TSUKUBA J. MATH.
Vol. 35 No. 1 (2011), 53–66

# THE STRUCTURE JACOBI OPERATOR FOR REAL HYPERSURFACES IN THE COMPLEX PROJECTIVE PLANE AND THE COMPLEX HYPERBOLIC PLANE

By

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**Abstract.** Recently, we investigated real hypersurfaces in a $n$-dimentional complex projective space and complex hyperbolic space with respect to various structure Jacobi operator conditions. However these results necessitates dimension assumption $n \geq 3$. The purpose of this paper is to study such real hypersurfaces in the complex projective plane and the complex hyperbolic plane.

## 1. Introduction

A complex $n$-dimensional Kähler manifold with Kähler structure $J$ of constant holomorphic sectional curvature $4c$ is called a complex space form, which is denoted by $M_n(c)$. As is well-known, a connected complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}$ or a complex hyperbolic space $H_n\mathbf{C}$ according as $c > 0$, $c = 0$ or $c < 0$.

The study of real hypersurfaces in complex projective space $P_n\mathbf{C}$ was initiated by Takagi [12], who proved that all homogeneous real hypersurfaces in $P_n\mathbf{C}$ could be devided into six types which are said to be of type $A_1$, $A_2$, $B$, $C$, $D$ and $E$.

In the case of complex hyperbolic space $H_n\mathbf{C}$, the classification of homogeneous real hypersurfaces in $H_n\mathbf{C}$ is obtained by Berndt and Tamaru [2]. In particular, real hypersurfaces in $H_n\mathbf{C}$, which are said to be of type $A_0$, $A_1$ and $A_2$ were treated by Montiel and Romero [9]. Real hypersurfaces in $P_n\mathbf{C}$ and $H_n\mathbf{C}$

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2000 Mathematics Subject Classification: Primary 53B25; Secondary 53C15, 53C25.
Keywords: complex projective plane, complex hyperbolic plane, real hypersurface, structure Jacobi operator.
Received July 2, 2010.
Revised January 11, 2011.

have been studied by several authors (cf. Cecil and Ryan [3], Okumura [8], Montiel and Romero [7]).

Let $M$ be a real hypersurface in $M_n(c)$, $c \neq 0$ and $\nu$ a unit normal vector field on $M$. Then a tangent vector field $\xi := -J\nu$ to $M$ is called the *structure vector field* on $M$. $M$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ induced from $J$. We denote $\nabla$ and $S$, the Levi-Civita connection and the Ricci tensor of $M$, respectively. If the structure vector is a principal vector, then $M$ is called a *Hopf hypersurface*. It is known that the principal curvature $\alpha$ is locally constant (Maeda, Y. [9], Ki and Suh [6]).

On the other hand, the Jacobi operator field with respect to $X$ in a Riemannian mannifold $M$ is defined by $R_X = R(\cdot, X)X$, where $R$ denotes the Riemannian curvature tensor of $M$. We will call the Jacobi operator on $M$ with respect to $\xi$ the *structure Jacobi operator* on $M$. The structure Jacobi operator $R_\xi$ is said to be *cyclic-parallel* if it satisfies

$$\mathfrak{S} R'_\xi(X, Y, Z) = \mathfrak{S} g(\nabla_X R_\xi(Y), Z) = 0$$

for any vector fields $X$, $Y$ and $Z$, where $\mathfrak{S}$ denote the cyclic sum. The structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ has a fundamental role in contact geometry. Ortega, Pérez and Santos [10] have proved that there are no real hypersurfaces in $P_n\mathbf{C}$, $n \geq 3$ with parallel structure Jacobi operator $\nabla R_\xi = 0$. More generally, such a result has been extended by [11] due to them. Recently, author et al. have some classification results with respect to the structure Jacobi operator for real hypersurfaces in $M_n(c)$, $c \neq 0$ [4, 5].

Theorem 1 (Ki and Kurihara (in preparation)). *Let $M$ be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$ which satisfies $\nabla_\xi R_\xi = 0$. Then $M$ holds $R_\xi \phi S = R_\xi S \phi$ if and only if $\alpha = 0$ or $M$ is locally congruent to one of real hypersurfaces of type $A_1$, $A_2$ of $P_n\mathbf{C}$ or of type $A_0$–$A_2$ of $H_n\mathbf{C}$.*

Theorem 2 ([5]). *Let $M$ be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$ which satisfies $\nabla_\xi R_\xi = 0$. Then $R_\xi \phi S = S \phi R_\xi$ if and only if $M$ is locally congruent to one of real hypersurfaces of type $A_1$, $A_2$ of $P_n\mathbf{C}$ with $\alpha \neq 0$ or of type $A_0$–$A_2$ of $H_n\mathbf{C}$.*

