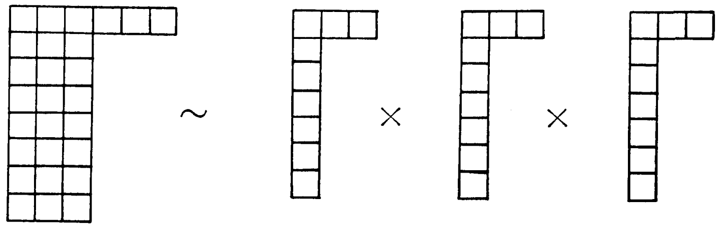
This implies that there exist 8×8 symmetric matrices $\phi(x)$ (resp. $\phi^*(x)$) whose entries are homogeneous polynomials of degree 6 (resp. 10) satisfying $\phi(\rho(g)x)=(\det g)^2\cdot g\phi(x){}^tg$ (resp. $\phi^*(\rho(g)x)=(\det g)^4\cdot {}^tg^{-1}\phi^*(x)g^{-1}$) for $g\in GL(8, \boldsymbol{C})$ (See Example 2.2). Put $\Phi(x)=\phi(x)\phi^*(x)$. Then we have $\Phi(\rho(g)x)=(\det g)^6\cdot g\Phi(x)g^{-1}$ for $g\in GL(8, \boldsymbol{C})$ and $f(x)=\mathrm{tr}\,\Phi(x)$. In fact, $\Phi(x)$ is a non-zero scalar matrix $\Phi(x)=\frac{1}{8}f(x)\cdot I_8$ (See [3]).

Applying Proposition 4.3 for $m=4$ and $r=16$, we have $k=6\equiv 0 \mod(m-1)$, and hence we can construct a relative invariant of degree 24 on $\overset{3}{\wedge}\boldsymbol{C}^9$.

EXAMPLE (IV) (Relative invariants on $\overset{3}{\wedge}\boldsymbol{C}^9$). It is known that there exist four independent relative invariants $f_k(x)$ of degree k ($k=12, 18, 24, 30$) which generate all relative invariants on $\overset{3}{\wedge}\boldsymbol{C}^9$ (See [4]). We have constructed $f_{12}(x)$ by Proposition 4.1 for $m=3$. We have also constructed a relative invariant $f(x)$ of degree 24 on $\overset{3}{\wedge}\boldsymbol{C}^9$ by Proposition 4.3 (See Remark 4.6). For $x_0=u_1\wedge u_2\wedge u_3+u_4\wedge u_5\wedge u_6+u_7\wedge u_8\wedge u_9$, we have $f^{ijk}(x_0)=0$ for all i, j, k, and $f^{ij}_{ij}(x_0)=f^{ji}_{ij}(x_0)=-f^{ij}_{ji}(x_0)=-f^{ji}_{ji}(x_0)=1$ for $i\in A_1=\{1, 2, 3\}$, $j\in A_2=\{4, 5, 6\}$ or $i\in A_2$, $j\in A_3=\{7, 8, 9\}$ or $i\in A_3$, $j\in A_1$, and $f^{ij}_{kl}(x_0)=0$ for other indices. Hence, by calculation, we have $f(x_0)=0$ and $f_{12}(x_0)\neq 0$, i.e., $f(x)=f_{24}(x)$ and $f(x)\neq cf_{12}(x)^2$ for any $c\in\boldsymbol{C}^\times$. Now let us consider the following polynomials.

$$F^{i_1i_2i_2}_{j_1j_2j_3}(x)=\sum_{k_1,k_2,k_3} f^{i_1k_1}_{j_2k_3}(x)\cdot f^{i_2k_2}_{j_2k_1}(x)\cdot f^{i_3k_3}_{j_3k_2}(x). \tag{4.18}$$

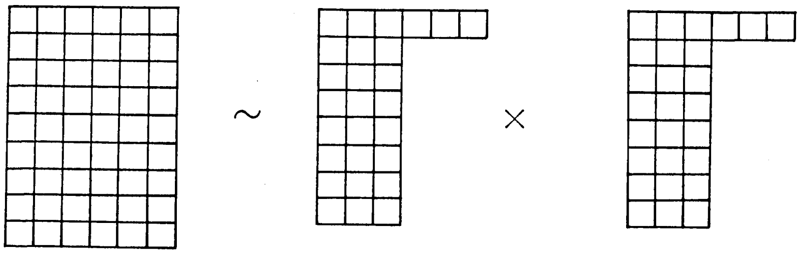
By Proposition 3.1, these polynomials correspond to the following decomposition.

(4.19) 

Then, we have

$$f_{18}(x)=\sum_{\substack{i_1,i_2,i_3\\ j_1,j_2,j_3}} F^{i_1i_2i_3}_{j_1j_2j_3}(x)\cdot F^{j_1j_2j_3}_{i_1i_2i_3}(x)\,. \tag{4.20}$$

This corresponds to the following decomposition.

