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Bayesian Inference of C-AR(1) Time Series Model with Structural Break

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Abstract. A variable may be affected by some associated variables which may influence the estimation and testing procedures and also not much important to model separately, such types of variables are called covariates. The present paper dealt the covariate autoregressive (C-AR(1)) time series model with structural break in mean and variance under Bayesian framework. Parameters of the model have been estimated considering appropriate prior assumptions and compared with maximum likelihood estimator. A simulation study has been carried out to validate the theoretical results, and then the same implemented on the monthly REER time series of SAARC countries. Both studies, empirical and simulation justify our findings. A unit root hypothesis is also tested for the model under study and gets satisfactory result.

Key words: Autoregressive Model; Bayesian Inference; Covariate; Structural break; Gibbs Sampler; Posterior Probability
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Résumé. Une variable peut être affectée par certaines autres variables associées qui peuvent influencer les procédures d'estimation et de test et qui ne sont pas non plus très importantes pour le modèle lorsque'elle sont prices séparément. De telles de variables sont dites covariables. Le présent article porte sur le modèle de séries chronologiques autorégressives (C-AR (1)) en présence de covariables, avec une rupture structurelle de la moyenne et de la variance dans le cadre bayésien. Les paramètres du modèle ont été estimés en avec des hypothèses appropriées sur la distribution à priori des paramètres. Les résuluats sont comparés avec ceux de l'estimation du maximum de vraisemblance. Une étude de simulation est réalisée pour valider les résultats théoriques. Les résultats sont ensuite appliqués à des données réelles.

1. Introduction

A time series is a sequence of observations recorded in a chronological order that are acquired over a time interval. In usual statistical investigation, most observations may be considered independent but in case of time series, it is rarely possible. If the dependency is linear and present observation depends only on the observation just before then it is called AR(1) process (Box and Jenkins (1970)). If there is any more variable associated with the series and is contributing significantly in model called covariate. The main motive behind the study of time series model with covariate is to make correct inferences about the impact of covariates on the response series.

There are very few papers explored with covariate. Hansen (1995) developed unit root test with some stationary covariates for autoregressive parameter. He proposed covariate augmented Dickey-Fuller (CADF) unit root test and obtained the asymptotic local power function of CADF statistic. This CADF test was further extended to a point-optimal covariate (POC) unit root test by Elliot and Jansoon (2003), power function, which is tangent to Gaussian power envelope at some point of the alternative. Juhl and Xiao (2003) explored the POC test by introducing the standard of optimality proposed by Cox and Hinkley (1974). Costantini and Lupi (2011) developed panel data model with stationary covariate which is the extension of Hansen (1995) model. Moreover, Chang *et al.* (2017) developed bootstrap unit root tests with covariate method to the CADF test to deal with the nuisance parameter dependency and provided a valid basis for inference based on the CADF test.

Sometimes unobserved variation known as structural break affects the structure of the model other than covariate, and if not taken into account it gives misleading conclusion. A vast amount of literature has been developed by many researchers for making inference in respect to structural break. Chaturvedi and Kumar (2007) described autoregressive model with single known break point in the trend component under Bayesian approach and derived posterior odds ratio (POR) to test unit root hypothesis. Fossati (2013) extended covariate unit root tests in the presence of structural break in trend function and contributed inference of unit root hypothesis that improves the power of correlated stationary covariates. Tsong *et al.* (2013) developed a covariate Fourier F-type unit root testing procedure under structural change in STAR dynamics model and considered the number of breaks and its location as unknown and estimating by the method describe in Becker *et al.* (2004) and Enders and Lee (2012). Tongkhow and Kantanantha (2013) studied the covariate in association with seasonality, trend and outlier to forecast the model. In their study, they used the Gibbs sampling and Markov Chain Monte Carlo (MCMC) algorithm for the parameters estimation. Garcia-Mora*et al.* (2016) explored the statistical flowgraph models involving covariate using both frequentist and Bayesian approaches and proposed an easy way to perform a Bayesian approach with covariate in reference to bladder carcinoma data.

The present paper deals with Covariate Autoregressive of order one (C-AR(1)) time series model. The parameters of the models are estimated under Bayesian framework. AR process is usually used if it is non-stationary, and non-stationarity may occur due to unit root. Therefore, unit root hypothesis is tested before the estimation. The posterior odds ratio has been derived under appropriate prior assumption and then the parameter of the model is estimated. The main advantage of the present model is that it allows for the break in mean, error variance, as well as in covariate. Simulation and empirical studies of the model are also used to identify the stationarity or non-stationarity series.