Theorem 3 ([4]). *Let $M$ be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$. If the structure Jacobi operator is cyclic-parallel, then $M$ is locally congruent to one real hypersurfaces of type $A_1$, $A_2$ and a tube of radius $r$*

*over complex quadric* $Q_{n-1}$, *where* $\cot r = (\sqrt{2c+4} + \sqrt{2c})/2$ *of* $P_n\mathbf{C}$ *or of type* $A_0$–$A_2$ *of* $H_n\mathbf{C}$.

However these results are proved for $n \geq 3$ and the methods of proofs depend on this. In this paper we invistigate corresponding results for $n = 2$ (Theorem 3–7 in Section 4–6).

All manifolds in this paper are assumed to be connected and of class $C^\infty$ and the real hypersurfaces are supposed to be oriented.

## 2. Preliminaries

### 2.1. Real Hypersurfaces in $M_n(c)$, $c \neq 0$

We denote by $M_n(c)$, $c \neq 0$ be a nonflat complex space form with the Fubini-Study metric $\tilde{g}$ of constant holomorphic sectional curvature $4c$ and Levi-Civita connection $\tilde{\nabla}$. For an immersed $(2n-1)$-dimensional Riemannian manifold $\tau : M \to M_n(c)$, the Levi-Civita connection $\nabla$ of induced metric and the shape operator $H$ of the immersion are characterized

$$\tilde{\nabla}_X Y = \nabla_X Y + g(HX, Y)v, \quad \tilde{\nabla}_X v = -HX$$

for any vector fields $X$ and $Y$ on $M$, where $g$ denotes the Riemannian metric of $M$ induced from $\tilde{g}$ and $v$ a unit normal vector on $M$. In the sequel the indeces $i, j, k, l, \dots$ run over the range $\{1, 2, \dots, 2n-1\}$ unless otherwise stated. For a local orthonormal frame field $\{e_i\}$ of $M$, we denote the dual 1-forms by $\{\theta_i\}$. Then the connection forms $\theta_{ij}$ are defined by

$$d\theta_i + \sum_j \theta_{ij} \wedge \theta_j = 0, \quad \theta_{ij} + \theta_{ji} = 0.$$

Then we have

$$\nabla_{e_i} e_j = \sum_k \theta_{kj}(e_i) e_k = \sum_k \Gamma_{kij} e_k,$$

where we put $\theta_{ij} = \sum_k \Gamma_{ijk} \theta_k$. The almost contact metric structure $(\phi = (\phi_{ij})$, $\xi = \sum_i \xi_i e_i)$ is induced on $M$ by following equation:

$$J(e_i) = \sum_j \phi_{ji} e_j + \xi_i v.$$

The structure tensor $\phi = \sum_i \phi_i e_i$ and the structure vector $\xi = \sum_i \xi_i e_i$ satisfy

$$\sum_k \phi_{ik}\phi_{kj} = \xi_i\xi_j - \delta_{ij}, \quad \sum_j \xi_j\phi_{ij} = 0, \quad \sum_i \xi_i^2 = 1, \quad \phi_{ij} + \phi_{ji} = 0,$$

$$d\phi_{ij} = \sum_k (\phi_{ik}\theta_{kj} - \phi_{jk}\theta_{ki} - \xi_i h_{jk}\theta_k + \xi_j h_{ik}\theta_k), \tag{2.1}$$

$$d\xi_i = \sum_j \xi_j\theta_{ji} - \sum_{j,k} \phi_{ji}h_{jk}\theta_k.$$

We denote the components of the shape operator or the second fundamental tensor $H$ of $M$ by $h_{ij}$. The components $h_{ij;k}$ of the covariant derivative of $H$ are given by $\sum_k h_{ij;k}\theta_k = dh_{ij} - \sum_k h_{ik}\theta_{kj} - \sum_k h_{jk}\theta_{ki}$. Then we have the equation of Gauss and Codazzi

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.2}$$

$$h_{ij;k} - h_{ik;j} = c(\xi_k\phi_{ij} - \xi_j\phi_{ik} + 2\xi_i\phi_{kj}), \tag{2.3}$$

respectively.

