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# SEMILINEAR DEGENERATE ELLIPTIC BOUNDARY VALUE PROBLEMS VIA CRITICAL POINT THEORY

By

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Dedicated to the memory of Professor Seizô Itô (1927–2011)

**Abstract.** The purpose of this paper is to study a class of semilinear elliptic boundary value problems with *degenerate* boundary conditions which include as particular cases the Dirichlet and Robin problems. The approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of partial differential equations. By making use of a variant of the Ljusternik–Schnirelman theory of critical points, we prove very exact results on the number of solutions of our problem. The results here extend earlier theorems due to Castro–Lazer to the degenerate case.

## 1. Statement of Main Results

Let $\Omega$ be a bounded domain of Euclidean space $\mathbf{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$; its closure $\overline{\Omega} = \Omega \cup \partial\Omega$ is an $N$-dimensional, compact smooth manifold with boundary. Let $A$ be a second-order, elliptic differential operator with real coefficients such that

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{N} a^{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u. \tag{1.1}$$

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Here:

(1) $a^{ij} \in C^\infty(\overline{\Omega})$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{\Omega}$ and $1 \leq i, j \leq N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2 \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbf{R}^N.$$

(2) $c \in C^\infty(\overline{\Omega})$ and $c(x) \geq 0$ in $\Omega$.

Let $B$ be a first-order, boundary condition with real coefficients such that

$$Bu = a(x')\frac{\partial u}{\partial \nu} + b(x')u. \tag{1.2}$$

Here:

(3) $a \in C^\infty(\partial\Omega)$ and $a(x') \geq 0$ on $\partial\Omega$.
(4) $b \in C^\infty(\partial\Omega)$ and $b(x') \geq 0$ on $\partial\Omega$.
(5) $\partial/\partial\nu$ is the conormal derivative associated with the operator $A$:

$$\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{N} a^{ij}(x')n_j\frac{\partial}{\partial x_i},$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ is the unit exterior normal to the boundary $\partial\Omega$.

Our fundamental hypotheses on the boundary condition $B$ are the following:

(H.1) $a(x') + b(x') > 0$ on $\partial\Omega$.
(H.2) $b(x') \not\equiv 0$ on $\partial\Omega$.

It should be noticed that if $a(x') \equiv 0$ and $b(x') \equiv 1$ on $\partial\Omega$ (resp. $a(x') \equiv 1$ on $\partial\Omega$), then the boundary condition $B$ is the Dirichlet condition (resp. Robin condition). Moreover, it is easy to see that the boundary condition $B$ is non-degenerate (or coercive) if and only if either $a(x') > 0$ on $\partial\Omega$ or $a(x') \equiv 0$ and $b(x') > 0$ on $\partial\Omega$. Therefore, our boundary condition $B$ is a *degenerate* boundary value problem from an analytical point of view (cf. [17]). Amann [3] studied the boundary condition $B$ in the non-degenerate case where the boundary $\partial\Omega$ is the disjoint union of the two closed subsets $M = \{x' \in \partial\Omega : a(x') = 0\}$ and $\partial\Omega \backslash M = \{x' \in \partial\Omega : a(x') > 0\}$, each of which is an $(N-1)$-dimensional, compact smooth manifold.

The intuitive meaning of condition (H.1) is that the absorption phenomenon occurs at each point of the set $M$, while the reflection phenomenon occurs at each point of the set $\partial\Omega \backslash M$ (see [24]). On the other hand, condition (H.2) implies that the boundary condition $B$ is not equal to the purely Neumann condition (see Remark 1.1).

In this paper we study the following semilinear *non-homogeneous* elliptic boundary value problem: Let $g(t)$ be a real-valued function defined on $\mathbf{R}$. Given a function $h(x)$ in $\Omega$, find a function $u(x)$ in $\Omega$ such that

$$\begin{cases} -Au + g(u) = h & \text{in } \Omega, \\ Bu = a(x') \dfrac{\partial u}{\partial \nu} + b(x')u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In order to study problem (1.3), we consider the linear elliptic boundary value problem

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

in the framework of the Hilbert space $L^2(\Omega)$. We associate with problem (1.4) a densely defined, closed linear operator

$$\mathfrak{A} : L^2(\Omega) \to L^2(\Omega)$$

as follows:

(1) $\mathscr{D}(\mathfrak{A}) = \{u \in W^{2,2}(\Omega) : Bu = 0 \text{ on } \partial\Omega\}$.
(2) $\mathfrak{A}u = Au$ for every $u \in \mathscr{D}(\mathfrak{A})$.

Here and in the following $W^{k,p}(\Omega)$ denotes the usual Sobolev space for $k \in \mathbf{N}$ and $1 < p < \infty$.

Then we have the following fundamental spectral results (i), (ii), (iii) and (iv) of the operator $\mathfrak{A}$ (see [25, Theorem 5.1]):

(i) The operator $\mathfrak{A}$ is positive and selfadjoint in $L^2(\Omega)$.
(ii) Let $\lambda_j$ be the eigenvalues of the operator $\mathfrak{A}$ that are arranged in an increasing sequence

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \ldots,$$

each eigenvalue being repeated according to its multiplicity. The first eigenvalue $\lambda_1$ is *positive* and *algebraically simple*, and its corresponding eigenfunction $\varphi_1 \in C^\infty(\overline{\Omega})$ may be chosen to be *strictly positive* in $\Omega$.
(iii) *No* other eigenvalues $\lambda_j$, $j \geq 2$, have positive eigenfunctions.
(iv) The family $\{\varphi_j\}_{j=1}^\infty$ of eigenfunctions of $\mathfrak{A}$ forms a *complete* orthonormal system of $L^2(\Omega)$.

REMARK 1.1. If the boundary condition $B$ is equal to the *purely Neumann condition*, then the first eigenvalue $\lambda_1$ is equal to zero. This is the reason why we study the semilinear elliptic boundary value problem (1.3) under condition (H.2).

In this paper we consider problem (1.3) under the assumption that the range of $g'(t)$ contains eigenvalues $\lambda_j$ of $\mathfrak{A}$, and prove non-uniqueness results for problem (1.3).

First, the next existence theorem is a generalization of Castro–Lazer [10, Theorem A] to the degenerate case:

THEOREM 1.1. *Assume that* $g \in C^1(\mathbf{R})$ *with* $g(0) = 0$ *and that* $g'(t)$ *is bounded on* $\mathbf{R}$. *Then we have the following two assertions* (I) *and* (II):

(I) *If there exist an integer* $J \in \mathbf{N}$ *and constants* $\gamma > 0$, $\gamma' > 0$ *such that*

$$\begin{cases} \lambda_J < \gamma < \gamma' < \lambda_{J+1}, \\ g'(t) \leq \gamma' \quad \textit{for all } t \in \mathbf{R} \end{cases} \tag{A}$$

*and that*

$$\inf_{t \in \mathbf{R}} \left[ \int_0^t g(s)\, ds - \frac{\gamma t^2}{2} \right] > -\infty, \tag{B}$$

*and if the condition*

$$g'(0) < \lambda_J \tag{C}$$

*is satisfied, then the homogeneous problem*

$$\begin{cases} -Au + g(u) = 0 & \textit{in } \Omega, \\ Bu = 0 & \textit{on } \partial\Omega \end{cases} \tag{1.5}$$

*has at least* two *solutions—one trivial solution and at least one non-trivial solution* $u \in C^{2+\alpha}(\overline{\Omega})$ *with exponent* $0 < \alpha < 1$.

(II) *Let* $h \in C^{\alpha}(\overline{\Omega})$ *with exponent* $0 < \alpha < 1$. *If, in addition to condition* (C), *the function* $g(t)$ *satisfies the condition*

$$g'(0) \neq \lambda_j \quad \textit{for all } j = 1, 2, \ldots, \tag{D}$$

*then the non-homogeneous problem* (1.3)

$$\begin{cases} -Au + g(u) = h & \textit{in } \Omega, \\ Bu = 0 & \textit{on } \partial\Omega \end{cases}$$

*has at least* three *solutions* $u_1, u_2, u_3 \in C^{2+\alpha}(\overline{\Omega})$ *provided that* $\|h\|_{L^2(\Omega)}$ *is sufficiently small. In particular*, *the homogeneous problem* (1.5) *has one trivial solution and at least two non-trivial solutions.*

Example 1.1. A simple example of the nonlinear term $g(t)$ is given by the formula

$$g(t) = \begin{cases} \frac{\lambda_1+\lambda_2}{2}\left(t + \frac{1}{2t} - \frac{5}{4}\right) & \text{for } t > 1, \\ \left(\frac{\lambda_1+\lambda_2}{8}\right)t^2 & \text{for } 0 \leq t \leq 1, \\ -\left(\frac{\lambda_1+\lambda_2}{8}\right)t^2 & \text{for } -1 \leq t \leq 0, \\ \frac{\lambda_1+\lambda_2}{2}\left(t + \frac{1}{2t} + \frac{5}{4}\right) & \text{for } t < -1. \end{cases}$$

It is easy to verify that this function $g(t)$ satisfies conditions (A), (B), (C) and (D) for $J = 1$:

$$\gamma' = \frac{\lambda_1 + \lambda_2}{2}, \quad \gamma = \frac{3\lambda_1 + \lambda_2}{4},$$

$$g'(0) = 0 < \lambda_1 < g'(\pm\infty) = \frac{\lambda_1 + \lambda_2}{2} < \lambda_2.$$

The next corollary is a simplified version of Theorem 1.1 with $J := n + k$:

Corollary 1.2. *Let* $h \in C^{\alpha}(\overline{\Omega})$ *with exponent* $0 < \alpha < 1$. *Assume that* $g \in C^1(\mathbf{R})$ *with* $g(0) = 0$ *and that* $g'(t)$ *is bounded on* $\mathbf{R}$. *If the finite limits* $g'(\pm\infty) = \lim_{t\to\pm\infty} g'(t)$ *exist and if there exist two positive integers* $n$ *and* $k$ *such that*

$$\lambda_n < g'(0) < \lambda_{n+1} \leq \cdots \leq \lambda_{n+k} < g'(\pm\infty) < \lambda_{n+k+1}, \tag{E}$$

*then the non-homogeneous problem* (1.3) *has at least* three *solutions* $u_1, u_2, u_3 \in C^{2+\alpha}(\overline{\Omega})$ *provided that* $\|h\|_{L^2(\Omega)}$ *is sufficiently small.*

Rephrased, Corollary 1.2 asserts that the non-homogeneous problem (1.3) has at least three solutions provided that $g'(t)$ crosses eigenvalues $\lambda_j$ of $\mathfrak{A}$ if $|t|$ goes from 0 to $\infty$.

Remark 1.2. Ambrosetti–Prodi [6] considered the case where the range of $g'(t)$ contains only the first eigenvalue $\mu_1$ of the Dirichlet problem, and studied the non-homogeneous problem (1.3) in the framework of singularity theory in Banach spaces ([22, Chapter 6]). They characterized completely the solution structure of the non-homogeneous problem (1.3) ([6, Theorem 3.1], [7, Chapter 4, Theorem 2.4], [8, Theorem 3]). Their result is generalized to the degenerate case by Taira ([26, Theorem 1.1]).

With stronger assumptions on $g(t)$, we can give the exact number of solutions. In fact, the next existence theorem is a generalization of Castro–Lazer [10, Theorem B] to the degenerate case (see also [5, Theorem 1.2]):

THEOREM 1.3. *Let* $h \in C^{\alpha}(\overline{\Omega})$ *with exponent* $0 < \alpha < 1$. *Assume that* $g \in C^2(\mathbf{R})$ *with* $g(0) = 0$ *and that*

$$tg''(t) > 0 \quad \textit{for all } t \neq 0. \tag{1.6}$$

*If the finite limits* $g'(\pm\infty) = \lim_{t\to\pm\infty} g'(t) = \lim_{t\to\pm\infty} g(t)/t$ *exist and if there exists a positive integer* $J$ *such that*

$$\lambda_{J-1} < g'(0) < \lambda_J < g'(\pm\infty) < \lambda_{J+1}, \tag{F}$$

*then there exists a constant* $r > 0$ *such that the non-homogeneous problem* (1.3) *has exactly* three *solutions* $u_1, u_2, u_3 \in C^{2+\alpha}(\overline{\Omega})$ *provided that* $\|h\|_{L^2(\Omega)}$ *is smaller than* $r$. *In particular, the homogeneous problem* (1.5) *has one trivial solution and exactly two non-trivial solutions.*

EXAMPLE 1.2. A simple example of the nonlinear term $g(t)$ is given by the formula

$$g(t) = \begin{cases} \frac{\lambda_1+\lambda_2}{2}\left(t + \frac{1}{2t} - \frac{4}{3}\right) & \text{for } t > 1, \\ \left(\frac{\lambda_1+\lambda_2}{12}\right)t^3 & \text{for } -1 \le t \le 1, \\ \frac{\lambda_1+\lambda_2}{2}\left(t + \frac{1}{2t} + \frac{4}{3}\right) & \text{for } t < -1. \end{cases}$$

It is easy to verify that this function $g(t)$ satisfies condition (F) for $J = 1$:

$$g'(0) = 0 < \lambda_1 < g'(\pm\infty) = \frac{\lambda_1 + \lambda_2}{2} < \lambda_2.$$

If the nonlinear term $g(t)$ is an odd function of $t$, then we can improve assertion (I) of Theorem 1.1. The next existence theorem is a generalization of Castro–Lazer [10, Theorem C] to the degenerate case (see also [16, Theorem 2]; [32, Theorem 1]):

Theorem 1.4. *Let $g(t)$ be a function as in assertion* (I) *of Theorem* 1.1. *Moreover, if $g(t)$ is an* odd *function of $t$ and if $K$ is a positive integer such that $K \leq J$ and*

$$\lambda_{K-1} < g'(0) < \lambda_K \leq \lambda_J, \tag{G}$$

*then the homogeneous problem* (1.5) *has at least* $2(J-K+1)$ *non-trivial solutions in* $C^{2+\alpha}(\overline{\Omega})$ *with exponent* $0 < \alpha < 1$.

Example 1.3. A simple example of the nonlinear term $g(t)$ is given by the formula

$$g(t) = \begin{cases} \frac{\lambda_J+\lambda_{J+1}}{2}\left(t+\frac{1}{2t}-\frac{5}{4}\right) & \text{for } t > 1, \\ \left(\frac{\lambda_J+\lambda_{J+1}}{8}\right)t^2 & \text{for } 0 \leq t \leq 1, \\ -\left(\frac{\lambda_J+\lambda_{J+1}}{8}\right)t^2 & \text{for } -1 \leq t \leq 0, \\ \frac{\lambda_J+\lambda_{J+1}}{2}\left(t+\frac{1}{2t}+\frac{5}{4}\right) & \text{for } t < -1. \end{cases}$$

It is easy to verify that this function $g(t)$ satisfies conditions (A), (B), (C) and (G) for $K = 1$:

$$\gamma' = \frac{\lambda_J + \lambda_{J+1}}{2}, \quad \gamma = \frac{3\lambda_J + \lambda_{J+1}}{4},$$
$$g'(0) = 0 < \lambda_1 < \lambda_J.$$

The next corollary is a simplified version of Theorem 1.4 with $J := n+k$ and $K := n+1$:

Corollary 1.5. *Assume that $g \in C^1(\mathbf{R})$ is an* odd *function of $t$ with $g(0) = 0$ and that $g'(t)$ is bounded on $\mathbf{R}$. If the finite limits $g'(\pm\infty) = \lim_{t\to\pm\infty} g'(t)$ exist and if condition* (E) *is satisfied, then the homogeneous problem* (1.5) *has at least* $2k$ *non-trivial solutions in* $C^{2+\alpha}(\overline{\Omega})$.

Our method of proving Theorems 1.1, 1.3 and 1.4 consists of reducing a certain infinite dimensional problem to a finite dimensional problem and then applying finite dimensional critical point theory as in Castro–Lazer [10]. The approach here is based on the extensive use of the ideas and techniques characteristic of the recent developments in the theory of semilinear elliptic boundary value problems with degenerate boundary conditions ([26]–[31]).