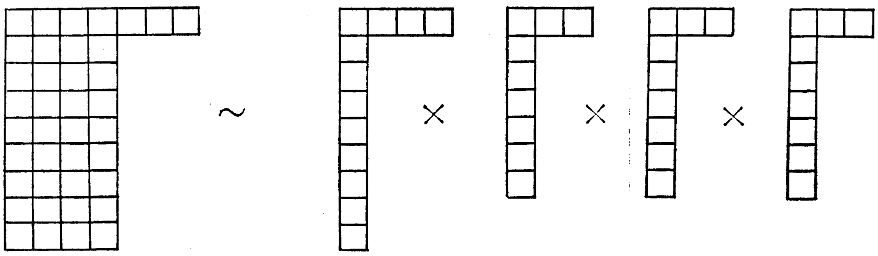
(4.21) 

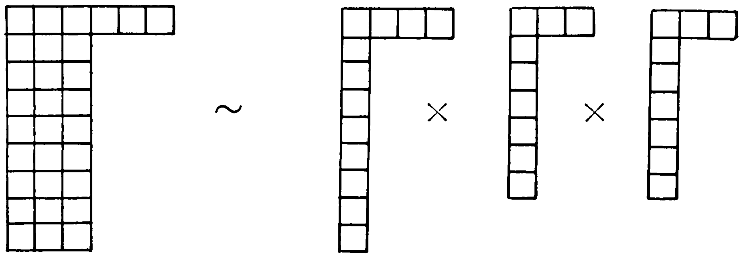
One can check that $f_{18}(x_0)=-2\cdot 3^5$.
Next let us consider the following $F^{i_1i_2i_3}(x)$ and $F_*^{i'_1i'_2i'_3}(x)$.

$$F^{i_1i_2i_3}(x)=\sum_{\substack{k_1k_2k_3\\ j_1j_2j_3}} f^{k_1k_2k_3}(x)\cdot f^{i_1j_2}_{k_1j_1}(x)\cdot f^{i_2j_3}_{k_2j_2}(x)\cdot f^{i_3j_1}_{k_3j_3}(x)\,. \tag{4.22}$$

$$F_*^{i'_1i'_2i'_3}(x)=\sum_{\substack{k'_2,k'_3\\ l,l'}} f^{i'_1k'_2k'_3}(x)\cdot f^{i'_2l}_{k'_2l'}(x)\cdot f^{i'_3l'}_{k'_3l}(x)\,. \tag{4.23}$$

By Proposition 3.1, they correspond to the following decompositions respectively.

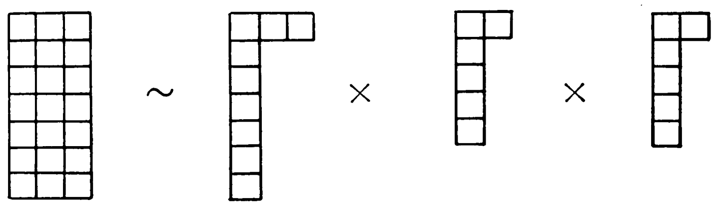
(4.24) 

(4.25) 

The existence of a relative invariant of degree 7 on $\overset{3}{\wedge} C^7$ corresponds to the following polynomials for $n=9$.

$$F_{i_1 i_1', i_2 i_2', i_3 i_3'}(x) = \sum_{\substack{i_2'' i_3'' \\ st}} f^{i_2'' i_3''}_{i_1 i_1'} f^{s}_{i_2 i_2' i_2'' t} f^{t}_{i_3 i_3' i_3'' s}(x). \tag{4.26}$$

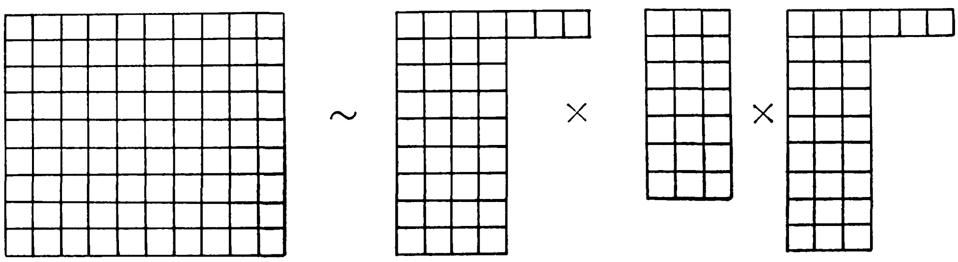
This corresponds to the following decomposition.

(4.27) 

Then, we have

$$\begin{aligned} f_{30}(x) &= \sum_{\substack{i_1 i_2 i_3 \\ i_1' i_2' i_3'}} F^{i_1 i_2 i_3}(x) \cdot F_{*}^{i_1' i_2' i_3'}(x) \cdot F_{i_1 i_1', i_2 i_2', i_3 i_3'}(x). \\ &= \sum f^{k_1 k_2 k_3} f^{i_1' k_2' k_3'} f^{i_1 j_2}_{k_1 j_1} f^{i_2 j_3}_{k_2 j_2} f^{i_3 j_1}_{k_3 j_3} f^{j_2' l}_{k_2' l'} f^{i_3' l'}_{k_3' l} f^{i_2'' i_3''}_{i_1 i_1'} f^{s}_{i_2 i_2' i_2'' t} f^{t}_{i_3 i_3' i_3'' s}(x). \end{aligned} \tag{4.28}$$

In fact, one can check easily that $f_{30}(x_0)=0$ and $f_{12}(x_0) f_{18}(x_0) \neq 0$. This (4.28) corresponds to the following decomposition.

(4.29) 

REMARK 4.7. Formally, we can construct, for example,

$$f(x) = \sum_{\substack{i_1 i_2 i_3 \\ i_1' i_2' i_3'}} F^{i_1 i_2 i_3}(x) \cdot F^{i_1' i_2' i_3'}(x) \cdot F_{i_1 i_1', i_2 i_2', i_3 i_3'}(x).$$

But it is identically zero since $F_{i_1 i_1', i_2 i_2', i_3 i_3'}(x) = -F_{i_1' i_1, i_2' i_2, i_3' i_3}(x)$.

References

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