From (2.2) the structure Jacobi operator $R_\xi = (\Xi_{ij})$ is given by

$$\Xi_{ij} = \sum_{k,l} h_{ik}h_{jl}\xi_k\xi_l - \sum_{k,l} h_{ij}h_{kl}\xi_k\xi_l + c\xi_i\xi_j - c\delta_{ij}. \tag{2.4}$$

From (2.2) the Ricci tensor $S = (S_{ij})$ is given by

$$S_{ij} = (2n+1)c\delta_{ij} - 3c\xi_i\xi_j + hh_{ij} - \sum_k h_{ik}h_{kj}, \tag{2.5}$$

where $h = \sum_i h_{ii}$.

First we remark

Lemma 1 ([5]). *Let $U$ be an open set in $M$ and $F$ a smooth function on $U$. We put $dF = \sum_i F_i\theta_i$. Then we have*

$$F_{ij} - F_{ji} = \sum_k F_k\Gamma_{kij} - \sum_k F_k\Gamma_{kji}.$$

### 2.2. The Case Where $n = 2$

In this section, we treat the case where $n = 2$.

Now we retake a local orthonormal frame field $\{e_1, e_2, e_3\}$ in such a way that

- $e_1 = \xi$,
- $e_2$ is in the direction of $h_{12}e_2 + h_{13}e_3$,
- $e_3 = \phi e_2$.

Then we have

$$\xi_1 = 1, \quad \xi_2 = \xi_3 = 0 \quad \text{and} \quad \phi_{32} = 1. \tag{2.6}$$

We put $\alpha := h_{11}$, $\beta := h_{12}$, $\gamma := h_{22}$, $\varepsilon := h_{23}$ and $\delta := h_{33}$. Then the shape operator $H$ and the structure tensor $\phi$ are represented by matrices

$$H = \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & \varepsilon \\ 0 & \varepsilon & \delta \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.7}$$

respectively.

Since $d\xi_i = 0$, we have

$$\theta_{12} = \varepsilon\theta_2 + \delta\theta_3, \tag{2.8}$$

$$\theta_{13} = -\beta\theta_1 - \gamma\theta_2 - \varepsilon\theta_3. \tag{2.9}$$

We put

$$\theta_{23} = X_1\theta_1 + X_2\theta_2 + X_3\theta_3. \tag{2.10}$$

The equations (2.4) and (2.5) are rewritten as

$$\Xi_{ij} = -\alpha h_{ij} + h_{1i}h_{1j} + c\delta_{i1}\delta_{j1} - c\delta_{ij}, \tag{2.11}$$

$$S_{ij} = (\alpha + \gamma + \delta)h_{ij} - \sum_{k=1}^{3} h_{ik}h_{jk} - 3c\delta_{i1}\delta_{j1} + 5c\delta_{ij}, \tag{2.12}$$

respectively, where $i \in \{1, 2, 3\}$.

Now, a fundamental property are stated for later use.

THEOREM 4 (Okumura [8], Montiel and Romero [7]). *Let $M$ be a real hypersurface in $P_2\mathbf{C}$ or $H_2\mathbf{C}$. If the shape operator is commuts with the structure tensor, then $M$ is locally congruent to one of the following*:

- *in case that $P_2\mathbf{C}$,*
  $(A_1)$ *a geodesic hypersphere of radius $r$, where $0 < r < \pi/\sqrt{4c}$,*
- *in case that $H_2\mathbf{C}$,*
  $(A_0)$ *a horosphere,*
  $(A_1)$ *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_1\mathbf{C}$.*

## 3. Real Hypersurfaces with the Condition $\nabla_\xi R_\xi = 0$

Hereafter the indeces $i, j, k, l$ run over the range $\{1,2,3\}$ unless otherwise stated.

In this section we assume that $\nabla_\xi R_\xi = 0$. The components $\Xi_{ij;k}$ of the covariant derivativation of $R_\xi = (\Xi_{ij})$ is given by

$$\sum_k \Xi_{ij;k}\theta_k = d\Xi_{ij} - \sum_k \Xi_{kj}\theta_{ki} - \sum_k \Xi_{ik}\theta_{kj}.$$

Substituting (2.11) into the above equation, we have

$$\begin{aligned}\sum_k \Xi_{ij;k}\theta_k = &-(d\alpha)h_{ij} - \alpha dh_{ij} + (dh_{1i})h_{1j} + h_{1i}(dh_{1j}) \\ &+ \alpha\sum_k h_{kj}\theta_{ki} - \alpha h_{1j}\theta_{1i} - \beta h_{1j}\theta_{2i} - c\delta_{j1}\theta_{1i} \\ &+ \alpha\sum_k h_{ik}\theta_{kj} - \alpha h_{1i}\theta_{1j} - \beta h_{1i}\theta_{2j} - c\delta_{i1}\theta_{1j}.\end{aligned} \tag{3.1}$$

In the following, we assume that $\beta \neq 0$.