The rest of this paper is organized as follows. In Section 2 we discuss some preliminary material such as differential calculus in Banach spaces, Brouwer degree, the index theorem (Theorem 2.4) and the three-solution theorem (Theorem 2.5) in finite dimensional critical point theory which will be used throughout the paper. In Section 3 we introduce the notion of weak solutions of problem (1.3), and prove that any weak solutions of problem (1.3) is a classical solution in the usual sense. This section is the heart of the subject. In Subsection 3.1 we introduce an underlying Hilbert space $\mathscr{H}$ for the study of problem (1.3) (Theorems 3.1 and 3.2). The crucial point in our variational approach is how to use the theory of fractional powers of analytic semigroups developed in [23]. In Subsection 3.2 we prove that any weak solutions of problem (1.3) is a classical solution (Theorem 3.3). The proof of Theorem 3.3 is essentially based on the regularity, existence and uniqueness theorems for the linear elliptic boundary value problem (1.4) ([24]). Section 4 is devoted to the proof of Theorem 1.1. By virtue of Theorem 3.3, we have only to prove Theorem 1.1 for weak solutions. Subsection 4.1 is devoted to an abstract theorem on Hilbert space functionals (Theorem 4.1) essentially due to Castro–Lazer [10] which will play an important role in the proof of Theorems 1.1, 1.3 and 1.4. In Subsection 4.2 we prove that if conditions (A), (B) and (C) of Theorem 1.1 are satisfied, then the homogeneous problem (1.5) has at least two weak solutions. If we introduce an energy functional $F$ on the Hilbert space $\mathscr{H}$, then we find that the weak solutions of the homogeneous problem (1.5) coincide with the critical points of $F$. We verify all the conditions for assertion (I) of Theorem 4.1 (Proposition 4.2). In Subsection 4.3 we prove that if conditions (B), (C) and (D) of Theorem 1.1 are satisfied, then the non-homogeneous problem (1.3) has at least three weak solutions provided that $\|h\|_{L^2(\Omega)}$ is sufficiently small. First, by using the inverse mapping theorem we construct a weak solution $\phi$ of problem (1.3). Moreover, if we introduce a new energy functional $F_1$ on $\mathscr{H}$, then we find that the weak solutions of the non-homogeneous problem (1.3) coincide with the critical points of $F_1$. We verify all the conditions for assertion (II) of Theorem 4.1 (Proposition 4.4), and construct two weak solutions $\phi + u_0$, $\phi + u_2$ of problem (1.3) different from $\phi$. Section 5 is devoted to the proof of Theorem 1.3. The proof is carried out in a series of several lemmas (Lemmas 5.1 through 5.6). In the proof of Theorem 1.3 we make essential use of the comparison property of eigenvalues of degenerate elliptic boundary value problems with indefinite weights (Lemma 5.3). The last Section 6 is devoted to the proof of Theorem 1.4. Our proof is based on a result of Clark [12] concerning the Ljusternik–Schnirelman theory of critical points (Theorem 6.1). More precisely, we mention that the notion of category introduced by

Ljusternik–Schnirelman [19] is a topological invariant for the estimate of the lower bound of the number of critical points (see [11, Chapter 5, Section 5.2]).

## 2. Preliminaries

In this section we discuss some preliminary material such as differential calculus in Banach spaces, Brouwer degree and finite dimensional critical point theory. The results of this section will be used in the proof of assertion (II) of Theorem 1.1 and in the proof of Theorem 1.3 (Theorems 2.4, 2.5 and 2.6).

### 2.1. Differentiability and the Inverse Mapping Theorem

In this subsection we give an outline of differential calculus in Banach spaces (see [1], [13]; [21]). The next proposition generalizes the usual notion of symmetry of the second partial derivatives of a function $f : \mathbf{R}^n \to \mathbf{R}$:

Proposition 2.1. *Let $X$ and $Y$ be Banach spaces. If $f \in C^2(X, Y)$, then the second derivative $d^2f(x)$ of $f$ at $x \in X$ is* symmetric, *that is, we have the formula*

$$d^2f(x)(u, v) = d^2f(x)(v, u) \quad \text{for all } u, v \in X.$$

The inverse mapping theorem provides a criterion for a map to be a local $C^r$-diffeomorphism in terms of its derivative:

Theorem 2.2 (the inverse mapping theorem). *Let $X$ and $Y$ be Banach spaces, and let $f$ be a $C^r$-map ($r \geq 1$) of an open subset $U$ of $X$ into $Y$. Assume that the derivative $df(x_0) : X \to Y$ is an algebraic and topological* isomorphism *at a point $x_0$ of $U$. Then the map $f$ is a $C^r$-diffeomorphism of some neighborhood of $x_0$ onto some neighborhood of $f(x_0)$.*

The next theorem is one of the most important applications of Theorem 2.2:

Theorem 2.3 (the implicit function theorem). *Let $X$, $Y$, $Z$ be Banach spaces, and let $f$ be a $C^r$-map ($r \geq 1$) of an open subset $U \times V$ of $X \times Y$ into $Z$. Assume that the partial derivative $d_yf(x_0, y_0) : Y \to Z$ is an algebraic and topological* isomorphism *at a point $(x_0, y_0)$ of $U \times V$. Then there exist neighborhoods $U_0$ of $x_0$ and $W_0$ of $f(x_0, y_0)$ and a unique $C^r$ map $g : U_0 \times W_0 \to V$ such that*

$$f(x, g(x, w)) = w \quad \text{for all } (x, w) \in U_0 \times W_0.$$

### 2.2. Functionals and Critical Points

Let $X$ be a real Banach space. A *functional* on $X$ is a continuous, real-valued map $F : X \to \mathbf{R}$. A point $u \in X$ is called a *critical point* of $F$ if $F$ is Fréchet differentiable at $u$ and if $dF(u) = 0$, that is, if we have, for all $v \in X$,

$$dF(u)(v) = 0.$$

Let $H$ be a real Hilbert space with inner product $(\cdot, \cdot)_H$. If $F \in C^1(H, \mathbf{R})$ and $u \in H$, then it follows from an application of the Riesz representation theorem ([33, Chapter III, Section 6, Theorem]) that there exists a unique element $\nabla F(u)$ of $H$ such that

$$dF(u)(v) = (\nabla F(u), v)_H \quad \text{for all } v \in H.$$

The element $\nabla F(u)$ of $H$ is called the *gradient* of $F$ at $u$. We can identify $dF(u)$ with $\nabla F(u)$. It should be noticed that a critical point $u$ of $F$ is a solution of the equation $\nabla F(u) = 0$.

Moreover, if $F \in C^2(H, \mathbf{R})$, we can define the derivative $D^2F(u)$ of $\nabla F$ at $u$ by the formula

$$d^2F(u)(v, w) = (D^2F(u)v, w)_H \quad \text{for all } v, w \in H. \tag{2.1}$$

By virtue of Proposition 2.1, we find that the linear operator $D^2F(u)$ is selfadjoint on $H$.

### 2.3. Brouwer Degree and the Index Theorem

In this subsection we consider the following (see [20]):

(a) $\Omega$ is a bounded open set in $\mathbf{R}^n$ with boundary $\partial\Omega$.
(b) $f = (f_1, \dots, f_n) : \overline{\Omega} \to \mathbf{R}^n$ is a continuous map.
(c) $p$ is a point of $\mathbf{R}^n$ such that $f(x) \neq p$ for all $x \in \partial\Omega$.

For each triplet $(f, \Omega, p)$, we can define an integer-valued function $\deg(f, \Omega, p)$. The integer $\deg(f, \Omega, p)$ is called the *Brouwer degree* of the map $f$ with respect to the set $\Omega$ and the point $p$.

Since the Brouwer degree $\deg(f, \Omega, p)$ enjoys the excision property, we can define the index of an isolated solution of the equation $f(x) = p$ as follows: Let $x_0$ be a point of $\Omega$ such that $f(x_0) = p$. If there exists a constant $r > 0$ such that

$$f(x) \neq p \quad \text{for all } x \in \overline{B_r(x_0)} \setminus \{x_0\},$$

then it follows from an application of the excision property that

$$\deg(f, B_\rho(x_0), p) = \deg(f, B_r(x_0), p) \quad \text{for all } \rho \in (0, r).$$

Thus we can define an integer $i(f, x_0)$ by the formula

$$i(f, x_0) = \lim_{\rho \to 0} \deg(f, B_\rho(x_0), p), \quad p = f(x_0).$$

The integer $i(f, x_0)$ is called the *index* of the map $f$ with respect to the point $x_0$.

The next theorem will play an important role in the proof of Theorem 1.3 in Section 6 (see [21, Theorem 2.8.1]):

THEOREM 2.4 (the index theorem). *Let $f \in C^1(\Omega, \mathbf{R}^n) \cap C(\overline{\Omega}, \mathbf{R}^n)$. If $x_0$ is a point of $\Omega$ such that $J_f(x_0) \neq 0$, then we have the formula*

$$i(f, x_0) = (-1)^\beta, \tag{2.2}$$

*where $J_f(x_0)$ is the Jacobian determinant of $f$ at $x_0$ and $\beta$ is the sum of the algebraic multiplicities of the negative eigenvalues of the derivative $Df(x_0)$.*

### 2.4. Finite Dimensional Critical Point Theory

Let $f \in C^1(\mathbf{R}^n, \mathbf{R})$. If $\bar{x}$ is a point of $\mathbf{R}^n$ such that $\nabla f(\bar{x}) = 0$, then we say that $\bar{x}$ is a *non-degenerate critical point* of $f$ if the Hessian matrix $D^2 f(\bar{x})$ of $f$ at $\bar{x}$ is non-singular.

The next theorem will play an important role in the proof of Theorem 4.1 (see [10, Theorem 3]):

THEOREM 2.5 (the three-solution theorem). *Let $f \in C^2(\mathbf{R}^n, \mathbf{R})$. Assume that the following three conditions* (i), (ii) *and* (iii) *are satisfied*:

(i) $f(x) \to \infty$ *as* $\|x\| \to \infty$.
(ii) *There exists a point $x_0$ of $\mathbf{R}^n$ such that $f(x_0) = \min_{x \in \mathbf{R}^n} f(x)$.*
(iii) *There exists a non-degenerate critical point $x_1$ of $f$ such that $x_1 \neq x_0$.*

*Then the map $f$ has at least* three *distinct critical points.*

The next theorem will play an important role in the proof of Theorem 1.3 in Section 6 (see [4, Corollary 1]):

THEOREM 2.6. *Let $f \in C^1(\mathbf{R}^n, \mathbf{R})$. If $f(x) \to +\infty$ as $\|x\| \to \infty$ and if the set of solutions of $\nabla f(x) = 0$ is a* finite *set $\{x_0, x_1, x_2, \ldots, x_k\}$, then we have the formula*

$$\sum_{j=0}^{k} i(\nabla f, x_j) = 1.$$

## 3. Regularity of Weak Solutions

In this section we introduce the notion of weak solutions of problem (1.3), and prove that any weak solutions of problem (1.3) is a classical solution in the usual sense. This section is the heart of the subject. In Subsection 3.1 we introduce an underlying Hilbert space $\mathscr{H}$ for the study of problem (1.3) (Theorems 3.1 and 3.2). The crucial point in our variational approach is how to use the theory of fractional powers of analytic semigroups developed in [23]. In Subsection 3.2 we prove that any weak solutions of problem (1.3) is a classical solution (Theorem 3.3). The proof of Theorem 3.3 is essentially based on the regularity, existence and uniqueness theorems for the linear elliptic boundary value problem (1.4) ([24]).

### 3.1. Hilbert Space $\mathscr{H}$

In this subsection we introduce an underlying Hilbert space $\mathscr{H}$ for the study of problem (1.3). Since the operator $\mathfrak{A}$ is positive and selfadjoint in the Hilbert space $L^2(\Omega)$, we can define its square root

$$\mathscr{C} = \mathfrak{A}^{1/2}$$

as follows ([23]):

$$\mathscr{C}u = \sum_{m=1}^{\infty} \sqrt{\lambda_m}(u, \varphi_m)_{L^2(\Omega)}\varphi_m \quad \text{in } L^2(\Omega). \tag{3.1}$$

Here we recall that the family $\{\varphi_m\}_{m=1}^{\infty}$ of eigenfunctions of $\mathfrak{A}$

$$\begin{cases} A\varphi_m = \lambda_m\varphi_m & \text{in } \Omega, \\ B\varphi_m = 0 & \text{on } \partial\Omega \end{cases}$$

forms a *complete* orthonormal system of $L^2(\Omega)$.

Moreover, we can introduce an underlying Hilbert space $\mathscr{H}$ with inner product $(\cdot,\cdot)_{\mathscr{H}}$ as follows:

$$\mathscr{H} = \text{the domain } D(\mathscr{C}) \text{ with the inner product}$$

$$(u,v)_{\mathscr{H}} = (\mathscr{C}u, \mathscr{C}v)_{L^2(\Omega)} \quad \text{for all } u, v \in D(\mathscr{C}).$$

The next theorem gives a more concrete and useful characterization of the Hilbert space $\mathscr{H}$ (see [26, Theorem 3.1]):

Theorem 3.1. *The Hilbert space* $\mathscr{H}$ *coincides with the* completion *of the domain*

$$D(\mathfrak{A}) = \{u \in W^{2,2}(\Omega) : Bu = 0 \;\; on \;\; \partial\Omega\}$$

*with respect to the inner product*

$$\begin{aligned}(u, v)_{\mathscr{H}} &= (\mathfrak{A}u, v)_{L^2(\Omega)} \\ &= \sum_{i,j=1}^{N} \int_\Omega a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_\Omega c(x) u \cdot v \, dx \\ &\quad + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} u \cdot v \, d\sigma \quad \text{for all } u, v \in D(\mathfrak{A}). \end{aligned} \tag{3.2}$$

*Here the last term on the right-hand side is an inner product of the Hilbert space* $L^2(\partial\Omega)$ *with respect to the surface measure* $d\sigma$ *of* $\partial\Omega$.

Our approach is based on the following imbedding result for the Hilbert space $\mathscr{H}$ (see [26, Corollary 3.2]):

Theorem 3.2. *We have the inclusions*

$$D(\mathfrak{A}) \subset \mathscr{H} \subset W^{1,2}(\Omega) \tag{3.3}$$

*with continuous injections.*

Remark 3.1. The following diagram gives a bird's eye view of the right Hilbert space $\mathscr{H}$ for the variational approach (see [15, Theorems 1 and 2]):

| $B$ | $\mathscr{H}$ | $a(x')$ and $b(x')$ |
|---|---|---|
| The Dirichlet case | $W_0^{1,2}(\Omega)$ | $a(x') \equiv 0$ and $b(x') \equiv 1$ |
| The Robin case | $W^{1,2}(\Omega)$ | $a(x') \equiv 1$ and $b(x') \not\equiv 0$ |
| The degenerate case | $D(\mathfrak{A}^{1/2})$ | (H.1) and (H.2) |

First, we have, by formula (3.1),

$$(u, u)_{\mathscr{H}} = \sum_{m=1}^{\infty} \lambda_m (u, \varphi_m)^2_{L^2(\Omega)}. \tag{3.4}$$

Indeed, it suffices to note the following:

$$\begin{aligned}
(u,u)_{\mathscr{H}} &= (\mathscr{C}u, \mathscr{C}u)_{L^2(\Omega)} \\
&= \left( \sum_{m=1}^{\infty} \sqrt{\lambda_m} (u, \varphi_m)_{L^2(\Omega)} \varphi_m, \sum_{\ell=1}^{\infty} \sqrt{\lambda_\ell} (u, \varphi_\ell)_{L^2(\Omega)} \varphi_\ell \right)_{L^2(\Omega)} \\
&= \sum_{m=1}^{\infty} \lambda_m (u, \varphi_m)^2_{L^2(\Omega)}. \qquad (3.5)
\end{aligned}$$

Secondly, since we have the Fourier series expansion formula

$$u = \sum_{m=1}^{\infty} (u, \varphi_m)_{L^2(\Omega)} \varphi_m \quad \text{in } L^2(\Omega),$$

it follows that

$$\begin{aligned}
(u,u)_{L^2(\Omega)} &= \left( \sum_{m=1}^{\infty} (u, \varphi_m)_{L^2(\Omega)} \varphi_m, \sum_{\ell=1}^{\infty} (u, \varphi_\ell)_{L^2(\Omega)} \varphi_\ell \right)_{L^2(\Omega)} \\
&= \sum_{m=1}^{\infty} (u, \varphi_m)^2_{L^2(\Omega)}. \qquad (3.6)
\end{aligned}$$

Thirdly, we have, by formulas (3.5) and (3.6),

$$(u,u)_{L^2(\Omega)} \leq \frac{1}{\lambda_1} (u,u)_{\mathscr{H}}. \qquad (3.7)$$

If $J$ is the positive integer as in Theorem 1.1, we let

$$X = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_J\},$$

and

$$Y = X^\perp = \{v \in \mathscr{H} : (v,u)_{\mathscr{H}} = 0 \text{ for all } u \in X\}.$$

In other words, $X^\perp$ is the set of all those elements of $\mathscr{H}$ which are orthogonal to every element of $X$.