2. Structural break model

Let us assume that$\{y_t; t = 1, 2, \dots, T\}$ is a time series considering structural break in mean at a single time point T_B where mean of the series changed from μ_1 to μ_2

$$y_t = \begin{cases} \mu_1 + u_t & \text{for } t = 1, 2, ..., T_B \\ \mu_2 + u_t & \text{for } t = T_B + 1, T_B + 2, ..., T \end{cases} \tag{1}$$

where error the term ut follows AR(1) process serially correlated with a stationary covariate $\{w_t\}$

$$u_t = \begin{cases} \rho u_{t-1} + \sum\limits_{j=-r_1+1}^{p_1} \lambda_j w_{t-j} + \sigma_1 \varepsilon_t & \text{for } t \leq T_B \\ \rho u_{t-1} + \sum\limits_{j=-r_2+1}^{p_2} \lambda_j w_{t-j} + \sigma_2 \varepsilon_t & \text{for } t > T_B \end{cases} \tag{2}$$

Errors process is also having shifted both error variance and covariate's coefficients at same time point T_B. Here $\{\varepsilon_t; t = 1, 2, \dots, T\}$ are disturbance variable having mean zero and unknown variance. Model (1) can be written by using equation (2) as

$$y_t = \begin{cases} \rho y_{t-1} + (1-\rho)\mu_1 + \sum\limits_{j=-r_1+1}^{p_1} \lambda_j w_{t-j} + \sigma_1 \varepsilon_t & \text{for } t \leq T_B \\ \rho y_{t-1} + (1-\rho)\mu_2 + \sum\limits_{j=-r_2+1}^{p_2} \lambda_j w_{t-j} + \sigma_2 \varepsilon_t & \text{for } t > T_B \end{cases} \tag{3}$$

We are also interested to test the unit root hypothesis for the above model, $H_0 : \rho = 1$ against the alternative $H_1 : \rho \,\epsilon\, S$ with $S = \{a < \rho < 1; a > -1\}$. Under the null hypothesis of unit root, the model reduces to

$$\Delta y_t = \begin{cases} \sum\limits_{j=-r_1+1}^{p_1} \lambda_j w_{t-j} + \sigma_1 \varepsilon_t & \text{for } t \leq T_B \\ \sum\limits_{j=-r_2+1}^{p_2} \lambda_j w_{t-j} + \sigma_2 \varepsilon_t & \text{for } t > T_B \end{cases} \tag{4}$$

The model is given in equation (3) and (4) may be written in matrix notation

$$\begin{aligned} y_{T_B} &= \rho y_{-1}^{T_B} + (1-\rho)\,\mu_1 l_{T_B} + W_{T_B}\Lambda_1 + \sigma_1 \xi_{T_B} && \text{for } t \le T_B \\ y_{T-T_B} &= \rho y_{-1}^{T-T_B} + (1-\rho)\,\mu_2 l_{T-T_B} + W_{T-T_B}\Lambda_2 + \sigma_2 \xi_{T-T_B} && \text{for } t > T_B \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Delta y_{T_B} &= W_{T_B}\Lambda_1 + \sigma_1 \xi_{T_B} && \text{for } t \le T_B \\ \Delta y_{T-T_B} &= W_{T-T_B}\Lambda_2 + \sigma_2 \xi_{T-T_B} && \text{for } t > T_B \end{aligned} \tag{6}$$

where

$$\begin{aligned} y_{T_B} &= (y_1\ y_2\ \ldots\ y_{T_B})', \\ y_{-1}^{T_B} &= (y_0\ y_1\ \ldots\ y_{T_B-1})', \\ y_{T-T_B} &= (y_{T_B+1}\ y_{T_B+2}\ \ldots\ y_T)', \\ y_{-1}^{T-T_B} &= (y_{T_B}\ y_{T_B+1}\ \ldots\ y_{T-1})', \\ l_{T_B} &= (1\ 1\ \ldots\ 1)', \\ l_{T-T_B} &= (1\ 1\ \ldots\ 1)', \\ \Lambda_1 &= \left(\lambda_{-r_1+1}^{(1)}\ \lambda_{-r_1+2}^{(1)}\ \ldots\ \lambda_{p_1}^{(1)}\right)', \\ \Lambda_2 &= \left(\lambda_{-r_2+1}^{(2)}\ \lambda_{-r_2+2}^{(2)}\ \ldots\ \lambda_{p_2}^{(2)}\right)', \\ \xi_{T_B} &= (\varepsilon_1\ \varepsilon_2\ \ldots\ \varepsilon_{T_B})', \\ \xi_{T-T_B} &= (\varepsilon_{T_B+1}\ \varepsilon_{T_B+2}\ \ldots\ \varepsilon_T)', \\ W_{T_B} &= \begin{pmatrix} w_{r_1} & w_{r_1-1} & \cdots & w_{1-p_1} \\ w_{r_1-1} & w_{r_1} & \cdots & w_{2-p_1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{T_B+r_1-1} & w_{T_B+r_1-2} & \cdots & w_{T_B-p_1} \end{pmatrix}, \\ W_{T-T_B} &= \begin{pmatrix} w_{T_B+r_2} & w_{T_B+r_2-1} & \cdots & w_{T_B+1-p_2} \\ w_{T_B+r_2+1} & w_{T_B+r_2} & \cdots & w_{T_B+2-p_2} \\ \vdots & \vdots & \ddots & \vdots \\ w_{T_B+r_2-1} & w_{T_B+r_2-2} & \cdots & w_{T_B-p_2} \end{pmatrix}. \end{aligned} \tag{7}$$