Our assumption $\nabla_\xi R_\xi = 0$ is equivalent to $\Xi_{ij;1} = 0$, which can be stated as follows:

$$\varepsilon = 0, \quad \alpha\delta + c = 0, \tag{3.2}$$

$$(\beta^2 - \alpha\gamma)_1 = 0, \tag{3.3}$$

$$(\beta^2 - \alpha\gamma - c)X_1 = 0. \tag{3.4}$$

In the following, using the notion of Lemma 1, we write as follows:

$$\alpha_i = h_{11;i},\ \ \beta_i = h_{12;i},\ \ \gamma_i = h_{22;i},\ \ \delta_i = h_{33;i}\ \ (1 \le i \le 3).$$

Now, we denote the equation (2.3) by $(ijk)$ simply. Then from (2.3) we have following equations (112)–(323):

$$\alpha_2 - \beta_1 = 0, \tag{112}$$

$$\beta_2 - \gamma_1 = 0, \tag{212}$$

$$(\alpha - \delta)\gamma - \beta^2 + (\gamma - \delta)X_1 - \beta X_2 = -c, \tag{312}$$

$$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0, \tag{113}$$

$$\beta_3 - \alpha\delta + \gamma\delta - \beta^2 + (\gamma - \delta)X_1 = c, \tag{213}$$

(313) $$\delta_1 + \beta X_3 = 0,$$

(223) $$\gamma_3 - 2\beta\delta - \beta\gamma + (\gamma - \delta)X_2 = 0,$$

(323) $$\delta_2 + (\gamma - \delta)X_3 = 0.$$

Remark 1. Above equations (112)–(323) may not use equations (3.3) and (3.4).

## 4. The Condition $\nabla_\xi R_\xi = 0$ and $R_\xi \phi S = R_\xi S \phi$

Let $M$ be a real hypersurface in $P_2\mathbf{C}$ or $H_2\mathbf{C}$, which satisfies $\nabla_\xi R_\xi = 0$ and $R_\xi \phi S = R_\xi S \phi$. Under the assumption $\nabla_\xi R_\xi = 0$, it follows from (2.7), (2.11) and (2.12) that the condition $R_\xi \phi S = R_\xi S \phi$ is equivalent to the following equation

(4.1) $$\beta^2 - \alpha\gamma - c = 0.$$

Then taking account of the coefficient of $\theta_3$ in the exterior derivative of (4.1), we have

(4.2) $$2\beta\beta_3 - \gamma\alpha_3 - \alpha\gamma_3 = 0.$$

From (312), (113), (213), (223) and (4.1) we have the following:

(4.3) $$\delta\gamma - (\gamma - \delta)X_1 + \beta X_2 = 0,$$

(4.4) $$\alpha_3 + 3\beta\delta - \alpha\beta + \beta X_1 = 0,$$

(4.5) $$\beta_3 + \gamma\delta - \alpha\gamma - c + (\gamma - \delta)X_1 = 0,$$

(4.6) $$\gamma_3 - 2\beta\delta - \beta\gamma + (\gamma - \delta)X_2 = 0.$$

Substituting of (4.4)–(4.6) into (4.2), we have

$$\beta\delta(X_1 - 4\alpha) = 0,$$

by virtue of (4.3). If $\delta = 0$, then by (3.2) we have a contradiction and hence

(4.7) $$X_1 = 4\alpha.$$

Substituting of this equation into (4.3)–(4.5), we have

(4.8) $$\beta X_2 = 4\alpha(\gamma - \delta) - \delta\gamma,$$

(4.9) $$\alpha_3 + 3\beta\delta + 3\alpha\beta = 0,$$

(4.10) $$\beta_3 + 3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0.$$

It follows from (223), (4.1) and (4.8) that

$$\alpha\gamma_3 + \beta(3\alpha\gamma - 6\alpha\delta - \gamma\delta) = 0. \tag{4.11}$$

Remark 2. we have already obtained above equations in [5], page 53.

We may put $\lambda := \alpha_1/\alpha = \beta_1/\beta$. In fact, eliminating $X_3$ from (313) and (323), we have $\beta\delta_2 + (\delta - \gamma)\delta_1 = 0$ which, together with (3.2) and (112), implies $\alpha_1/\alpha = \beta_1/\beta$.