From formulas (3.4) and (3.6), we obtain the inequality

$$(v,v)_{\mathscr{H}} \geq \lambda_{J+1} (v,v)_{L^2(\Omega)} \quad \text{for all } v \in Y. \qquad (3.8)$$

Indeed, it follows that

$$
\begin{aligned}
(v, v)_{\mathscr{H}} &= \sum_{m=1}^{\infty} \lambda_m (v, \varphi_m)_{L^2(\Omega)}^2 = \sum_{m=J+1}^{\infty} \lambda_m (v, \varphi_m)_{L^2(\Omega)}^2 \\
&\geq \lambda_{J+1} \sum_{m=J+1}^{\infty} (v, \varphi_m)_{L^2(\Omega)}^2 = \lambda_{J+1} \sum_{m=1}^{\infty} (v, \varphi_m)_{L^2(\Omega)}^2 \\
&= \lambda_{J+1} (v, v)_{L^2(\Omega)} \quad \text{for all } v \in Y.
\end{aligned}
$$

Similarly, we have the inequality

$$
(u, u)_{\mathscr{H}} \leq \lambda_J (u, u)_{L^2(\Omega)} \quad \text{for all } u \in X. \tag{3.9}
$$

Indeed, it follows that

$$
\begin{aligned}
(u, u)_{\mathscr{H}} &= \sum_{m=1}^{\infty} \lambda_m (u, \varphi_m)_{L^2(\Omega)}^2 = \sum_{m=1}^{J} \lambda_m (u, \varphi_m)_{L^2(\Omega)}^2 \\
&\leq \lambda_J \sum_{m=1}^{J} (u, \varphi_m)_{L^2(\Omega)}^2 = \lambda_J \sum_{m=1}^{\infty} (u, \varphi_m)_{L^2(\Omega)}^2 \\
&= \lambda_J (u, u)_{L^2(\Omega)}^2 \quad \text{for all } u \in X.
\end{aligned}
$$

### 3.2. Weak Solutions of Problem (1.3)

In this subsection we prove that any weak solutions of problem (1.3) is a classical solution. The proof of Theorem 3.3 is essentially based on the regularity, existence and uniqueness theorems for the linear elliptic boundary value problem (1.4) ([24]).

A function $u \in \mathscr{H}$ is called a *weak solution* of problem (1.3) if it satisfies the condition

$$
\begin{aligned}
&(u, w)_{\mathscr{H}} - \int_{\Omega} g(u) w \, dx + \int_{\Omega} h \cdot w \, dx \\
&\quad = \sum_{i,j=1}^{J} \int_{\Omega} a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx + \int_{\Omega} c(x) u \cdot w \, dx \\
&\qquad + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} u \cdot w \, d\sigma - \int_{\Omega} g(u) w \, dx + \int_{\Omega} h \cdot w \, dx \\
&\quad = 0 \quad \text{for all } w \in \mathscr{H}.
\end{aligned} \tag{3.10}
$$

The next theorem asserts that any weak solution $u$ of problem (1.3) is a classical solution:

Theorem 3.3. *Let $g(t)$ be a function in $C^1(\mathbf{R})$ such that the derivative $g'(t)$ is bounded on $\mathbf{R}$, and let $h \in C^\alpha(\overline{\Omega})$ with exponent $0 < \alpha < 1$. If $u \in \mathscr{H}$ is a weak solution of problem* (1.3), *then it follows that*

$$u \in C^{2+\alpha}(\overline{\Omega})$$

*with exponent $0 < \alpha < 1$. In particular, $u$ is a* classical solution.

Proof. The proof of Theorem 3.3 is based on the regularity theorem and the existence and uniqueness theorem for the linear elliptic boundary value problem (1.4) ([24, Theorem 8.2 and Theorem 9.1]). We make use of a standard "bootstrap argument".

Assume that a function $u \in \mathscr{H}$ satisfies condition (3.10). Then we have, for all $w \in D(\mathfrak{A}) \subset D(\mathfrak{A}^{1/2}) = \mathscr{H}$,

$$(u, \mathfrak{A}w)_{L^2(\Omega)} = (u, w)_{\mathscr{H}} = (g(u) - h, w)_{L^2(\Omega)}.$$

This proves that

$$\begin{cases} u \in D(\mathfrak{A}), \\ \mathfrak{A}u = g(u) - h, \end{cases}$$

since the operator $\mathfrak{A}$ is selfadjoint in $L^2(\Omega)$. In particular, it follows from assertion (3.3) that

$$u \in W^{1,2}(\Omega) \subset L^2(\Omega).$$

Now we assume that $u \in L^q(\Omega)$ for some $q \geq 2$. Since $g'(t)$ is bounded and $h(x) \in C^\alpha(\overline{\Omega})$, we obtain that

$$f(x) := g(u(x)) - h(x) \in L^q(\Omega).$$

Therefore, since $u$ is a weak solution of the linear boundary value problem

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

if follows from an application of the regularity theorem ([24, Theorem 8.2]) that

$$u \in W^{2,q}(\Omega).$$

(a) If $2q \geq N$, then it follows from the Sobolev imbedding theorem (see [2, Theorem 4.12, Part I]) that

$$u \in L^r(\Omega) \quad \text{for all } r \geq 1.$$

(b) If $2q < N$, then it follows that

$$u \in L^r(\Omega) \quad \text{for } r = \frac{Nq}{N-2q} > q.$$

Repeating this procedure, we have, after a finite number of steps,

$$u \in W^{2,r}(\Omega) \quad \text{for } r \text{ so large that } \frac{N}{r} < 1 - \alpha,$$

so that

$$u \in W^{2,r}(\Omega) \subset C^{1+\beta}(\overline{\Omega})$$

with exponent

$$\beta = 1 - \frac{N}{r} > \alpha.$$

Since $g'(t)$ is continuous and bounded on $\mathbf{R}$, it follows that

$$f(x) = g(u(x)) - h(x) \in C^{\alpha}(\overline{\Omega}).$$

Therefore, by applying the existence and uniqueness theorem ([24, Theorem 9.1]) we can find a unique classical solution $v \in C^{2+\alpha}(\overline{\Omega})$ of the boundary value problem

$$\begin{cases} Av = f & \text{in } \Omega, \\ Bv = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Since $u$ and $v$ are both solutions of problem (3.11) in $W^{2,r}(\Omega)$, by applying the uniqueness theorem ([24, Theorem 8.6]) we obtain that

$$u = v \in C^{2+\alpha}(\overline{\Omega}).$$

Summing up, we have proved that any weak solution $u$ of problem (1.3) is a classical solution.

The proof of Theorem 3.3 is complete. □

## 4. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. By virtue of Theorem 3.3, we have only to prove Theorem 1.1 for weak solutions. Subsection 4.1 is devoted to an abstract theorem on Hilbert space functionals (Theorem 4.1) essentially due to Castro–Lazer [10] which will play an important role in the proof of Theorems 1.1, 1.3 and 1.4. In Subsection 4.2 we prove that if conditions (A), (B) and (C) of Theorem 1.1 are satisfied, then the homogeneous problem (1.5) has at least two weak solutions. If we introduce an energy functional $F$ on the Hilbert space $\mathscr{H}$, then we find that the weak solutions of the homogeneous problem (1.5) coincide with the critical points of $F$. We verify all the conditions for assertion (I) of Theorem 4.1 (Proposition 4.2). In Subsection 4.3 we prove that if conditions (B), (C) and (D) of Theorem 1.1 are satisfied, then the non-homogeneous problem (1.3) has at least three weak solutions provided that $\|h\|_{L^2(\Omega)}$ is sufficiently small. First, by using the inverse mapping theorem (Theorem 2.2) we construct a weak solution $\phi$ of problem (1.3) (Lemma 4.3). Moreover, if we introduce a new energy functional $F_1$ on the Hilbert space $\mathscr{H}$, then we find that the weak solutions of the non-homogeneous problem (1.3) coincide with the critical points of $F_1$. We verify all the conditions for assertion (II) of Theorem 4.1 (Proposition 4.4), and construct two weak solutions $\phi + u_0$, $\phi + u_2$ of problem (1.3) different from $\phi$.

### 4.1. An Abstract Theorem on Hilbert Space Functionals

Let $H$ be a real Hilbert space. If $F \in C^2(H, \mathbf{R})$, then, by using the Riesz representation theorem ([33, Chapter III, Section 6, Theorem]) we can define a $C^1$ map

$$\begin{aligned} \nabla F : H &\to H \\ u &\mapsto \nabla F(u) \end{aligned}$$

by the formula

$$dF(u)(w) = \frac{d}{dt} F(u + tw)|_{t=0} = (\nabla F(u), w)_H \quad \text{for all } w \in H.$$

The element $\nabla F(u)$ of $H$ is the gradient of $F$ at $u \in H$.

Moreover, the derivative $D^2 F(u)$ of $\nabla F$ at $u \in H$ can be defined by the formula

$$\begin{aligned} d^2 F(u)(v, w) &= \frac{d}{dt}(dF(u + tv)(w))|_{t=0} = \frac{d}{dt}(\nabla F(u + tv), w)_H|_{t=0} \\ &= (D^2 F(u)v, w)_H \quad \text{for all } v, w \in H. \end{aligned}$$

We recall that $D^2 F(u)$ is a selfadjoint operator on $H$.

The next theorem is adapted from Castro–Lazer [10, Theorem 4] (see also [9]; [18]):

THEOREM 4.1 (Castro–Lazer). *Let* $F \in C^2(H, \mathbf{R})$. *We assume that the following two conditions* (a) *and* (b) *are satisfied*:

(a) $\nabla F(0) = 0$ *and there exist closed subspaces* $X_1$ *and* $Y_1$ *of* $H$ *and a constant* $m_1 > 0$ *such that*

(i) $H = X_1 \oplus Y_1$.
(ii) $\dim X_1 < \infty$.
(iii) $(D^2F(0)x, x)_H \leq 0$ *for all* $x \in X_1$.
(iv) $(D^2F(0)y, y)_H \geq m_1\|y\|_H^2$ *for all* $y \in Y_1$.

(b) *There exist closed subspaces* $X$ *and* $Y$ *of* $H$ *and a constant* $m > 0$ *such that*

(v) $H = X \oplus Y$.
(vi) $\dim X_1 < \dim X < \infty$.
(vii) $(F|X)(x) \to -\infty$ *as* $\|x\|_H \to \infty$, *where* $F|X$ *is the restriction of* $F$ *to* $X$.
(viii) $(D^2F(u)y, y)_H \geq m\|y\|_H^2$ *for all* $y \in Y$ *and all* $u \in H$.

*Then we have the following two assertions* (I) *and* (II):

(I) *There exists a non-zero element* $u_0$ *of* $H$ *such that* $\nabla F(u_0) = 0$. *Moreover, we have the formula*

$$F(u_0) = \max_{x \in X} \min_{y \in Y} F(x + y).$$

(II) *If condition* (iii) *is replaced by the condition*

(iii*) $(D^2F(0)x, x)_H < 0$ *if* $x$ *is a non-zero element of* $X_1$,

*then there exists a non-zero element* $u_2$ *with* $u_2 \neq u_0$ *such that* $\nabla F(u_2) = 0$.

We remark that the proof of Theorem 4.1 is based on the three-solution theorem (Theorem 2.5).

### 4.2. Proof of Theorem 1.1, Part I

In this subsection we prove that if conditions (A), (B) and (C) of Theorem 1.1 are satisfied, then the homogeneous problem (1.5) has at least *two* weak solutions. We verify all the conditions of Theorem 4.1. The proof is divided into two steps.

**Step 1**: By condition (C), we can choose a positive integer $K \le J$ such that

$$\lambda_{K-1} \le g'(0) < \lambda_K \le \lambda_J, \tag{4.1}$$

where $\lambda_0 = -\infty$. We let

$$X = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_J\}, \quad Y = X^{\perp},$$

$$X_1 = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_{K-1}\}, \quad Y_1 = X_1^{\perp}.$$

We remark that

$$\dim X_1 = K - 1 \le J - 1 < J = \dim X. \tag{4.2}$$

Now we define an energy functional

$$F : \mathscr{H} \to \mathbf{R}$$

by the formula

$$\begin{aligned} F(u) &= \frac{1}{2}(u,u)_{\mathscr{H}} - \int_\Omega \Gamma(u(x))\,dx \\ &= \frac{1}{2}\sum_{i,j=1}^N \int_\Omega a^{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx + \frac{1}{2}\int_\Omega c(x)u^2\,dx \\ &\quad + \frac{1}{2}\int_{\{a(x')\neq 0\}} \frac{b(x')}{a(x')}u^2\,d\sigma - \int_\Omega \left(\int_0^{u(x)} g(s)\,ds\right) dx \quad \text{for all } u \in \mathscr{H}, \end{aligned} \tag{4.3}$$

where

$$\Gamma(t) = \int_0^t g(s)\,ds.$$

The next claim asserts that $u \in \mathscr{H}$ is a weak solution of the homogeneous problem (1.5) if and only if it is a critical point of the energy functional $F$ (cf. [18]):

CLAIM 4.1. *If* $g \in C^1(\mathbf{R})$ *and* $g'(t)$ *is bounded on* $\mathbf{R}$, *then we have the following two assertions* (i) *and* (ii):

(i) $F \in C^2(\mathscr{H}, \mathbf{R})$.
(ii) *The weak solutions of the homogeneous problem* (1.5) *coincide with the critical points of* $F$.

Proof. (i) First, we recall (assertion (3.3)) that

$$\mathscr{H} \subset W^{1,2}(\Omega)$$

with continuous injection. Moreover, it follows from an application of the Sobolev imbedding theorem (see [2, Theorem 4.12, Part I]) that

$$W^{1,2}(\Omega) \subset \begin{cases} L^{2^*}(\Omega) & \text{for } 2^* = 2N/(N-2) \text{ if } N \geq 3, \\ L^r(\Omega) & \text{for all } r \geq 1 \text{ if } N = 2. \end{cases}$$

Therefore, we have the continuous injections

$$\mathscr{H} \subset W^{1,2}(\Omega) \subset \begin{cases} L^{2^*}(\Omega) & \text{for } 2^* = 2N/(N-2) \text{ if } N \geq 3, \\ L^r(\Omega) & \text{for all } r \geq 1 \text{ if } N = 2. \end{cases} \tag{4.4}$$

By virtue of assertion (4.4), since $g \in C^1(\mathbf{R})$ and $g'(t)$ is bounded on $\mathbf{R}$ we can prove the following formulas (4.5) and (4.6) (see [7, Chapter 1, Theorem 2.9]):

$$\begin{aligned}(\nabla F(u), w)_{\mathscr{H}} &= \frac{d}{dt} F(u + tw)|_{t=0} \\ &= (u, w)_{\mathscr{H}} - \int_\Omega g(u(x)) w \, dx \\ &= \sum_{i,j=1}^N \int_\Omega a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx + \int_\Omega c(x) u \cdot w \, dx \\ &\quad + \int_{\{a(x') \neq 0\}} \frac{b(x')}{a(x')} u \cdot w \, d\sigma - \int_\Omega g(u(x)) w \, dx \quad \text{for all } w \in \mathscr{H},\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}(D^2 F(u) v, w)_{\mathscr{H}} &= \frac{d}{dt} (\nabla F(u + tv), w)_{\mathscr{H}}|_{t=0} \\ &= (v, w)_{\mathscr{H}} - \int_\Omega g'(u(x)) v \cdot w \, dx \quad \text{for all } v, w \in \mathscr{H}.\end{aligned} \tag{4.6}$$

Therefore, we obtain from formulas (4.5) and (4.6) that $F \in C^2(\mathscr{H}, \mathbf{R})$.