3. Bayesian Analysis

Let us assume prior for mean term (μ_i) and error variance (σ_i^2) as conjugate normal prior $N(\mu_i^{'}, \sigma_i^2)$ and inverse gamma prior $IG(a_i, b_i)$ respectively and uniform prior for autoregressive coefficient (ρ) and covariate coefficient (λ_i). Then the joint prior distribution of model parameters is

$$\begin{aligned} P(\Theta) = \frac{b_1^{a_1} b_2^{a_2} \left(\sigma_1^2\right)^{-a_1-\frac{3}{2}} \left(\sigma_2^2\right)^{-a_2-\frac{3}{2}}}{2\pi\Gamma(a_1)\Gamma(a_2)(1-a)} exp\Bigg[&-\frac{1}{2\sigma_1^2}\left\{\left(\mu_1-\mu_1^{'}\right)^2 + 2b_1\right\} \\ &-\frac{1}{2\sigma_2^2}\left\{\left(\mu_2-\mu_2^{'}\right)^2 + 2b_2\right\}\Bigg] \end{aligned} \tag{8}$$

where i=1,2. First, we have obtained the posterior odds ratio to test the unit root hypothesis and then estimated the parameters of model. Let us define the following notations to derive the posterior odds ratio:

$$
\begin{aligned}
\Sigma_1 =& I - W_{T_B}^{'}\left(W_{T_B}^{'}W_{T_B}\right)^{-1}W_{T_B},\\
\Sigma_2 =& I - W_{T-T_B}^{'}\left(W_{T-T_B}^{'}W_{T-T_B}\right)^{-1}W_{T-T_B},\\
A(\rho) =& \left(y_{T_B}-\rho y_{-1}^{T_B}\right)^{'}\Sigma_1\left(y_{T_B}-\rho y_{-1}^{T_B}\right)+\left(\mu_1^{'}\right)^2+2b_1\\
&-\left[l_{T_B}^{'}(1-\rho)\Sigma_1\left(y_{T_B}-\rho y_{-1}^{T_B}\right)+\mu_1^{'}\right]^{'}\left[l_{T_B}^{'}(1-\rho)^2\Sigma_1 l_{T_B}+1\right]^{-1}\\
&\left[l_{T_B}^{'}(1-\rho)\Sigma_1\left(y_{T_B}-\rho y_{-1}^{T_B}\right)+\mu_1^{'}\right],\\
B(\rho) =& \left(y_{T-T_B}-\rho y_{-1}^{T-T_B}\right)^{'}\Sigma_2\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}\right)+\left(\mu_2^{'}\right)^2+2b_2\\
&-\left[l_{T-T_B}^{'}(1-\rho)\Sigma_2\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}\right)+\mu_2^{'}\right]^{'}\left[l_{T-T_B}^{'}(1-\rho)^2\Sigma_2 l_{T-T_B}+1\right]^{-1}\\
&\left[l_{T-T_B}^{'}(1-\rho)\Sigma_2\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}\right)+\mu_2^{'}\right].
\end{aligned}
\tag{9}
$$

Theorem: For testing the unit root hypothesis $H_0 : \rho = 1$against the alternative hypothesis $H_1 : \rho \ \epsilon \ S; S = \{a < \rho < 1; a > -1\}$ with prior odds ratio $\frac{p_0}{1-p_0}$, the posterior odds ratio denoted by β_{01} for the covariate model is derived as

$$
\begin{aligned}
\beta_{01} =& \frac{p_0}{1-p_0}\int\limits_a^1\left[\frac{|1+(1-\rho)^2 l_{T_B}^{'}\Sigma_1 l_{T_B}|^{-\frac{1}{2}}|1+(1-\rho)^2 l_{T-T_B}^{'}\Sigma_2 l_{T-T_B}|^{-\frac{1}{2}}}{A(\rho)^{\frac{T_B}{2}+a_1-\frac{p_1+r_1}{2}}B(\rho)^{\frac{T-T_B}{2}+a_2-\frac{p_2+r_2}{2}}}\right]^{-1}d\rho\\
&\left[\frac{(1-a)}{\left(\Delta y_{T_B}^{'}\Sigma_1\Delta y_{T_B}+2b_1\right)^{\frac{T_B}{2}+a_1-\frac{p_1+r_1}{2}}\left(\Delta y_{T-T_B}^{'}\Sigma_2\Delta y_{T-T_B}+2b_2\right)^{\frac{T-T_B}{2}+a_2-\frac{p_2+r_2}{2}}}\right]
\end{aligned}
\tag{10}
$$