From (3.2) and (4.1) we have

$$\delta = -\frac{c}{\alpha}, \quad \gamma = \frac{\beta^2 - c}{\alpha}. \tag{4.12}$$

Using above two equations, we can express $X_2$ and $X_3$ by three smooth functions $\alpha$, $\beta$ and $\lambda$. From (3.2), (212) and (4.1) two equations (4.8) and (323) are rewritten as

$$X_2 = \frac{1}{\alpha^2\beta}(4\alpha^2\beta^2 + \beta^2 c - c^2), \tag{4.13}$$

$$X_3 = \frac{\delta_2}{\delta - \gamma} = \frac{c\alpha\delta_2}{c\alpha(\delta - \gamma)} = \frac{-c\alpha_2}{\alpha^2(\gamma - \delta)} = \frac{-c\beta_1}{\alpha(\alpha\gamma + c)} = -\frac{c}{\alpha\beta}\lambda, \tag{4.14}$$

respectively.

On the other hand, taking account of the coefficient of $\theta_1 \wedge \theta_2$ in the exterior derivative of (2.10), we have

$$-X_{1,2} + X_{2,1} + \gamma X_3 + X_1 X_3 = 0. \tag{4.15}$$

Again taking account of the coefficient of $\theta_1$ in the exterior derivative of (4.13), we have

$$X_{2,1} = 4\beta_1 + c\lambda\frac{3c - \beta^2}{\alpha^2\beta},$$

and therefore the equation (4.15) implies

$$\lambda(2\alpha^2 + \beta^2 - 2c) = 0. \tag{4.16}$$

*The case where* $\lambda = 0$. Then we have $\alpha_1 = \beta_1 = 0$. Thus from (313) we have $X_3 = 0$ and therefore $\alpha_2 = \delta_2 = 0$ because of (112). Hence, taking account of the coefficient of $\theta_1$ in the exterior derivative of (4.1), we have $\gamma_1 = 0$, and so $\beta_2 = 0$.

Now put $F = \alpha$ and $\beta$ in Lemma 1. Then we have

$$\alpha_3(\gamma + X_1) = 0, \quad \beta_3(\gamma + X_1) = 0.$$

If $\gamma + X_1 \neq 0$, then we have $\alpha_3 = \beta_3 = 0$, which implies $\alpha$, $\beta$ and $\delta$ are constant. Furthermore, by (4.1) we see that $\gamma$ is constant. Thus from (4.9)–(4.11) we have

$$\alpha + \delta = 0, \tag{4.17}$$

$$3\alpha\gamma - 3\alpha\delta + \gamma\delta = 0, \tag{4.18}$$

$$3\alpha\gamma - 6\alpha\delta - \gamma\delta = 0. \tag{4.19}$$

Hence, by (3.2) and (4.7) we have $\alpha^2 - c = 0$. Moreover eliminating $\gamma\delta$ from (4.18) and (4.19), we have $2\beta^2 + c = 0$ because of (3.2) and (4.1), which is a contradiction. Therefore $X_1 = -\gamma$, which, together with (4.7), implies $\gamma = -X_1 = -4\alpha$. Thus it follows from (4.9) that $\gamma_3 = -4\alpha_3 = 12\beta(\delta + \alpha)$. Hence from (4.11) this contradics $\alpha\delta = 0$.

*The case where* $\lambda \neq 0$. Then from (4.16) we have

$$2\alpha^2 + \beta^2 = 2c. \tag{4.20}$$

Taking account of the coefficient of $\theta_1$ in the exterior derivative of this equation, we have $\lambda(2\alpha^2 + \beta^2) = 0$ and so $2\alpha^2 + \beta^2 = 0$. It follows from (4.20) that $c = 0$, which is a contradiction. Therefore we have $\beta = 0$.

Since (2.5) and $\beta = 0$, we see that $\alpha$ is constant in $M$ (see [6]). Thus from (3.1) our assumption $\Xi_{ij;1} = 0$ is equivalent to $\alpha h_{ij;1} = 0$. Put $j = 1$ in (2.3). Then by above equation we have $\alpha h_{i1;k} = -c\alpha\phi_{ik}$. Therefore since (2.1) and $d\xi_i = 0$, we have

$$\alpha \sum_{k,l} h_{ik}\phi_{lk}h_{kj} + \alpha^2 \sum_k \phi_{ki}h_{kj} = -\alpha h_{i1;j} = c\alpha\phi_{ij},$$

which implies that $\alpha^2(\phi H - H\phi) = 0$. Hence owing to Theorem 4, we complete the proof of following Theorem 5.