(ii) By formula (4.5), we find from formula (3.10) with $h := 0$ that the weak solutions $u$ of the homogeneous problem (1.5) coincide with the critical points of $F$. Indeed, it suffices to note that

$$(u, w)_{\mathscr{H}} - \int_\Omega g(u(x)) w \, dx = 0 \quad \text{for all } w \in \mathscr{H} \Leftrightarrow \nabla F(u) = 0.$$

The proof of Claim 4.1 is complete. □

The next proposition is an essential step in the proof of Theorem 1.1:

Proposition 4.2. *Assume that conditions* (A), (B) *and* (C) *are satisfied. Then the function* $F(u)$ *satisfies all conditions* (i) *through* (viii) *of Theorem* 4.1, *where*

$$X = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_J\}, \quad Y = X^{\perp},$$

$$X_1 = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_{K-1}\}, \quad Y_1 = X_1^{\perp}.$$

Proof. (1) Conditions (i), (ii) and (v) are trivially satisfied.

(2) Condition (viii): We have, by formula (4.5),

$$(D^2F(u)v, w)_{\mathscr{H}} = (v, w)_{\mathscr{H}} - \int_{\Omega} g'(u(x))v \cdot w \, dx \quad \text{for all } v, w \in \mathscr{H}.$$

Thus we obtain from inequality (3.8) and condition (A) of Theorem 1.1 that we have, for all $v \in Y$,

$$\begin{aligned} (D^2F(u)v, v)_{\mathscr{H}} &\geq (v, v)_{\mathscr{H}} - \gamma'(v, v)_{L^2(\Omega)} \\ &\geq \left(1 - \frac{\gamma'}{\lambda_{J+1}}\right)(v, v)_{\mathscr{H}} = m\|v\|_{\mathscr{H}}^2, \end{aligned} \tag{4.7}$$

with

$$m = 1 - \frac{\gamma'}{\lambda_{J+1}} > 0.$$

Hence, condition (viii) of Theorem 4.1 is satisfied.

(3) Condition (vii): If $u \in X$, it follows from condition (B) of Theorem 1.1 that there exists a constant $c_0$ such that

$$\int_0^{u(x)} g(s) \, ds - \frac{\gamma}{2} u(x)^2 \geq c_0 \quad \text{for all } x \in \Omega.$$

Hence we have the inequality

$$F(u) = \frac{1}{2}(u, u)_{\mathscr{H}} - \int_{\Omega} \left( \int_0^{u(x)} g(s) \, ds \right) dx \leq \frac{1}{2}(u, u)_{\mathscr{H}} - \frac{\gamma}{2}(u, u)_{L^2(\Omega)} - c_0|\Omega|,$$

where $|\Omega|$ denotes the volume of $\Omega$. By using inequality (3.9) and condition (A) of Theorem 1.1, we have, for some constant $c$,

$$F(u) \leq \frac{1}{2}(u, u)_{\mathscr{H}} - \frac{\gamma}{2}(u, u)_{L^2(\Omega)} + c \leq \frac{1}{2}\left(1 - \frac{\gamma}{\lambda_J}\right)\|u\|_{\mathscr{H}}^2 + c \quad \text{for all } u \in X,$$

with

$$\frac{1}{2}\left(1 - \frac{\gamma}{\lambda_J}\right) < 0.$$

Therefore, we obtain that the restriction $F|X$ of $F$ to $X$ satisfies condition (vii) of Theorem 4.1.

(4) Conditions (iii) and (iv): From the definitions of $X_1$ and $Y_1$ and formulas (3.4) and (3.6), we have the inequalities

$$(r, r)_{\mathscr{H}} \leq \lambda_{K-1}(r, r)_{L^2(\Omega)} \quad \text{for all } r \in X_1$$

and

$$(s, s)_{\mathscr{H}} \geq \lambda_K(s, s)_{L^2(\Omega)} \quad \text{for all } s \in Y_1 = X_1^{\perp}.$$

Hence, by using condition (4.1) and formula (4.6) we obtain that

$$\begin{aligned}(D^2F(0)r, r)_{\mathscr{H}} &= (r, r)_{\mathscr{H}} - g'(0)(r, r)_{L^2(\Omega)} \leq (r, r)_{\mathscr{H}} - \lambda_{K-1}(r, r)_{L^2(\Omega)} \\ &\leq 0 \quad \text{for all } r \in X_1,\end{aligned}$$

and that

$$\begin{aligned}(D^2F(0)s, s)_{\mathscr{H}} &= (s, s)_{\mathscr{H}} - g'(0)(s, s)_{L^2(\Omega)} \\ &\geq \left(1 - \frac{g'(0)}{\lambda_K}\right)(s, s)_{\mathscr{H}} = m_1\|s\|_{\mathscr{H}}^2 \quad \text{for all } s \in Y_1,\end{aligned}$$

with

$$m_1 = 1 - \frac{g'(0)}{\lambda_K} > 0.$$

Therefore, we find that conditions (iii) and (iv) of Theorem 4.1 are satisfied.

(5) Finally, we have only to note that

$$\dim X_1 = K - 1 \leq J - 1 < J = \dim X.$$

This verifies condition (vi).

The proof of Proposition 4.2 is complete. □

**Step 2**: By applying assertion (I) of Theorem 4.1, we obtain that conditions (A), (B) and (C) of Theorem 1.1 imply the existence of at least two solutions of the homogeneous problem (1.5).

The proof of Theorem 1.1, Part I is complete. □

### 4.3. Proof of Theorem 1.1, Part II

In this subsection we prove that if conditions (B), (C) and (D) of Theorem 1.1 are satisfied, then the non-homogeneous problem (1.3) has at least *three* weak solutions provided that $\|h\|_{L^2(\Omega)}$ is sufficiently small. We verify all the conditions of Theorem 4.1 including condition (iii$^*$). The proof is divided into three steps.

**Step 1**: Now we assume that condition (D) of Theorem 1.1 is satisfied. In this case we obtain from condition (C) that

$$\lambda_{K-1} < g'(0) < \lambda_K. \tag{4.8}$$

First, we construct a weak solution $\phi$ of the non-homogeneous problem (1.3). More precisely, we prove the following:

Lemma 4.3. *There exist constants $r > 0$ and $\delta_1 > 0$ such that if $h \in L^2(\Omega)$ with $\|h\|_{L^2(\Omega)} < \sqrt{\lambda_1}r$, then the non-homogeneous problem* (1.3)

$$\begin{cases} -Au + g(u) = h & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

*has a unique weak solution $\phi \in D(\mathfrak{A})$ such that $\|\phi\|_{\mathscr{H}} < \delta_1$.*

Proof. (1) If we introduce a linear operator $T : \mathscr{H} \to \mathscr{H}$ by the formula

$$T = \mathfrak{A}^{-1}|_{\mathscr{H}} : \mathscr{H} \longrightarrow L^2(\Omega) \xrightarrow{\mathfrak{A}^{-1}} \mathscr{H}, \tag{4.9}$$

then we obtain that $T$ is a *compact* operator. Indeed, it suffices to note the following three assertions:

(a) The injection

$$\mathscr{H} \hookrightarrow W^{1,2}(\Omega)$$

is continuous (see assertion (3.3)).

(b) The injection

$$W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$$

is compact (the Rellich–Kondrachov theorem (see [2, Theorem 6.3])).

(c) The resolvent

$$\mathfrak{A}^{-1} : L^2(\Omega) \to \mathscr{H}$$

is continuous.

Moreover, we have the formula

$$(Tv, w)_{\mathscr{H}} = (v, w)_{L^2(\Omega)} \quad \text{for all } w \in \mathscr{H}. \tag{4.10}$$

Indeed, it follows from formula (4.9) that

$$(Tv, w)_{\mathscr{H}} = (\mathfrak{A}^{-1}v, w)_{\mathscr{H}} = (\mathfrak{A}(\mathfrak{A}^{-1}v), w)_{L^2(\Omega)} = (v, w)_{L^2(\Omega)} \quad \text{for all } w \in \mathscr{H}.$$

(2) Secondly, by combining formulas (4.5) and (4.10) we obtain that

$$\begin{aligned}(\nabla F(u), w)_{\mathscr{H}} &= (u, w)_{\mathscr{H}} - \int_\Omega g(u(x))w\,dx = (u, w)_{\mathscr{H}} - (g(u), w)_{L^2(\Omega)} \\ &= (u, w)_{\mathscr{H}} - (T(g(u)), w)_{\mathscr{H}} = (u - T(g(u)), w)_{\mathscr{H}} \quad \text{for all } w \in \mathscr{H}.\end{aligned}$$

This proves that

$$\nabla F(u) = u - T(g(u)) \quad \text{for all } u \in \mathscr{H}. \tag{4.11}$$

Similarly, we have, by formulas (4.6) and (4.10),

$$D^2F(u) = I - T(g'(u)) \quad \text{for all } u \in \mathscr{H}. \tag{4.12}$$

In particular, we have the formula

$$D^2F(0) = I - g'(0)T. \tag{4.13}$$

(3) Thirdly, we show that if condition (4.8) is satisfied, then the continuous operator

$$D^2F(0) = I - g'(0)T : \mathscr{H} \to \mathscr{H}$$

is bijective. To do this, we have only to show the injectivity of $D^2F(0)$, since formula (4.9) implies that the Fredholm alternative holds true for the operator $D^2F(0)$.

Assume that $v \in \mathscr{H}$ and $D^2F(0)v = 0$. Then it follows from formulas (4.9) and (4.13) that

$$v = g'(0)Tv = g'(0)\mathfrak{A}^{-1}v.$$

This proves that

$$\begin{cases} v \in D(\mathfrak{A}), \\ \mathfrak{A}v = g'(0)v. \end{cases}$$

However, we see from condition (4.8) that $v = 0$, since $g'(0)$ is not an eigenvalue of the operator $\mathfrak{A}$.

(4) Since the Fréchet derivative

$$D^2F(0) : \mathscr{H} \to \mathscr{H}$$

of $\nabla F$ at 0 is bijective and since $\nabla F(0) = 0$, it follows from an application of the inverse mapping theorem (Theorem 2.2) that there exists an open neighborhood $U$ of the origin 0 in $\mathscr{H}$ such that:

(i) The restriction of $\nabla F$ to $U$ is bijective.
(ii) $\nabla F(U)$ is an open neighborhood of 0 in $\mathscr{H}$.
(iii) $\nabla F$ restricted to $U$ has a $C^1$ inverse map.

Without loss of generality, we may assume that

$$U \subset B(0, \delta_1) = \{u \in \mathscr{H} : \|u\|_{\mathscr{H}} < \delta_1\} \quad \text{for some constant } \delta_1 > 0,$$

and that

$$B(0, r) = \{v \in \mathscr{H} : \|v\|_{\mathscr{H}} < r\} \subset \nabla F(U) \quad \text{for some constant } r > 0.$$

(5) We show that if $h \in L^2(\Omega)$ and $\|h\|_{L^2(\Omega)} < \sqrt{\lambda_1}r$, then there exists a unique weak solution $\phi$ of the non-homogeneous problem

$$\begin{cases} -A\phi + g(\phi) = h & \text{in } \Omega, \\ B\phi = 0 & \text{on } \partial\Omega \end{cases}$$

such that $\|\phi\|_{\mathscr{H}} < \delta_1$.

To see this, we note that the linear functional

$$\mathscr{H} \ni w \mapsto -(h, w)_{L^2(\Omega)}$$

represents a continuous linear functional on $\mathscr{H}$. Hence it follows from an application of the Riesz representation theorem ([33, Chapter III, Section 6, Theorem]) that there exists a unique function $v \in \mathscr{H}$ such that

$$-(h, w)_{L^2(\Omega)} = (v, w)_{\mathscr{H}} \quad \text{for all } w \in \mathscr{H}. \tag{4.14}$$

By using the Schwarz inequality and inequality (3.7), we obtain that

$$\begin{aligned} \|v\|_{\mathscr{H}}^2 = (v, v)_{\mathscr{H}} &= |(h, v)_{L^2(\Omega)}| \leq \|h\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq \|h\|_{L^2(\Omega)} \frac{1}{\sqrt{\lambda_1}} \|v\|_{\mathscr{H}} < r\|v\|_{\mathscr{H}}. \end{aligned}$$

This proves that

$$v \in B(0, r) \subset \nabla F(U).$$

Since we can find a unique function $\phi \in U$ such that

$$\nabla F(\phi) = v,$$

we have, by formula (4.14),

$$(\nabla F(\phi), w)_{\mathscr{H}} = (v, w)_{\mathscr{H}} = -(h, w)_{L^2(\Omega)} \quad \text{for all } u \in \mathscr{H}.$$

Therefore, we obtain from formula (4.5) that

$$(\phi, w)_{\mathscr{H}} - (g(\phi), w)_{L^2(\Omega)} = (\nabla F(\phi), w)_{\mathscr{H}} = -(h, w)_{L^2(\Omega)} \quad \text{for all } w \in \mathscr{H}.$$

This proves that $\phi$ is a weak solution of the non-homogeneous problem (1.3). Moreover, since we have, for all $w \in D(\mathfrak{A}) \subset D(\mathfrak{A}^{1/2}) = \mathscr{H}$,

$$(\phi, \mathfrak{A}w)_{L^2(\Omega)} = (\phi, w)_{\mathscr{H}} = (g(\phi) - h, w)_{L^2(\Omega)}$$

and since the operator $\mathfrak{A}$ is selfadjoint in $L^2(\Omega)$, we obtain that

$$\begin{cases} \phi \in D(\mathfrak{A}), \\ \mathfrak{A}\phi = g(\phi) - h. \end{cases}$$

The proof of Lemma 4.3 is now complete. $\square$

**Step 2**: We find two weak solutions $\phi + u_0$, $\phi + u_2$ of the non-homogeneous problem (1.3) different from $\phi$ constructed in Step 1. To do this, we fix $h \in L^2(\Omega)$ and $\phi \in \mathscr{H}$, and introduce a new energy functional

$$F_1 : \mathscr{H} \to \mathbf{R}$$

by the formula

$$\begin{aligned} F_1(u) &= F(u + \phi) + (h, u + \phi)_{L^2(\Omega)} \\ &= \frac{1}{2}(u + \phi, u + \phi)_{\mathscr{H}} - \int_\Omega \Gamma(u + \phi)\, dx + \int_\Omega h \cdot (u + \phi)\, dx \quad \text{for all } u \in \mathscr{H}. \end{aligned}$$

Then we obtain that $u + \phi \in \mathscr{H}$ is a weak solution of the non-homogeneous problem (1.3) if and only if $u$ is a critical point of the energy functional $F_1$. Indeed, since we have, for all $w \in \mathscr{H}$,

$$\begin{aligned} (\nabla F_1(u), w)_{\mathscr{H}} &= \frac{d}{dt} F_1(u + tw)|_{t=0} \\ &= (u + \phi, w)_{\mathscr{H}} - \int_\Omega (g(u + \phi)w - h \cdot w)\, dx \\ &= (u + \phi, w)_{\mathscr{H}} - (g(u + \phi) - h, w)_{L^2(\Omega)}, \end{aligned} \tag{4.15}$$

it follows that

$$(u+\phi, w)_{\mathscr{H}} - \int_{\Omega} (g(u+\phi) - h) w \, dx = 0 \quad \text{for all } w \in \mathscr{H}$$

$$\Leftrightarrow \nabla F_1(u) = 0.$$

In this case, we have the assertions

$$\begin{cases} u+\phi \in D(\mathfrak{A}), \\ \mathfrak{A}(u+\phi) = g(u+\phi) - h. \end{cases}$$

Now we show that $F_1(u)$ satisfies all the conditions of Theorem 4.1, with condition (iii) replaced by condition (iii$^*$). The next proposition is an essential step in the proof of Theorem 1.1:

Proposition 4.4. *Assume that conditions* (B), (C) *and* (D) *are satisfied. Then the function $F_1(u)$ satisfies all conditions* (i) *through* (viii) *and* (iii$^*$) *of Theorem* 4.1, *where*

$$X = \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_J\}, \quad Y = X^{\perp},$$

$$X_1 = \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_{K-1}\}, \quad Y_1 = X_1^{\perp}.$$

Proof. (1) First, conditions (i), (ii) and (v) are trivially satisfied.

(2) Now we recall the following two inequalities

$$\begin{aligned}(D^2F(0)s, s)_{\mathscr{H}} &= (s, s)_{\mathscr{H}} - g'(0)(s, s)_{L^2(\Omega)} \\ &\geq \left(1 - \frac{g'(0)}{\lambda_K}\right)(s, s)_{\mathscr{H}} = m_1 \|s\|_{\mathscr{H}}^2 \quad \text{for all } s \in Y_1 \end{aligned} \tag{4.16}$$

and

$$\begin{aligned}(D^2F(0)r, r)_{\mathscr{H}} &= (r, r)_{\mathscr{H}} - g'(0)(r, r)_{L^2(\Omega)} \\ &\leq \left(1 - \frac{g'(0)}{\lambda_{K-1}}\right)(r, r)_{\mathscr{H}} = -m_2 \|r\|_{\mathscr{H}}^2 \quad \text{for all } r \in X_1. \end{aligned} \tag{4.17}$$

Here it follows from condition (4.8) that

$$m_1 = 1 - \frac{g'(0)}{\lambda_K} > 0, \quad m_2 = \frac{g'(0)}{\lambda_{K-1}} - 1 > 0.$$

Since $D^2F$ is continuous, there exists a constant $\delta_1 > 0$ such that

$$\|D^2F(u) - D^2F(0)\| < \min\left\{\frac{m_1}{2}, \frac{m_2}{2}\right\} \quad \text{if} \ \|u\|_{\mathscr{H}} < \delta_1.$$

Hence, by inequalities (4.17) and (4.16) it follows that if $\|u\|_{\mathscr{H}} < \delta_1$, then we have two inequalities

$$\begin{aligned}(D^2F(u)r, r)_{\mathscr{H}} &= (D^2F(0)r, r)_{\mathscr{H}} + (D^2F(u)r - D^2F(0)r, r)_{\mathscr{H}} \\ &\leq (D^2F(0)r, r)_{\mathscr{H}} + \|D^2F(u) - D^2F(0)\|(r, r)_{\mathscr{H}} \\ &\leq -m_2\|r\|_{\mathscr{H}}^2 + \frac{m_2}{2}\|r\|_{\mathscr{H}}^2 = -\frac{m_2}{2}\|r\|_{\mathscr{H}}^2 \quad \text{for all} \ r \in X_1, \qquad (4.18)\end{aligned}$$

and

$$\begin{aligned}(D^2F(u)s, s)_{\mathscr{H}} &= (D^2F(0)s, s)_{\mathscr{H}} + (D^2F(u)s - D^2F(0)s, s)_{\mathscr{H}} \\ &\geq (D^2F(0)s, s)_{\mathscr{H}} - \|D^2F(u) - D^2F(0)\|(s, s)_{\mathscr{H}} \\ &\geq m_1\|s\|_{\mathscr{H}}^2 - \frac{m_1}{2}\|s\|_{\mathscr{H}}^2 = \frac{m_1}{2}\|s\|_{\mathscr{H}}^2 \quad \text{for all} \ s \in Y_1. \qquad (4.19)\end{aligned}$$

(3) Condition (viii): Since we have, by formula (4.15),

$$\begin{aligned}(D^2F_1(u)v, w)_{\mathscr{H}} &= \frac{d}{dt}(\nabla F_1(u + tv), w)_{\mathscr{H}}|_{t=0} \\ &= (v, w)_{\mathscr{H}} - \int_{\Omega} g'(u + \phi)v \cdot w \, dx \quad \text{for all} \ v, w \in \mathscr{H},\end{aligned}$$

it follows from formula (4.12) that

$$D^2F_1(u) = D^2F(u + \phi) \quad \text{for all} \ u \in \mathscr{H}.$$

Consequently, we obtain from inequality (4.7) with $u := u + \phi$ that

$$(D^2F_1(u)v, v)_{\mathscr{H}} \geq m\|v\|_{\mathscr{H}}^2 \quad \text{for all} \ v \in Y \ \text{and all} \ u \in \mathscr{H}.$$

This verifies condition (viii).

(4) Conditions (iii$^*$) and (iv): Since $\|\phi\|_{\mathscr{H}} < \delta_1$, we see from inequalities (4.18) and (4.19) that

$$(D^2F_1(0)r, r)_{\mathscr{H}} = (D^2F(\phi)r, r)_{\mathscr{H}} \leq -\frac{m_2}{2}\|r\|_{\mathscr{H}}^2 \quad \text{for all} \ r \in X_1,$$

and that

$$(D^2F_1(0)s,s)_{\mathscr{H}} = (D^2F(\phi)s,s)_{\mathscr{H}} \geq \frac{m_1}{2}\|s\|_{\mathscr{H}}^2 \quad \text{for all } s \in Y_1.$$

Hence we find that conditions (iii$^*$) and (iv) are satisfied.

(5) Condition (vii): Let $u$ be an arbitrary element of $X$. By inequality (3.9) and condition (B) of Theorem 1.1, we have, for some constants $c$ and $c'$,

$$\begin{aligned}
F_1(u) &= \frac{1}{2}(u+\phi, u+\phi)_{\mathscr{H}} - \int_\Omega \Gamma(u+\phi)\,dx + (h, u+\phi)_{L^2(\Omega)} \\
&\leq \frac{1}{2}\|u+\phi\|_{\mathscr{H}}^2 - \frac{\gamma}{2}\|u+\phi\|_{L^2(\Omega)}^2 + c + \|h\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \\
&\leq \frac{1}{2}[\|u\|_{\mathscr{H}}^2 - \gamma\|u\|_{L^2(\Omega)}^2] + \frac{1}{2}\|\phi\|_{\mathscr{H}}^2 - \frac{\gamma}{2}\|\phi\|_{L^2(\Omega)}^2 + \gamma\|u\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \\
&\quad + \|\phi\|_{\mathscr{H}}\|u\|_{\mathscr{H}} + c + \|h\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \\
&\leq \frac{1}{2}\left(1 - \frac{\gamma}{\lambda_J}\right)\|u\|_{\mathscr{H}}^2 + \|\phi\|_{\mathscr{H}}\|u\|_{\mathscr{H}} \\
&\quad + \frac{1}{\sqrt{\lambda_1}}(\|h\|_{L^2(\Omega)} + \gamma\|\phi\|_{L^2(\Omega)})\|u\|_{\mathscr{H}} + c' \quad \text{for all } u \in X,
\end{aligned}$$

with

$$\frac{1}{2}\left(1 - \frac{\gamma}{\lambda_J}\right) < 0.$$

Hence we obtain that the restriction $F_1|X$ of $F_1$ to $X$ satisfies the condition

$$(F_1|X)(u) \to -\infty \quad \text{as } \|u\|_{\mathscr{H}} \to \infty.$$

This verifies condition (vii).

(6) Finally, we have only to note that

$$\dim X_1 = K - 1 \leq J - 1 < J = \dim X.$$

This verifies condition (vi).

The proof of Proposition 4.4 is complete. □

**Step 3**: By Proposition 4.4, we can apply assertion (II) of Theorem 4.1 to obtain two distinct non-trivial functions $u_0$ and $u_2$ such that

$$\nabla F_1(u_k) = \nabla F(\phi + u_k) = 0, \quad k = 0, 2.$$

Summing up, we have proved that the non-homogeneous problem (1.3) has three distinct weak solutions $\phi$, $\phi + u_0$ and $\phi + u_2$.

Now the proof of Theorem 1.1, Part II, and hence that of Theorem 1.1, is complete. □

## 5. Proof of Theorem 1.3

In this section we prove Theorem 1.3 in a series of several lemmas (Lemma 5.1 through Lemma 5.6). By virtue of Theorem 3.3, we have only to prove Theorem 1.3 for weak solutions. In the proof of Theorem 1.3 we make use of the comparison property of eigenvalues of degenerate elliptic boundary value problems with indefinite weights ([24] and [25]). The proof is divided into seven steps.

**Step 1**: Let $H$ be a real Hilbert space and let $F \in C^2(H, \mathbf{R})$. Assume that $F(u)$ satisfies conditions (v), (vii) and (viii) of Theorem 4.1 with $\dim X < \infty$. Then we can define a map $\varphi : X \to Y$ as follows: For a given element $x \in X$, $\varphi(x)$ is the unique element of $Y$ such that

$$(\nabla F(x + \varphi(x)), k)_H = 0 \quad \text{for all } k \in Y, \tag{5.1}$$

and that

$$F(x + \varphi(x)) = \min_{y \in Y} F(x + y). \tag{5.2}$$

By using the implicit function theorem (Theorem 2.3), we obtain from condition (viii) that the map $\varphi$ is of class $C^1$ (see [18, pp. 597–598] for the details). Moreover, we have the following:

Claim 5.1. *If we define a function*

$$G : X \to \mathbf{R}$$

*by the formula*

$$G(x) = F(x + \varphi(x)), \quad x \in X,$$

*then it follows that $G$ is of class $C^2$ on $X$.*

The next lemma is essentially obtained in the proof of Theorem 4.1:

Lemma 5.1. *Assume that $F(u)$ satisfies conditions* (v), (vii) *and* (viii) *of Theorem* 4.1 *with* $\dim X < \infty$. *Then $\nabla F(u) = 0$ for $u \in H$ if and only if $u = x + \varphi(x)$ for some $x \in X$ and $\nabla G(x) = 0$.*

PROOF. (1) The "if" part: Indeed, it follows from the formula

$$\begin{aligned}(\nabla G(x), h)_H &= (\nabla F(x+\varphi(x)), h+\varphi'(x)(h))_H \\ &= (\nabla F(x+\varphi(x)), h)_H \quad \text{for all } h \in X \end{aligned} \tag{5.3}$$

that

$$(\nabla F(u), h)_H = (\nabla F(x+\varphi(x)), h)_H = (\nabla G(x), h)_H = 0 \quad \text{for all } h \in X.$$

On the other hand, we have, by formula (5.1),

$$(\nabla F(u), k)_H = (\nabla F(x+\varphi(x)), k)_H = 0 \quad \text{for all } k \in Y.$$

Therefore, we obtain from condition (v) that

$$(\nabla F(u), v)_H = 0 \quad \text{for all } v \in H = X \oplus Y,$$

so that

$$\nabla F(u) = 0, \quad u = x + \varphi(x).$$

(2) The "only if" part: Assume that

$$\begin{cases} \nabla F(u) = 0, \\ u = x + y \in H = X \oplus Y. \end{cases}$$

Then we find from the proof of Theorem 4.1 that if we have, for all $k \in Y$,

$$(\nabla F(x+y), k)_H = (\nabla F(u), k)_H = 0,$$

then it follows that $y = \varphi(x)$. Hence we have the formula

$$u = x + \varphi(x).$$

Therefore, we obtain from formula (5.3) that

$$(\nabla G(x), h)_H = (\nabla F(x+\varphi(x)), h)_H = (\nabla F(u), h)_H = 0 \quad \text{for all } h \in X.$$

This proves that

$$\nabla G(x) = 0.$$

The proof of Lemma 5.1 is complete. □

**Step 2**: We prove the following:

LEMMA 5.2. *If the function* $g(t)$ *satisfies the conditions of Theorem* 1.3, *then the function* $F(u)$, *defined by formula* (4.3), *satisfies all conditions* (i), (ii), (iii$^*$), (iv) *through* (viii) *of Theorem* 4.1 *where*

$$X = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_J\}, \quad Y = X^{\perp}$$

*and*

$$X_1 = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_{J-1}\}, \quad Y_1 = X_1^{\perp}.$$

PROOF. Assume that the function $g(t)$ satisfies the conditions of Theorem 1.3. Let $\gamma$ and $\gamma'$ be any numbers satisfying the condition

$$\lambda_J < \gamma < \min\{g'(-\infty), g'(\infty)\} \le \max\{g'(-\infty), g'(\infty)\} \le \gamma' < \lambda_{J+1}.$$

Then we can find a constant $t_0 > 0$ such that

$$\gamma \le \frac{g(t)}{t} \le \gamma' \quad \text{for all } |t| \ge t_0.$$

Hence we have the inequality

$$\int_0^t g(s)\, ds - \frac{\gamma t^2}{2} \ge c_0 \quad \text{for all } t \in \mathbf{R},$$

where

$$c_0 = \min_{|t| \le t_0} \left\{ \int_0^t g(s)\, ds - \frac{\gamma t^2}{2} \right\}.$$

This verifies condition (B) of Theorem 1.1.

Since we have, by condition (1.6),

$$t g''(t) > 0 \quad \text{for all } t \ne 0,$$

it follows from condition (F) that

$$\lambda_{J-1} < g'(0) \le g'(t) \le \max\{g'(-\infty), g'(\infty)\} \le \gamma' < \lambda_{J+1}. \tag{5.4}$$

This verifies conditions (C) and (D) of Theorem 1.1.

Therefore, we obtain from Proposition 4.2 and inequality (4.17) that the function $F$ satisfies all the conditions (i), (ii), (iii$^*$), (iv) through (viii) of Theorem 4.1.

The proof of Lemma 5.2 is complete. □

**Step 3**: The next lemma is an essential step in the proof of Theorem 1.3:

Lemma 5.3. *Assume that $\nabla F(u_0) = 0$ for $u_0 \in \mathscr{H}$, and further (by Lemma 5.1) that $u_0 = v_0 + \varphi(v_0)$ with $v_0 \in X$ and $\nabla G(v_0) = 0$. Then it follows that $v_0$ is a non-degenerate critical point of $G$. More precisely, we have the formula*

$$\operatorname{sgn} \det D^2 G(v_0) = \begin{cases} (-1)^J & \text{if } v_0 \neq 0, \\ (-1)^{J-1} & \text{if } v_0 = 0. \end{cases} \tag{5.5}$$

*Here it should be noticed that $u_0 = v_0 + \varphi(v_0) \neq 0$ if and only if $v_0 \neq 0$.*

Proof. The proof of formula (5.5) is based on the index theorem (Theorem 2.4), and is divided into two steps.

**Step 3-1**: We consider the case where $\nabla F(u_0) = 0$ for $u_0 \neq 0$. Then it follows that $u_0$ is a weak solution of the homogeneous problem problem

$$\begin{cases} -Au_0 + g(u_0) = 0 & \text{in } \Omega, \\ Bu_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

as is shown in the proof of Theorem 1.1.

By Theorem 3.3, we remark that $u_0$ is a classical solution of problem (5.6), that is,

$$u_0 \in C^{2+\alpha}(\overline{\Omega}).$$

We introduce a bounded, continuous function $\psi(t)$ defined on $\mathbf{R}$ by the formula

$$\psi(t) = \begin{cases} \frac{g(t)}{t} & \text{if } t \neq 0, \\ g'(0) & \text{if } t = 0. \end{cases}$$

We consider the eigenvalue problem with the weight $\psi(u_0(x))$

$$\begin{cases} Aw = \alpha\psi(u_0(x))w & \text{in } \Omega, \\ Bw = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.7}$$

and the eigenvalue problem with the weight $g'(u_0(x))$

$$\begin{cases} Aw = \beta g'(u_0(x))w & \text{in } \Omega, \\ Bw = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

We let

$$\alpha_1(\psi(u_0)) < \alpha_2(\psi(u_0)) \leq \cdots \leq \alpha_k(\psi(u_0)) \leq \alpha_{k+1}(\psi(u_0)) \leq \cdots,$$

and

$$\beta_1(g'(u_0)) < \beta_2(g'(u_0)) \le \cdots \le \beta_k(g'(u_0)) \le \beta_{k+1}(g'(u_0)) \le \cdots$$

denote the eigenvalues of problems (5.7) and (5.8), respectively, each eigenvalue being repeated according to its multiplicity (see [25, Theorem 1.2]).