Theorem 5. *Let $M$ be a real hypersurface in $P_2\mathbf{C}$ or $H_2\mathbf{C}$, which satisfies $\nabla_\xi R_\xi = 0$. Then $M$ holds $R_\xi\phi S = R_\xi S\phi$ if and only if $H\xi = 0$ or $M$ is locally congruent to one of the following*:

- *in case that* $P_2\mathbf{C}$,

  $(A_1)$ *a geodesic hypersphere of radius* $r$, *where* $0 < r < \pi/2$ *and* $r \neq \pi/4$,
- *in case that* $H_2\mathbf{C}$,

  $(A_0)$ *a horosphere*,

  $(A_1)$ *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane* $H_1\mathbf{C}$.

## 5. The Condition $\nabla_\xi R_\xi = 0$ and $R_\xi \phi S = S \phi R_\xi$

Let $M$ be a real hypersurface in $P_2\mathbf{C}$ or $H_2\mathbf{C}$, which satisfies $\nabla_\xi R_\xi = 0$ and $R_\xi \phi S = S \phi R_\xi$. Under the assumption $\nabla_\xi R_\xi = 0$, it follows from (2.7), (2.11) and (2.12) that the condition $R_\xi \phi S = S \phi R_\xi$ is equivalent to the following equation

$$(\gamma\delta + 4c)(\beta^2 - \alpha\gamma - c) = 0. \tag{5.1}$$

If $\beta^2 - \alpha\gamma - c = 0$, then by the same argument as that in Section 4 we have $\beta = 0$, which is a contradiction. Therefore $\beta^2 - \alpha\gamma - c \neq 0$. Then from (3.4) and (5.1) we have

$$X_1 = 0, \quad \gamma\delta = -4c. \tag{5.2}$$

Now, taking account of the coefficient of $\theta_1 \wedge \theta_2$ in the exterior derivative of $\theta_{23} = X_2\theta_2 + X_3\theta_3$, we have

$$X_{2,1} + \gamma X_3 = 0. \tag{5.3}$$

From (5.2) the equation (312) is rewritten as

$$\beta X_2 = -(\beta^2 - \alpha\gamma - c) + 4c.$$

Therefore from (3.3) we have $(\beta X_2)_1 = 0$, which implies

$$\beta X_{2,1} = -\beta_1 X_2. \tag{5.4}$$

Hence, by (5.3) we have

$$\beta\gamma X_3 = \beta_1 X_2. \tag{5.5}$$

From (323), (5.5) and (112) it is easy to see that

$$\alpha_2((\delta - \gamma)X_2 + 4\beta\delta) = 0.$$

This, together with (223), gives

$$\alpha_2(\beta\gamma - 2\beta\delta - \gamma_3) = 0. \tag{5.6}$$

If $\alpha_2 \neq 0$, then we have

$$\gamma_3 = \beta\gamma - 2\beta\delta. \tag{5.7}$$

By (5.2) we have $(\gamma\delta)_3 = 0$, which implies that

$$\alpha\gamma_3 - \gamma\alpha_3 = 0.$$

Since $\beta \neq 0$ substituting of (113) and (5.7) into above equation, it is easy to show that $c = 0$, which is a contradiction. Hence we have $\alpha_2 = 0$. Then from (112), (3.3), (5.4) and (5.3) we have

$$\alpha_2 = \delta_2 = \beta_1 = (\alpha\gamma)_1 = X_{2,1} = X_3 = 0. \tag{5.8}$$

Taking account of the coefficient of $\theta_1 \wedge \theta_3$ and $\theta_2 \wedge \theta_3$ in the exterior derivative of $\theta_{23} = X_2\theta_2$, we have $X_2 = -2\beta$ and $\beta_3 - 2\beta^2 = \gamma\delta + 2c$, respectively. It follows from (213) that

$$\beta^2 = 8c, \tag{5.9}$$

which implies $\beta_3 = 0$. Thus from (213) we have $\beta^2 = -6c$, which contradicts (5.9).

Therefore $M$ is a Hopf hypersurface. Thus by the same argument as that in Section 4 we complete proof of following Theorem 6.

Theorem 6. *Let $M$ be a real hypersurface in $P_2\mathbf{C}$ or $H_2\mathbf{C}$, which satisfies $\nabla_\xi R_\xi = 0$. Then $M$ holds $R_\xi\phi S = S\phi R_\xi$ if and only if $H\xi = 0$ or $M$ is locally congruent to one of the following*:
- *in case that $P_2\mathbf{C}$,*
  $(A_1)$ *a geodesic hypersphere of radius $r$, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
- *in case that $H_2\mathbf{C}$,*
  $(A_0)$ *a horosphere,*
  $(A_1)$ *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_1\mathbf{C}$.*

## 6. Cyclic-Parallel Structure Jacobi Operator Condition

In this section we invistigate the condition "cyclic-parallel structure Jacobi operator" (see [4]). First in [4] the proof of Main Thorem suggests following proposition.