Then we have the following comparison property of eigenvalues $\beta_J(g'(u_0))$ and $\beta_{J+1}(g'(u_0))$ where $J$ is the positive integer given in Theorem 1.3:

Claim 5.2. *The eigenvalue problem* (5.8) *does not have* 1 *as eigenvalues. More precisely, we have the inequality*

$$\beta_J(g'(u_0)) < 1 < \beta_{J+1}(g'(u_0)) \quad \textit{for } u_0 \neq 0. \tag{5.9}$$

Proof. First, we have, by inequality (5.4),

$$\lambda_{J-1} < g'(0) \le g'(t) \le \gamma' < \lambda_{J+1} \quad \text{for all } t \in \mathbf{R}.$$

This implies that

$$\lambda_{J-1} < g'(u_0(x)) < \lambda_{J+1} \quad \text{for all } x \in \overline{\Omega}.$$

Hence it follows from an application of the comparison property of eigenvalues ([25, Corollary 3.6]) that

$$1 < \beta_{J+1}(g'(u_0)). \tag{5.10}$$

On the other hand, we have, by problem (5.6),

$$\begin{cases} Au_0 = \dfrac{g(u_0)}{u_0} \cdot u_0 = \psi(u_0)u_0 & \text{in } \Omega, \\ Bu_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

This proves that

$$\alpha_k(\psi(u_0)) = 1 \quad \text{for some } k \ge 1. \tag{5.11}$$

However, since we have, by inequality (5.4),

$$\lambda_{J-1} < \psi(u_0(x)) < \lambda_{J+1} \quad \text{for all } x \in \overline{\Omega},$$

it follows from an application of the comparison property of eigenvalues ([25, Corollary 3.6]) that

$$\alpha_{J-1}(\psi(u_0)) < 1 < \alpha_{J+1}(\psi(u_0)).$$

Hence we obtain from assertion (5.11) that

$$\alpha_J(\psi(u_0)) = 1. \tag{5.12}$$

Moreover, we have, by condition (1.6),

$$\psi(u_0(x)) = \int_0^1 g'(su_0(x))\, ds < g'(u_0(x)) \quad \text{for all } x \in \overline{\Omega},$$

it follows from assertion (5.12) that

$$\beta_J(g'(u_0)) < \alpha_J(\psi(u_0)) = 1. \tag{5.13}$$

Therefore, by combining assertions (5.10) and (5.13) we obtain the desired assertion (5.9) for $u_0 \neq 0$.

The proof of Claim 5.2 is complete. □

Let $\{\theta_k\}_{k=1}^{\infty}$ be a sequence of orthonormal eigenfunctions of problem (5.8). Namely, we have the assertions

$$\begin{cases} A\theta_k = \beta_k(g'(u_0))g'(u_0(x))\theta_k & \text{in } \Omega, \\ B\theta_k = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\int_\Omega g'(u_0(x))\theta_k(x)\theta_j(x)\, dx = \delta_{kj}.$$

If we let

$$V = \operatorname{span}\{\theta_1, \theta_2, \dots, \theta_J\}, \tag{5.14}$$

then it follows from a variational characterization formula of eigenvalues (see [26, Proposition 3.4]) that

$$(v, v)_{\mathscr{H}} \le \beta_J(g'(u_0)) \int_\Omega g'(u_0(x))v^2\, dx \quad \text{for all } v \in V.$$

Therefore, we conclude from formula (4.12) that

$$\begin{aligned} (D^2F(u_0)v, v)_{\mathscr{H}} &= (v, v)_{\mathscr{H}} - \int_\Omega g'(u_0(x))v^2\, dx \\ &\le \left(1 - \frac{1}{\beta_J(g'(u_0))}\right)(v, v)_{\mathscr{H}} \\ &= -\left(\frac{1}{\beta_J(g'(u_0))} - 1\right)\|v\|_{\mathscr{H}}^2 \quad \text{for all } v \in V. \end{aligned} \tag{5.15}$$

In order to apply formula (2.2) with $f := -\nabla G$, we show that all of the eigenvalues of the selfadjoint operator

$$D^2 G(v_0) : X \to X$$

are negative.

Assume, to the contrary, that there exists $h_1 \in X$ such that

$$(D^2 G(v_0)h_1, h_1)_{\mathscr{H}} \geq 0.$$

If we let

$$\bar{m} = h_1 + \varphi'(v_0)(h_1), \tag{5.16}$$

then it follows from the formula (see formula (5.1))

$$\begin{aligned}(D^2 G(x)h, h)_{\mathscr{H}} \\ = (D^2 F(x + \varphi(x))(h + \varphi'(x)(h)), h + \varphi'(x)(h))_{\mathscr{H}} \quad \text{for all } h \in X\end{aligned} \tag{5.17}$$

that

$$(D^2 F(u_0)\bar{m}, \bar{m})_{\mathscr{H}} = (D^2 G(v_0)h_1, h_1)_{\mathscr{H}} \geq 0. \tag{5.18}$$

Moreover, by using the formula

$$\begin{aligned}(D^2 F(x + \varphi(x))(h + \varphi'(x)(h)), k)_{\mathscr{H}} \\ = \frac{d}{dt}(\nabla F(x + th + \varphi(x + th)), k)_{\mathscr{H}}|_{t=0} = 0 \quad \text{for all } k \in Y,\end{aligned} \tag{5.19}$$

we obtain that

$$(D^2 F(u_0)\bar{m}, k)_{\mathscr{H}} = 0 \quad \text{for all } k \in Y. \tag{5.20}$$

We recall from Lemma 5.2 that there exists a constant $m > 0$ such that

$$(D^2 F(u_0)k, k)_{\mathscr{H}} \geq m\|k\|^2_{\mathscr{H}} \quad \text{for all } k \in Y. \tag{5.21}$$

Since $D^2 F(u_0)$ is selfadjoint in $\mathscr{H}$, we obtain from assertion (5.20) and inequalities (5.18) and (5.21) that

$$\begin{aligned}(D^2 F(u_0)(k + \alpha\bar{m}), k + \alpha\bar{m})_{\mathscr{H}} \\ = (D^2 F(u_0)k, k)_{\mathscr{H}} + \alpha^2 (D^2 F(u_0)\bar{m}, \bar{m})_{\mathscr{H}} \geq 0 \quad \text{for all } k \in Y \text{ and } \alpha \in \mathbf{R}.\end{aligned} \tag{5.22}$$

Thus, if $Z$ is a subspace of $\mathscr{H}$ defined by the formula

$$Z = \{z = k + \alpha\bar{m} \in \mathscr{H} : k \in Y, \alpha \in \mathbf{R}\}, \tag{5.23}$$

then we have, by inequality (5.22),

$$(D^2F(u_0)z, z)_{\mathscr{H}} \geq 0 \quad \text{for all } z \in Z. \tag{5.24}$$

Extend $h_1$ to a basis $\{h_1, h_2, \dots, h_J\}$ of $X$ and let

$$\hat{X} = \text{span}\{h_2, \dots, h_J\}.$$

Since $\mathscr{H} = X \oplus Y$, we obtain from formulas (5.16) and (5.23) that

$$\mathscr{H} = \hat{X} \oplus Z.$$

Consequently, it follows that

$$\theta_k = \ell_k + z_k, \quad \ell \in \hat{X},\ z_k \in Z,\ 1 \leq k \leq J.$$

Since $\dim \hat{X} = J - 1$, there exist constants $c_1, \dots, c_J$ such that

$$c_1\ell_1 + \cdots + c_J\ell_J = 0, \quad (c_1, \dots, c_J) \neq (0, \dots, 0).$$

Therefore, we obtain that

$$v = c_1\theta_1 + \cdots + c_J\theta_J = c_1z_1 + \cdots + c_Jz_J \in Z,$$

and from the independence of $\{\theta_1, \dots, \theta_J\}$ that

$$v = c_1\theta_1 + \cdots + c_J\theta_J \neq 0.$$

By inequality (5.24), it follows that

$$(D^2F(u_0)v, v)_{\mathscr{H}} \geq 0.$$

However, we have, by inequalities (5.15) and (5.9),

$$(D^2F(u_0)v, v)_{\mathscr{H}} \leq -\left(\frac{1}{\beta_J(g'(u_0))} - 1\right)\|v\|_{\mathscr{H}}^2 < 0.$$

This contradiction proves that all the eigenvalues of $D^2G(v_0)$ should be negative.

The proof of the first case where $\nabla G(v_0) = 0$ for $v_0 \neq 0$ is complete.

**Step 3-2**: We consider the case where $\nabla G(v_0) = 0$ for $v_0 = 0$.

In order to apply formula (2.2) with $f := -\nabla G$, we show that $D^2G(0)$ has one positive eigenvalue and $(J-1)$ negative eigenvalues. To do this, it suffices to prove the following three assertions:

(i) $D^2G(0)$ is non-singular.

(ii) The quadratic form associated with $D^2G(0)$ cannot be negative definite on all of $X$.

(iii) The quadratic form associated with $D^2G(0)$ cannot be positive definite on any two-dimensional subspace of $X$.

(a) Assertion (i) follows from Lemma 5.2, since condition (iii$^*$) of Theorem 4.1 implies that 0 is a non-degenerate critical point of $G$. Indeed, we have the following:

Claim 5.3. *If condition* (iii$^*$) *is satisfied, then it follows that* 0 *is a* non-degenerate *critical point of* $f(x) = -G(x)$.

Proof. First, we show that the kernel of $D^2F(0)$ is trivial. Assume that

$$D^2F(0)u = 0 \quad \text{for some } u = r + s \text{ with } r \in X_1 \text{ and } s \in Y_1.$$

Then it follows from the selfadjointness of $D^2F(0)$ that

$$\begin{aligned} 0 &= (r - s, D^2F(0)(r + s))_{\mathscr{H}} \\ &= (r, D^2F(0)r)_{\mathscr{H}} + (r, D^2F(0)s)_{\mathscr{H}} - (s, D^2F(0)r)_{\mathscr{H}} - (s, D^2F(0)s)_{\mathscr{H}} \\ &= (r, D^2F(0)r)_{\mathscr{H}} - (s, D^2F(0)s)_{\mathscr{H}}, \end{aligned}$$

so that

$$(r, D^2F(0)r)_{\mathscr{H}} = (s, D^2F(0)s)_{\mathscr{H}}.$$

However, we have, by conditions (iii) and (iv),

$$0 \geq (r, D^2F(0)r)_{\mathscr{H}} = (s, D^2F(0)s)_{\mathscr{H}} \geq m_1 \|s\|_{\mathscr{H}}^2 > 0 \quad \text{if } s \neq 0,$$

and, by conditions (iii$^*$) and (iv),

$$0 \leq m_1 \|s\|_{\mathscr{H}}^2 \leq (s, D^2F(0)s)_{\mathscr{H}} = (r, D^2F(0)r)_{\mathscr{H}} < 0 \quad \text{if } r \neq 0.$$

These contradictions prove that $u = r + s = 0$.

Now we assume that

$$D^2f(0)h_1 = -D^2G(0)h_1 = 0 \quad \text{for some } h_1 \in X.$$

Then it follows from formula (5.3) that we have, for all $h_2 \in X$,

$$\begin{aligned} 0 &= (D^2G(0)h_1, h_2)_{\mathscr{H}} = \frac{d}{dt}(\nabla G(th_1), h_2)_{\mathscr{H}}|_{t=0} \\ &= \frac{d}{dt}(\nabla F(th_1 + \varphi(th_1)), h_2)_{\mathscr{H}}|_{t=0} = (D^2F(0)(h_1 + \varphi'(0)h_1), h_2)_{\mathscr{H}}. \end{aligned} \tag{5.25}$$

On the other hand, it follows from formula (5.19) with $x := 0$ that

$$(D^2F(0)(h_1 + \varphi'(0)h_1), k)_{\mathscr{H}}$$
$$= (D^2F(0 + \varphi(0))(h_1 + \varphi'(0)h_1), k)_{\mathscr{H}} = 0 \quad \text{for all } k \in Y. \tag{5.26}$$

Since $\mathscr{H} = X \oplus Y$, we obtain from formulas (5.25) and (5.26) that

$$(D^2F(0)(h_1 + \varphi'(0)h_1), u)_{\mathscr{H}} = 0 \quad \text{for all } u = h_2 + k \in \mathscr{H}.$$

Hence we have the formula

$$D^2F(0)(h_1 + \varphi'(0)h_1) = 0.$$

However, since the kernel of $D^2F(0)$ is trivial, it follows that

$$h_1 + \varphi'(0)h_1 = 0, \quad h_1 \in X, \quad \varphi'(0)h_1 \in Y,$$

so that

$$h_1 = 0.$$

This proves that the Hessian matrix $D^2f(0)$ of $f$ at 0 is non-singular.

The proof of Claim 5.3 is complete. □

(b) To establish assertion (ii), we assume, to the contrary, that the quadratic form associated with $D^2G(0)$ is negative definite on all of $X$. Namely, we have the inequality

$$(D^2G(0)h, h)_{\mathscr{H}} < 0 \quad \text{for all non-zero elements } h \text{ of } X. \tag{5.27}$$

If we let

$$\hat{W} = \{\hat{w} = h + \varphi'(0)(h) : h \in X\},$$

then it follows that $\dim \hat{W} = \dim X = J$. Moreover, since $\varphi(0) = 0$, we obtain from formula (5.17) and inequality (5.27) that

$$(D^2F(0)\hat{w}, \hat{w})_{\mathscr{H}} = (D^2F(0)(h + \varphi'(0)(h)), h + \varphi'(0)(h))_{\mathscr{H}}$$
$$= (D^2G(0)h, h)_{\mathscr{H}} < 0 \quad \text{for all non-zero elements } \hat{w} \text{ of } \hat{W}.$$

Since $\operatorname{codim} Y_1 = J - 1$, we can find a non-zero element $w_1$ of $\hat{W} \cap Y_1$. Hence we have the inequality

$$(D^2F(0)w_1, w_1)_{\mathscr{H}} < 0.$$

However, we obtain from Lemma 5.2 and condition (iv) of Theorem 4.1 that

$$(D^2F(0)w_1, w_1)_{\mathscr{H}} \geq m_1\|w_1\|^2_{\mathscr{H}} > 0.$$

This contradiction proves that $D^2G(0)$ cannot be negative definite on all of $X$.

(c) To prove assertion (iii), we assume, to the contrary, that there exists a two-dimensional subspace $Q$ of $X$ such that the quadratic form associated with $D^2G(0)$ is positive definite on $Q$. Namely, we have the inequality

$$(D^2G(0)q, q)_{\mathscr{H}} > 0 \quad \text{for all non-zero elements } q \text{ of } Q. \tag{5.28}$$

If $\hat{Q}$ is a subspace of $\mathscr{H}$ defined by the formula

$$\hat{Q} = \{\hat{q} = q + \varphi'(0)(q) : q \in Q\},$$

then we have, by formula (5.17) and inequality (5.28),

$$\begin{aligned}(D^2F(0)\hat{q}, \hat{q})_{\mathscr{H}} &= (D^2F(0)(q + \varphi'(0)(q)), q + \varphi'(0)(q))_{\mathscr{H}} \\ &= (D^2G(0)q, q)_{\mathscr{H}} > 0 \quad \text{for all non-zero elements } \hat{q} \text{ of } \hat{Q}. \end{aligned} \tag{5.29}$$

On the other hand, we have, by formula (5.19),

$$(D^2F(0)\hat{q}, k)_{\mathscr{H}} = (D^2F(0)(q + \varphi'(0)(q)), k)_{\mathscr{H}} = 0 \quad \text{for all } k \in Y.$$

Therefore, we obtain from inequality (5.29) and condition (viii) of Theorem 4.1 that

$$\begin{aligned}&(D^2F(0)(\hat{q} + k), \hat{q} + k)_{\mathscr{H}} \\ &\quad = (D^2F(0)\hat{q}, \hat{q})_{\mathscr{H}} + (D^2F(0)k, k)_{\mathscr{H}} \\ &\quad > 0 \quad \text{for all } \hat{q} \in \hat{Q} \text{ and } k \in Y \text{ with } (\hat{q}, k) \neq (0, 0). \end{aligned} \tag{5.30}$$

This implies that

$$\hat{Q} \cap Y = \{0\}.$$

Moreover, since we have the formula

$$\operatorname{codim}(\hat{Q} \oplus Y) = \operatorname{codim} Y - \dim \hat{Q} = J - 2$$

and $\dim X_1 = J - 1$, we can find a non-zero element $z_1$ of $X_1 \cap (\hat{Q} \oplus Y)$.

Therefore, we obtain from inequality (5.30) with $\hat{q} + k := z_1$ and condition (iii$^*$) of Theorem 4.1 with $x := z_1$ that

$$0 < (D^2F(0)z_1, z_1)_{\mathscr{H}} < 0.$$

This contradiction proves that $D^2G(0)$ cannot have two positive eigenvalues.

Now the proof of Lemma 5.3 is complete. □

**Step 4**: By using the inverse mapping theorem (Theorem 2.2), we prove a local existence and uniqueness theorem for the non-homogeneous problem (1.3):

LEMMA 5.4. *If $u_0$ is a weak solution of problem* (1.5), *then there exist constants $\delta > 0$ and $\delta' > 0$ such that if $h \in L^2(\Omega)$ and $\|h\|_{L^2(\Omega)} < \delta$, then there exists a* unique *weak solution $u$ of problem* (1.3) *with $\|u - u_0\|_{\mathscr{H}} < \delta'$.*

PROOF. As a by-product of the proof of Lemma 5.3, we find that if $u_0$ is any solution of problem (1.5), then it follows from formulas (4.9) and (4.12) that the Fréchet derivative

$$D^2F(u_0) = I - T(g'(u_0)) : \mathscr{H} \to \mathscr{H}$$

of $\nabla F$ at $u_0$ corresponds to the linear eigenvalue problem with the weight $g'(u_0(x))$

$$\begin{cases} Aw = g'(u_0(x))w & \text{in } \Omega, \\ Bw = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.31}$$

However, problem (5.31) has only the trivial solution. Indeed, it suffices to note the following:

(a) If $u_0$ is not identically equal to zero, then 1 is not an eigenvalue of problem (5.31), since we have, by inequality (5.9),

$$\beta_J(g'(u_0)) < 1 < \beta_{J+1}(g'(u_0)).$$

(b) If $u_0$ is the trivial solution, then $g'(u_0(x)) = g'(0)$ is not en eigenvalue of the operator $\mathfrak{A}$, since we have, by condition (F),

$$\lambda_{J-1} < g'(0) < \lambda_J.$$

Hence we obtain from the Fredholm alternative for $D^2F(u_0)$ that $D^2F(u_0)$ is *bijective*. Therefore, it follows from an application of the inverse mapping theorem (Theorem 2.2) that there exists an open neighborhood $U(u_0)$ of $u_0$ such that:

(i) The restriction of $\nabla F$ to $U(u_0)$ is bijective.
(ii) $\nabla F(U(u_0))$ is an open neighborhood of the origin 0.
(iii) $\nabla F$ restricted to $U(u_0)$ has a $C^1$ inverse map.

Without loss of generality, we may assume that

$$U(u_0) \subset B(u_0, \delta') = \{u \in \mathscr{H} : \|u - u_0\|_{\mathscr{H}} < \delta'\} \quad \text{for some constant } \delta' > 0,$$

and that $\delta > 0$ is so small that $\|h\|_{L^2(\Omega)} < \delta$ for all $h \in \nabla F(U(u_0))$.

Summing up, we have proved that if $\|h\|_{L^2(\Omega)} < \delta$, then there exists a unique weak solution $u$ of problem (1.3) such that $\|u - u_0\|_{\mathscr{H}} < \delta'$.

The proof of Lemma 5.4 is complete. □

**Step 5**: The next lemma asserts that if $h \in L^2(\Omega)$ is bounded in $L^2(\Omega)$, then any weak solution $u$ of the non-homogeneous problem (1.3) is bounded in $\mathscr{H}$:

Lemma 5.5. *Given a number $r > 0$, there exists a constant $R(r) > 0$ such that if $h \in L^2(\Omega)$ with $\|h\|_{L^2(\Omega)} \le r$, then any weak solution $u$ of problem* (1.3) *satisfies the condition*

$$\|u\|_{\mathscr{H}} \le R(r).$$

Proof. Let $\gamma$ and $\gamma'$ be constants such that

$$\lambda_J < \gamma' \le \max\{g'(\infty), g'(-\infty)\} \le \gamma < \lambda_{J+1}. \tag{5.32}$$

Then there exists a constant $t_0 > 0$ such that

$$\gamma' \le \frac{g(t)}{t} \le \gamma \quad \text{for all } |t| \ge t_0.$$

We extend the restriction of $g(t)/t$ to $(-\infty, -t_0] \cup [t_0, \infty)$ to a continuous function $\gamma(t)$ on $\mathbf{R} = (-\infty, \infty)$ (for example, linearly between $-t_0$ and $t_0$) such that

$$\lambda_J < \gamma' \le \gamma(t) \le \gamma < \lambda_{J+1} \quad \text{for all } t \in \mathbf{R}. \tag{5.33}$$

Since the function $H(t) = g(t) - \gamma(t)t$ is continuous and has compact support, it is bounded on $\mathbf{R}$. Hence we have the formula

$$g(t) = \gamma(t)t + H(t), \quad |H(t)| \le L, \tag{5.34}$$

with some constant $L > 0$.

Assume that $h \in L^2(\Omega)$ with $\|h\|_{L^2(\Omega)} \le r$. Let $u \in \mathscr{H}$ be any weak solution of problem (1.3). Namely, we have, for all $z \in \mathscr{H}$,

$$(u, z)_{\mathscr{H}} - (g(u) - h, z)_{L^2(\Omega)} = 0. \tag{5.35}$$

If $u = v + w$ with $v \in X$ and $w \in Y = X^{\perp}$, then we let

$$z = w - v \in \mathscr{H}.$$

We remark that

$$\|z\|_{\mathscr{H}}^2 = \|v\|_{\mathscr{H}}^2 + \|w\|_{\mathscr{H}}^2 = \|u\|_{\mathscr{H}}^2.$$

Hence we obtain from formula (5.34) with $t := u$ and formula (5.35) that

$$\begin{aligned}
(w - v, w + v)_{\mathscr{H}} - \int_\Omega \gamma(u)(w^2 - v^2)\, dx &= (z, u)_{\mathscr{H}} - \int_\Omega \gamma(u) u \cdot z\, dx \\
&= (u, z)_{\mathscr{H}} - \int_\Omega (g(u) - H(u)) z\, dx \\
&= \int_\Omega (H(u) z - h \cdot z)\, dx.
\end{aligned}$$

By inequality (5.33), it follows from an application of the Schwarz inequality and inequality (3.7) that

$$\begin{aligned}
&\|w\|_{\mathscr{H}}^2 - \gamma\|w\|_{L^2(\Omega)}^2 + \gamma'\|v\|_{L^2(\Omega)}^2 - \|v\|_{\mathscr{H}}^2 \\
&\quad = \|w\|_{\mathscr{H}}^2 - \|v\|_{\mathscr{H}}^2 - \gamma\|w\|_{L^2(\Omega)}^2 + \gamma'\|v\|_{L^2(\Omega)}^2 \\
&\quad \le (w - v, w + v)_{\mathscr{H}} - \int_\Omega \gamma(u)(w^2 - v^2)\, dx \\
&\quad = \left| \int_\Omega (H(u(x)) z(x) - h(x) z(x))\, dx \right| \\
&\quad \le (L|\Omega|^{1/2} + \|h\|_{L^2(\Omega)}) \|z\|_{L^2(\Omega)} \le (L|\Omega|^{1/2} + r) \frac{1}{\sqrt{\lambda_1}} \|z\|_{\mathscr{H}} \\
&\quad = (L|\Omega|^{1/2} + r) \frac{1}{\sqrt{\lambda_1}} \|u\|_{\mathscr{H}}.
\end{aligned} \tag{5.36}$$

Moreover, by using inequalities (3.8) and (3.9) we obtain from inequality (5.36) that

$$\begin{aligned}\left(1-\frac{\gamma}{\lambda_{J+1}}\right)&\|w\|_{\mathscr{H}}^2+\left(\frac{\gamma'}{\lambda_J}-1\right)\|v\|_{\mathscr{H}}^2\\&\le\|w\|_{\mathscr{H}}^2-\gamma\|w\|_{L^2(\Omega)}^2+\gamma'\|v\|_{L^2(\Omega)}^2-\|v\|_{\mathscr{H}}^2\\&\le(L|\Omega|^{1/2}+r)\frac{1}{\sqrt{\lambda_1}}\|u\|_{\mathscr{H}}.\end{aligned}$$

Therefore, if we let

$$b=\min\left\{1-\frac{\gamma}{\lambda_{J+1}},\frac{\gamma'}{\lambda_J}-1\right\},$$

we have the inequality

$$b\|u\|_{\mathscr{H}}^2=b(\|v\|_{\mathscr{H}}^2+\|w\|_{\mathscr{H}}^2)\le(L|\Omega|^{1/2}+r)\frac{1}{\sqrt{\lambda_1}}\|u\|_{\mathscr{H}}.$$

This proves that

$$\|u\|_{\mathscr{H}}\le R(r),$$

where

$$R(r)=\frac{1}{\sqrt{\lambda_1}b}(L|\Omega|^{1/2}+r).$$

The proof of Lemma 5.5 is complete. □

**Step 6**: The next lemma proves that the homogeneous problem (1.5) has exactly three solutions:

Lemma 5.6. *Under the conditions of Theorem* 1.3, *there exist exactly three solutions*, *one trivial solution* 0 *and two non-trivial solutions* $v_1$, $v_2$ *of the homogeneous problem* (1.5).

Proof. By Lemma 5.1, it suffices to show that there are exactly three solutions of $\nabla G(v)=0$. If $\nabla G(v)=0$, then $u=v+\varphi(v)$ is a solution of the equation $\nabla F(u)=0$ or, equivalently, $u$ is a weak solution of problem (1.5). Hence we obtain from Lemma 5.5 with $r:=0$ that

$$\|u\|_{\mathscr{H}}\le R(0)=\frac{1}{\sqrt{\lambda_1}b}(L|\Omega|^{1/2}).$$

However, since $v$ and $\varphi(v)$ are orthogonal in $\mathscr{H}$, it follows that

$$\|v\|_{\mathscr{H}} \leq \|v + \varphi(v)\|_{\mathscr{H}} = \|u\|_{\mathscr{H}} \leq R(0).$$

Now, by virtue of Lemma 5.3 and the inverse mapping theorem (Theorem 2.2) we find that the solutions of $\nabla G(v) = 0$ are isolated. Hence there exist only a finite number of solutions of $\nabla G(v) = 0$. Let $v_1, v_2, \ldots, v_k$ denote the non-zero solutions of $\nabla G(v) = 0$. By Theorem 1.1, it follows that

$$k \geq 2.$$

Since the critical points of $G$ and $f = -G$ coincide, we have, by formula (5.5),

$$\operatorname{sgn} \det D^2 f(0) = (-1)^J \operatorname{sgn} \det D^2 G(0) = (-1)^{2J-1} = -1 \tag{5.37}$$

and

$$\begin{aligned} \operatorname{sgn} \det D^2 f(v_i) &= (-1)^J \operatorname{sgn} \det D^2 G(v_i) = (-1)^{2J} \\ &= 1 \quad \text{if } 1 \leq i \leq k. \end{aligned} \tag{5.38}$$

We remark that

$$F(x + \varphi(x)) \leq F(x) \quad \text{for all } x \in X.$$

By condition (vii), it follows that

$$G(x) = F(x + \varphi(x)) \to -\infty \quad \text{as } \|x\|_H \to \infty,$$

so that $f(v) = -G(v) \to +\infty$ as $\|v\|_{\mathscr{H}} \to \infty$.

Therefore, we have proved that $f$ satisfies all the conditions of Theorem 2.6. Since we have the formulas

$$i(\nabla f, v_j) = \operatorname{sgn} \det D^2 f(v_j), \quad j = 0, 1, \ldots, k,$$

it follows from an application of Theorem 2.6 and formulas (5.37) and (5.38) that

$$1 = \sum_{j=0}^{k} i(\nabla f, v_j) = -1 + (k \times 1) = k - 1.$$

This proves that

$$k = 2.$$

The proof of Lemma 5.6 is complete. $\square$

**Step 7**: To complete the proof of Theorem 1.3, let $u_0$, $u_1$ and $u_2$ be the three solutions of problem (1.5). By Lemma 5.4, we can choose constants $\delta > 0$ and $\delta' > 0$ such that if $h \in L^2(\Omega)$ and $\|h\|_{L^2(\Omega)} < \delta$, then there exist solutions $\tilde{u}_k$, $k = 0, 1, 2$ of problem (1.3) with

$$\|\tilde{u}_k - u_k\|_{\mathscr{H}} < \delta', \quad k = 0, 1, 2.$$

Since $\delta$ and $\delta'$ may be chosen to be arbitrarily small, these solutions are distinct provided that if $\|h\|_{L^2(\Omega)}$ is sufficiently small.

(1) Assume, to the contrary, that Theorem 1.3 does not hold true. Then there exists a sequence $\{h_m\}_{m=1}^{\infty}$ in $L^2(\Omega)$ such that

$$\|h_m\|_{L^2(\Omega)} \to 0 \quad \text{as } m \to \infty$$

and that there exist *four* distinct solutions $u_{\ell m}$, $\ell = 0, 1, 2, 3$, of the non-homogeneous problem (1.3) with $h := h_m$. Namely, we have, for all $w \in \mathscr{H}$,

$$(u_{\ell m}, w)_{\mathscr{H}} - \int_{\Omega} (g(u_{\ell m})w - h_m \cdot w)\, dx = 0, \quad \ell = 0, 1, 2, 3. \tag{5.39}$$

If we introduce a map $N : \mathscr{H} \to \mathscr{H}$ by the formula

$$N = \mathfrak{A}^{-1}(g(\cdot)) : \mathscr{H} \longrightarrow L^2(\Omega) \xrightarrow{g(\cdot)} L^2(\Omega) \xrightarrow{\mathfrak{A}^{-1}} \mathscr{H}, \tag{5.40}$$

then it follows that the map $N$ is *compact*. Indeed, it suffices to note the following four assertions:

(a) The injection

$$\mathscr{H} \hookrightarrow W^{1,2}(\Omega)$$

is continuous (assertion (3.3)).

(b) The injection

$$W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$$

is compact (the Rellich–Kondrachov theorem (see [2, Theorem 6.3])).

(c) The map

$$g(\cdot) : L^2(\Omega) \to L^2(\Omega)$$

is continuous, since $g(t)$ is Lipschitz continuous on $\mathbf{R}$.

(d) The resolvent

$$\mathfrak{A}^{-1} : L^2(\Omega) \to \mathscr{H}$$

is continuous.