Proposition 1. *Let $M$ be a Hopf real hypersurface in $P_2\mathbf{C}$ or $H_2\mathbf{C}$. If the structure Jacobi operator is cyclic-parallel, then $M$ is locally congruent to one of the following*:

- *in case that* $P_2\mathbf{C}$,
  ($A_1$) *a geodesic hypersphere of radius* $r$, *where* $0 < r < \pi/\sqrt{4c}$,
  ($B$) *a tube of radius* $r$ *over complex quadric* $Q_1$, *where* $\cot r = (\sqrt{2c+4} + \sqrt{2c})/2$,
- *in case that* $H_2\mathbf{C}$,
  ($A_0$) *a horosphere*,
  ($A_1$) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane* $H_1\mathbf{C}$.

We suppose that $M$ is a non-Hopf hypersurface in $P_2\mathbf{C}$ or $H_2\mathbf{C}$ satisfying $\mathfrak{S}g(\nabla_X R_\xi(Y), Z) = 0$ for any vector fields $X$, $Y$ and $Z$, where $\mathfrak{S}$ denote the cyclic sum. Then we have $\beta \neq 0$. Our assumption $\mathfrak{S}g(\nabla_X R_\xi(Y), Z) = 0$ for any vector fields $X$, $Y$ and $Z$ is equivalent to $\Xi_{ij;k} + \Xi_{jk;i} + \Xi_{ki;j} = 0$. This equation is rewritten as

$$
\begin{aligned}
(6.1)\qquad & \alpha_k h_{ij} + \alpha_i h_{jk} + \alpha_j h_{ki} + \alpha(h_{ijk} + h_{jki} + h_{kij}) \\
& - h_{1j}h_{1ik} - h_{1k}h_{1ji} - h_{1i}h_{1kj} - h_{1j}h_{1ki} - h_{1k}h_{1ij} - h_{1i}h_{1jk} \\
& + \alpha h_{1j}(\Gamma_{1ik} + \Gamma_{1ki}) + \alpha h_{1k}(\Gamma_{lji} + \Gamma_{1ij}) + \alpha h_{1i}(\Gamma_{1kj} + \Gamma_{1jk}) \\
& + \beta h_{1j}(\Gamma_{2ik} + \Gamma_{2ki}) + \beta h_{1k}(\Gamma_{2ji} + \Gamma_{2ij}) + \beta h_{1i}(\Gamma_{2kj} + \Gamma_{2jk}) \\
& + c\delta_{1j}(\Gamma_{1ik} + \Gamma_{1ki}) + c\delta_{1k}(\Gamma_{lji} + \Gamma_{1ij}) + c\delta_{1i}(\Gamma_{1kj} + \Gamma_{1jk}) \\
& - \alpha\sum_l h_{lj}(\Gamma_{lik} + \Gamma_{lkj}) - \alpha\sum_l h_{lk}(\Gamma_{lji} + \Gamma_{lij}) - \alpha\sum_l h_{li}(\Gamma_{lkj} + \Gamma_{ljk}) = 0,
\end{aligned}
$$

because of (3.1). Then the equation (6.1) can be stated as follows:

$$(6.2)\qquad \varepsilon = 0,$$

$$(6.3)\qquad \alpha\delta + c = 0,$$

$$(6.4)\qquad (\beta^2 - \alpha\gamma)_1 = 0,$$

$$(6.5)\qquad (\alpha\gamma)_3 + 2(\beta^2 - \alpha\gamma - c)X_2 = 0,$$

$$(6.6)\qquad (\beta^2 - \alpha\gamma - c)(X_1 - \delta) = 0,$$

$$(6.7)\qquad (\beta^2 - \alpha\gamma - c)X_3 = 0.$$

Hereafter we shall use (6.2) without quoting. Then from Remark 1 we have equations (112)–(323) in Section 3. If $\beta^2 - \alpha\gamma - c = 0$, then by the same argument as that in Section 4 we have $\beta = 0$, which is a contradiction. Therefore $\beta^2 - \alpha\gamma - c \neq 0$. Then equations (6.6) and (6.7) imply

$$X_3 = 0, \quad X_1 = \delta. \tag{6.8}$$

It follows from (112), (313), (323), (3.3) and (212) that

$$\alpha_1 = \delta_1 = \alpha_2 = \delta_2 = \beta_1 = \beta_2 = \gamma_1 = 0. \tag{6.9}$$

From (312), (113) and (6.8) we have the following

$$\beta X_2 + (\beta^2 - \alpha\gamma - c) + \delta^2 = 0, \tag{6.10}$$

$$\alpha_3 + 4\beta\delta - \alpha\beta = 0. \tag{6.11}$$

Taking account of the coefficient of $\theta_1 \wedge \theta_3$ in the exterior derivative of (6.10), we have

$$\delta_3 = -\beta\delta - 2X_2\delta, \tag{6.12}$$

which, together with (6.10) and (6.11), implies

$$-2\beta^2\delta + \alpha\delta^2 + \alpha(\beta^2 - \alpha\gamma - c) = 0.$$

Taking account of the coefficient of $\theta_2$ in the exterior derivative of above equation, we have $\gamma_2 = 0$.

Now put $F = \alpha, \gamma$ and $i = 1$, $j = 2$ in Lemma 1. Then, we have

$$\alpha_3(\gamma + \delta) = \gamma_3(\gamma + \delta) = 0.$$

If $\gamma + \delta \neq 0$, then from (6.5) and (6.12) we have a contradicton. Thus $\gamma + \delta = 0$, which also contradicts (6.5) and (6.12). Hence $M$ is a Hopf hypersurface. Therefore from Proposition 1 we complete proof of following Theorem 7.

Theorem 7. *Let $M$ be a real hypersurface in $P_2\mathbf{C}$ or $H_2\mathbf{C}$. If the structure Jacobi operator is cyclic-parallel, then $M$ is locally congruent to one of the following*:

- *in case that $P_2\mathbf{C}$,*
  - $(A_1)$ *a geodesic hypersphere of radius $r$, where $0 < r < \pi/\sqrt{4c}$,*
  - $(B)$ *a tube of radius $r$ over complex quadric $Q_1$, where $\cot r = (\sqrt{2c+4} + \sqrt{2c})/2$,*
- *in case that $H_2\mathbf{C}$,*
  - $(A_0)$ *a horosphere,*
  - $(A_1)$ *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_1\mathbf{C}$.*

## References

[1] Berndt, J., Real hypersurfaces with constant principal curvatures in complex hyperblic spaces, J. Reine Angew. Math. **395**, 132–141 (1989)

[2] Berndt, J. and Tamaru, H., Cohomogeneity one actions on noncompact symmetric spaces of rank one, Trans. Amer. Math. Soc. **359**, 3425–3438 (2007)

[3] Cecil, T. E. and Ryan, P. J., Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. **269**, 481–499 (1982)

[4] Ki, U-H. and Kurihara, H., Real hypersurfaces with cyclic-parallel structure Jacobi operators in a nonflat complex space form, Bull. Aust. Math. Soc. **81**, 260–273 (2010)

[5] Ki, U-H., Kurihara, H. and Takagi, R., Jacobi operators along the structure flow on real hypersurfaces in a nonflat complex space form, Tsukuba J. Math. **33**, 39–56 (2009)

[6] Ki, U-H. and Suh, Y. J., On real hypersurfaces of a complex space form, Math. J. Okayama Univ. **32**, 207–221 (1990)

[7] Montiel, S. and Romero, A., On some real hypersurfaces of a complex hyperblic space, Geom. Dedicata **20**, 245–261 (1986)

[8] Okumura, M., On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. **212**, 355–364 (1975)

[9] Maeda, Y., On real hypersurfaces in a complex projective space, J. Math. Soc. Japan **28**, 529–540 (1976)

[10] Ortega, M., Pérez, J. D. and Santos, F. G., Non-existence of real hypersurfaces with parallel structure Jacobi operator in nonflat complex space forms, Rocky Mountain J. Math. **36**, 1603–1613 (2006)

[11] Pérez, J. D., Santos, F. G. and Suh, Y. J., Real hypersurfaces in complex projective spaces whose structure Jacobi operator is $D$-parallel, Bull. Belg. Math. Soc. Simon Stevin **13**, 459–469 (2006)

[12] Takagi, R., On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. **19**, 495–506 (1973)

[13] Takagi, R., Real hypersurfaces in a complex projective space with constant principal curvatures, J. Math. Soc. Japan **15**, 43–53 (1975)

[14] Takagi, R., Real hypersurfaces in a complex projective space with constant principal curvatures II, J. Math. Soc. Japan **15**, 507–516 (1975)

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