Moreover, we obtain from formula (5.40) that

$$(N(u), w)_{\mathscr{H}} = (g(u), w)_{L^2(\Omega)} \quad \text{for all } w \in \mathscr{H}. \tag{5.41}$$

Indeed, it suffices to note that

$$\begin{aligned}(N(u), w)_{\mathscr{H}} &= (\mathfrak{A}^{-1}(g(u)), w)_{\mathscr{H}} = (\mathfrak{A}\mathfrak{A}^{-1}(g(u)), w)_{L^2(\Omega)} \\ &= (g(u), w)_{L^2(\Omega)} \quad \text{for all } w \in \mathscr{H}.\end{aligned}$$

(2) From the Riesz representation theorem ([33, Chapter III, Section 6, Theorem]), there exists a unique function $v_m \in \mathscr{H}$ such that

$$(h_m, w)_{L^2(\Omega)} = (v_m, w)_{\mathscr{H}} \quad \text{for all } w \in \mathscr{H}.$$

Then we have the inequalities (see inequality (3.7))

$$\|v_m\|_{\mathscr{H}}^2 \leq \|h_m\|_{L^2(\Omega)} \|v_m\|_{L^2(\Omega)} \leq \|h_m\|_{L^2(\Omega)} \frac{1}{\sqrt{\lambda_1}} \|v_m\|_{\mathscr{H}},$$

so that

$$\|v_m\|_{\mathscr{H}} \leq \frac{1}{\sqrt{\lambda_1}} \|h_m\|_{L^2(\Omega)}.$$

This proves that

$$\|v_m\|_{\mathscr{H}} \to 0 \quad \text{as } m \to \infty. \tag{5.42}$$

(3) By using formula (5.41), we can rewrite formula (5.39) in the form

$$\begin{aligned}(u_{\ell m}, w)_{\mathscr{H}} &= (N(u_{\ell m}), w)_{\mathscr{H}} - (v_m, w)_{\mathscr{H}} \\ &= (N(u_{\ell m}) - v_m, w)_{\mathscr{H}} \quad \text{for all } w \in \mathscr{H}.\end{aligned}$$

Hence we have the formula

$$u_{\ell m} = N(u_{\ell m}) - v_m, \quad \ell = 0, 1, 2, 3. \tag{5.43}$$

Since the sequence $\{h_m\}_{m=1}^{\infty}$ is bounded in $L^2(\Omega)$, it follows from Lemma 5.5 that the sequences $\{u_{\ell m}\}_{m=1}^{\infty}$ are bounded in $\mathscr{H}$. Thus, by using the local sequential weak compactness of Hilbert spaces ([33, Chapter V, Section 2, Theorem 1]) we can choose a subsequence $\{u_{\ell m_j}\}_{j=1}^{\infty}$ which converges *weakly* to some function $z_\ell$ in $\mathscr{H}$ for $0 \leq \ell \leq 3$:

$$u_{\ell m_j} \rightharpoonup z_\ell, \quad \ell = 0, 1, 2, 3. \tag{5.44}$$

However, we recall that the map

$$N(\cdot) = \mathfrak{A}^{-1}(g(\cdot)) : \mathscr{H} \to \mathscr{H}$$

is compact. This implies that the sequence $N(u_{\ell m_j})$ converges *strongly* to $N(z_\ell)$ for $0 \le \ell \le 3$:

$$N(u_{\ell m_j}) \to N(z_\ell), \quad \ell = 0, 1, 2, 3. \tag{5.45}$$

By passing to the limit in formula (5.43), we obtain from assertions (5.42), (5.44) and (5.45) that the sequence $\{u_{\ell m_j}\}$ converges strongly to $z_\ell$ and that

$$z_\ell = N(z_\ell), \quad \ell = 0, 1, 2, 3.$$

Therefore, we have the formula

$$(z_\ell, w)_{\mathscr{H}} = (N(z_\ell), w)_{\mathscr{H}} \quad \text{for all } w \in \mathscr{H}, \quad \ell = 0, 1, 2, 3,$$

or equivalently,

$$(z_\ell, w)_{\mathscr{H}} - \int_\Omega g(z_\ell) w \, dx = 0 \quad \text{for all } w \in \mathscr{H}, \quad \ell = 0, 1, 2, 3.$$

This proves that $z_\ell$ is a weak solution of the homogeneous problem (1.5).

Since each solution $z_\ell$, $\ell = 0, 1, 2, 3$, is equal to some solution $u_k$, $k = 0, 1, 2$. This implies that some two of the four sequences $\{u_{\ell m_j}\}$, $\ell = 0, 1, 2, 3$, should converge to the same weak solution of the homogeneous problem (1.5). However, we obtain that the *four* solutions $u_{\ell m_j}$, $\ell = 0, 1, 2, 3$, of the non-homogeneous problem (1.3) are distinct for each $j$ and $\|h_{m_j}\|_{L^2(\Omega)} \to 0$ as $j \to \infty$. This contradicts Lemma 5.4 for $j$ sufficiently large.

Now the proof of Theorem 1.3 is complete. □

## 6. Proof of Theorem 1.4

This last section is devoted to the proof of Theorem 1.4. By virtue of Theorem 3.3, we have only to prove Theorem 1.4 for weak solutions. The proof of Theorem 1.4 is divided into three steps.

**Step 1**: To prove Theorem 1.4, we make use of the following variant of the Ljusternik–Schnirelman theory due to Clark [12, Theorem 11]:

Theorem 6.1 (Clark). *Let $H$ be a real Hilbert space and let $f(x)$ be an* even, *real-valued $C^2$ function defined on $H$. Assume that $f(x)$ has the property that whenever $\{x_n\} \subset H$ is a bounded sequence such that $f(x_n) < 0$, $f(x_n)$ is bounded from below, and $\nabla f(x_n) \to 0$, then $\{x_n\}$ contains a convergent subsequence. Moreover, we assume that the following four conditions* (a) *through* (d) *are satisfied*:

(a) $f(0) = 0$.
(b) $f(x)$ *is bounded from below.*
(c) *There exists a subspace* $M$ *of* $H$ *of dimension* $\ell > 0$ *such that*

$$(D^2 f(0)x, x)_H < 0 \quad \textit{for all non-zero elements } x \textit{ of } M.$$

(d) $f(x) \geq 0$ *for* $\|x\|_H$ *sufficiently large.*

*Then there exist at last* $2\ell$ *non-zero solutions of the equation* $\nabla f(x) = 0$.

**Step 2**: If $g(t)$ is an odd function of $t$, then it follows from formula (4.3) that the energy function

$$F(u) = \frac{1}{2}(u, u)_{\mathscr{H}} - \int_\Omega \Gamma(u(x))\, dx = \frac{1}{2}(u, u)_{\mathscr{H}} - \int_\Omega \int_0^{u(x)} g(s)\, ds dx$$

is an even function of $u$ and from formula (4.11) that the gradient

$$\nabla F(u) = u - T(g(u))$$

is an odd function of $u$.

We recall that the function $F(u)$ satisfies all the hypotheses of Theorem 4.1, as is shown in the proof of Theorem 1.1.

**Step 3**: Now we obtain from condition (G) that inequality (4.1) holds true

$$\lambda_{K-1} < g'(0) < \lambda_K \leq \lambda_J$$

and that

$$X = \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_J\}, \quad \dim X = J,$$

$$Y = X^\perp,$$

$$X_1 = \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_{K-1}\}, \quad \dim X_1 = K - 1 \leq J - 1 < J = \dim X,$$

$$Y_1 = X_1^\perp.$$

**Step 3-1**: We have the following:

Claim 6.1. *If* $g(t)$ *is an odd function of* $t$, *then the function* $\varphi(v)$ *is an* odd *function of* $v$ *and the function*

$$G(v) = F(v + \varphi(v)), \quad v \in X,$$

*is an* even *function of* $v$.

Proof. First, we prove the oddness of $\varphi(v)$. By the oddness of $\nabla F(u)$, it follows from formula (5.1) that

$$(\nabla F(-v-\varphi(v)), k)_{\mathscr{H}} = -(\nabla F(v+\varphi(v)), k)_{\mathscr{H}} = 0 \quad \text{for all } k \in Y.$$

Since $\varphi(-v)$ is the unique element of $Y$ such that

$$(\nabla F(-v+\varphi(-v)), k)_{\mathscr{H}} = 0 \quad \text{for all } k \in Y,$$

we obtain that

$$\varphi(-v) = -\varphi(v) \quad \text{for all } v \in X.$$

This proves the oddness of $\varphi(v)$.

Secondly, since $\varphi(v)$ is odd and $F(u)$ is even, it follows that

$$\begin{aligned} G(-v) &= F(-v+\varphi(-v)) = F(-v-\varphi(v)) = F(v+\varphi(v)) \\ &= G(v) \quad \text{for all } v \in X. \end{aligned}$$

This proves the evenness of $G(v)$.

The proof of Claim 6.1 is complete. □

**Step 3-2**: We have the following:

Claim 6.2. *If condition* (G) *is satisfied, then the quadratic form associated with $D^2G(0)$ is positive definite on some subspace $M$ of $X$ of dimension $J-K+1$.*

Proof. Assume, to the contrary, that $D^2G(0)$ has at least $K$ non-positive eigenvalues. Then there exists a subspace $W$ of $X$ with $\dim W \geq K$ such that

$$(D^2G(0)w, w)_{\mathscr{H}} \leq 0 \quad \text{for all } w \in W. \tag{6.1}$$

If $\hat{W}$ is a subspace of $\mathscr{H}$ defined by the formula

$$\hat{W} = \{\hat{w} = w + \varphi'(0)w : w \in W\},$$

then, since $\varphi(0) = 0$, it follows from formula (5.17) and inequality (6.1) that

$$\begin{aligned} &(D^2F(0)(w+\varphi'(0)(w)), w+\varphi'(0)(w))_{\mathscr{H}} \\ &\quad = (D^2F(0+\varphi(0))(w+\varphi'(0)(w)), w+\varphi'(0)(w))_{\mathscr{H}} \\ &\quad = (D^2G(0)w, w)_{\mathscr{H}} \leq 0 \quad \text{for all } w \in W, \end{aligned}$$

so that

$$(D^2F(0)\hat{w}, \hat{w})_{\mathscr{H}} \le 0 \quad \text{for all } \hat{w} \in \hat{W}. \tag{6.2}$$

However, we have, by inequality (4.16),

$$(D^2F(0)s, s)_{\mathscr{H}} \ge m_1 \|s\|^2_{\mathscr{H}} \quad \text{for all } s \in Y_1, \tag{6.3}$$

with

$$m_1 = 1 - \frac{g'(0)}{\lambda_K} > 0,$$

and that

$$\operatorname{codim} Y_1 = K - 1.$$

Since $\dim \hat{W} = \dim W \ge K$, we can find a non-zero element $z$ of $\hat{W} \cap Y_1$.

By using inequality (6.3) with $s := z$ and inequality (6.2) with $\hat{w} := z$, we obtain that

$$0 < m_1 \|z\|^2_{\mathscr{H}} \le (D^2F(0)z, z)_{\mathscr{H}} \le 0.$$

This contradiction proves the existence of an $(J - K + 1)$-dimensional subspace $M$ of $X$ on which $D^2G(0)$ is positive definite.

The proof of Claim 6.2 is complete. □

If we let

$$f(v) = -G(v) = -F(v + \varphi(v)), \quad v \in X,$$

then it follows from Claim 6.2 that

$$(D^2f(0)v, v)_{\mathscr{H}} < 0 \quad \text{for all non-zero elements } v \text{ of } M.$$

This verifies condition (c) of Theorem 6.1 with $\ell := J - K + 1$.

Moreover, since we have, for all $v \in X$,

$$F(v + \varphi(v)) \le F(v),$$

we obtain from Proposition 4.2 (condition (vii) of Theorem 4.1) that

$$G(v) = F(v + \varphi(v)) \to -\infty \quad \text{as } \|v\|_{\mathscr{H}} \to \infty,$$

so that

$$f(v) = -G(v) \to +\infty \quad \text{as } \|v\|_{\mathscr{H}} \to \infty.$$

This verifies conditions (b) and (d) of Theorem 6.1.

Condition (a) of Theorem 6.1 is trivially satisfied.

**Step 3-3**: Since $X$ is finite-dimensional, it follows from an application of [14, Theorem 4.3.3] that every bounded sequence has a convergent subsequence.

Summing up, we have proved that the function

$$f(v) = -G(v) = -F(v + \varphi(v)), \quad v \in X,$$

satisfies all the conditions of Theorem 6.1 with $H := X$ and $\ell := J - K + 1$. Hence there exist at least $2(J - K + 1)$ non-zero solutions $v_1, v_2, \dots, v_{2(J-K+1)}$ of the equation

$$\nabla G(v) = 0.$$

Therefore, by applying Lemma 5.1 to our situation we can find at least $2(J - K + 1)$ non-zero solutions $u_1, u_2, \dots, u_{2(J-K+1)}$ of the equation

$$\nabla F(u) = 0$$

with

$$u_i = v_i + \varphi(v_i), \quad 1 \leq i \leq 2(J - K + 1).$$

This proves that there exist at least $2(J - K + 1)$ non-trivial solutions of problem (1.5), since critical points of $F$ are weak solutions of the homogeneous problem (1.5).

The proof of Theorem 1.4 is complete. $\square$

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## References

[1] Abraham, R., Marsden, J. E. and Ratiu, T., Manifolds, tensor analysis, and applications, Addison-Wesley Publishing Co., Reading, Mass., 1983.

[2] Adams, R. A. and Fournier, J. J. F., Sobolev spaces, second edition, Academic Press, Amsterdam Heidelberg New York Oxford, 2003.

[3] Amann, H., Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. **18** (1976), 620–709.

[4] Amann, H., A note on degree theory for gradient mappings, Proc. Amer. Math. Soc. **85** (1982), 591–595.

[5] Ambrosetti, A. and Mancini, G., Sharp nonuniqueness results for some nonlinear problems, Nonlinear Anal. **3** (1979), 635–645.

[6] Ambrosetti, A. and Prodi, G., On the inversion of some differentiable mappings with singularities between Banach spaces, Ann. Mat. Pura Appl. **93** (1972), 231–247.

[7] Ambrosetti, A. and Prodi, G., A primer of nonlinear analysis, Cambridge Stud. Adv. Math., Cambridge University Press, Cambridge, 1993.

[8] Berger, M. S. and Podolak, E., On the solutions of a nonlinear Dirichlet problem, Indiana Univ. Math. J. **24** (1975), 837–846.

[9] Castro, A. and Lazer, A. C., Applications of a max-min principle, Rev. Colombiana Mat. **10** (1976), 141–149.

[10] Castro, A. and Lazer, A. C., Critical point theory and the number of solutions of a nonlinear Dirichlet problem, Ann. Mat. Pura Appl. **120** (1979), 113–137.

[11] Chang, K. C., Methods in nonlinear analysis, Springer Monogr. Math., Springer-Verlag, Berlin Heidelberg New York, 2005.

[12] D. C. Clark, A variant of the Lusternik–Schnirelman theory, Indiana Univ. Math. J. **22** (1972), 65–74.

[13] Dieudonné, J., Foundations of modern analysis, Academic Press, New York London, 1969.

[14] Friedman, A., Foundations of modern analysis, Holt, Rinehart and Winston Inc., New York Montreal London, 1970.

[15] Fujiwara, D., Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order, Proc. Japan Acad. Ser. A Math. Sci., **43** (1967), 82–86.

[16] Hempel, J. A., Multiple solutions for a class of nonlinear boundary value problems, Indiana Univ. Math. J. **20** (1971), 983–996.

[17] Hörmander, L., The analysis of linear partial differential operators III, Pseudo-differential operators, 1994 edition, Grundlehren Math. Wiss., Springer-Verlag, Berlin Heidelberg New York Tokyo, 1994.

[18] Lazer, A. C., Landesman, E. M. and Meyers, D. R., On saddle point problems in the calculus of variations, the Ritz algorithm, and monotone convergence, J. Math. Anal. Appl. **33** (1975), 594–614.

[19] Ljusternik, L. and Schnirelman, L., Méthodes topologiques dans les problèmes variationelles, Gauthier-Villars, Paris, 1934.

[20] Milnor, J. W., Topology from the differentiable viewpoint, Princeton University Press, Princeton, NJ, 1997.

[21] Nirenberg, L., Topics in nonlinear functional analysis, Courant Institute of Mathematical Sciences, New York University, New York, 1974.

[22] Runst, T. and Sickel, W., Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, Walter de Gruyter, Berlin New York, 1996.

[23] Taira, K., Analytic semigroups and semilinear initial-boundary value problems, London Math. Soc. Lecture Note Ser. Vol. 223, Cambridge University Press, Cambridge, 1995.

[24] Taira, K., Semigroups, boundary value problems and Markov processes, Springer Monogr. Math., Springer-Verlag, Berlin Heidelberg New York, 2004.

[25] Taira, K., Degenerate elliptic boundary value problems with indefinite weights, Mediterr. J. Math. **5** (2008), 133–162.

[26] Taira, K., Degenerate elliptic boundary value problems with asymmetric nonlinearity, J. Math. Soc. Japan **62** (2010), 431–465.

[27] Taira, K., Semilinear degenerate elliptic boundary value problems at resonance, Ann. Univ. Ferrara Sez. VII Sci. Mat., **56** (2010), 369–392.

[28] Taira, K., Multiple solutions of semilinear degenerate elliptic boundary value problems, Math. Nachr. **284** (2011), 105–123.

[29] Taira, K., Multiple solutions of semilinear degenerate elliptic boundary value problems II, Math. Nachr. **284** (2011), 1554–1566.

[30] Taira, K., Degenerate elliptic boundary value problems with asymptotically linear nonlinearity, Rend. Circ. Mat. Palermo **60** (2011), 283–308.

[31] Taira, K., Multiple solutions of semilinear elliptic problems with degenerate boundary conditions, Mediterr. J. Math. **11** (2014), DOI:10.1007/s00009-012-0212-6.
[32] Thews, K., Multiple solutions for elliptic boundary value problems with odd nonlinearities, Math. Z. **163** (1978), 163–175.
[33] Yosida, K., Functional analysis, sixth edition, Grundlehren Math. Wiss., Springer-Verlag, Berlin Heidelberg New York, 1980.

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