Proof. See appendix for the proof of the theorem

In Bayesian framework, the estimation of parameters is obtained by the use of posterior probabilities. The posterior probability gives the information about the parameter(s) under the assumed prior and is obtained by integrating the joint distribution of the model. By using the mathematical manipulation, we get the posterior distribution of $\mu_1, \mu_2, \Lambda_1, \Lambda_2, \sigma_1, \sigma_2$ and ρ:

$$
\begin{aligned}
&\mu_1 \sim N\left(S_1M_1^{-1}, \sigma_1^2M_1^{-1}\right),\\
&\mu_2 \sim N\left(S_2M_2^{-1}, \sigma_2^2M_2^{-1}\right),\\
&\Lambda_1 \sim N\left(S_3M_3^{-1}, \sigma_1^2M_3^{-1}\right),\\
&\Lambda_2 \sim N\left(S_4M_4^{-1}, \sigma_2^2M_4^{-1}\right),\\
&\sigma_1^2 \sim IG(\alpha_1, \beta_1),\\
&\sigma_2^2 \sim IG(\alpha_2, \beta_2),\\
&\rho \sim TN\left(\left(\frac{S_5}{\sigma_1^2}+\frac{S_6}{\sigma_2^2}\right)\left(\frac{M_5}{\sigma_1^2}+\frac{M_6}{\sigma_2^2}\right)^{-1}\left(\frac{M_5}{\sigma_1^2}+\frac{M_6}{\sigma_2^2}\right)^{-1}, a, 1\right).
\end{aligned}
$$

where

$$\begin{aligned}
\alpha_1 &= \frac{T_B+1}{2}+a_1,\\
\alpha_2 &= \frac{T-T_B+1}{2}+a_2,\\
\beta_1 &= \frac{1}{2}\left[\left(y_{T_B}-\rho y_{-1}^{T_B}-(1-\rho)\mu_1 l_{T_B}-W_{T_B}\Lambda_1\right)^{'}\left(y_{T_B}-\rho y_{-1}^{T_B}-(1-\rho)\mu_1 l_{T_B}-W_{T_B}\Lambda_1\right)\right.\\
&\quad\left.+\left(\mu_1-\mu_1^{'}\right)^2+2b_1\right],\\
\beta_2 &= \frac{1}{2}\left[\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}-(1-\rho)\mu_2 l_{T-T_B}-W_{T-T_B}\Lambda_2\right)^{'}\right.\\
&\quad\left.\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}-(1-\rho)\mu_2 l_{T-T_B}-W_{T-T_B}\Lambda_2\right)+\left(\mu_2-\mu_2^{'}\right)^2+2b_2\right],\\
M_1 &= 1+(1-\rho)^2 l_{T_B}^{'} l_{T_B},\\
M_2 &= 1+(1-\rho)^2 l_{T-T_B}^{'} l_{T-T_B},\\
M_3 &= W_{T_B}^{'} W_{T_B},\\
M_4 &= W_{T-T_B}^{'} W_{T-T_B},\\
M_5 &= \left(y_{-1}^{T_B}-\mu_1 l_{T_B}\right)^{'}\left(y_{-1}^{T_B}-\mu_1 l_{T_B}\right),\\
M_6 &= \left(y_{-1}^{T-T_B}-\mu_1 l_{T-T_B}\right)^{'}\left(y_{-1}^{T-T_B}-\mu_2 l_{T-T_B}\right),\\
S_1 &= (1-\rho)\, l_{T_B}^{'}\left(y_{T_B}-\rho y_{-1}^{T_B}-W_{T_B}\Lambda_1\right)+\mu_1',\\
S_2 &= (1-\rho)\, l_{T-T_B}^{'}\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}-W_{T-T_B}\Lambda_2\right)+\mu_2',\\
S_3 &= W_{T_B}^{'}\left(y_{T_B}-\rho y_{-1}^{T_B}-(1-\rho)\mu_1 l_{T_B}\right),\\
S_4 &= W_{T-T_B}^{'}\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}-(1-\rho)\mu_2 l_{T-T_B}\right),\\
S_5 &= \left(y_{-1}^{T_B}-\mu_1 l_{T_B}\right)^{'}\left(y_{T_B}-\mu_1 l_{T_B}-W_{T_B}\Lambda_1\right),\\
S_6 &= \left(y_{-1}^{T-T_B}-\mu_2 l_{T-T_B}\right)^{'}\left(y_{T-T_B}-\mu_2 l_{T-T_B}-W_{T-T_B}\Lambda_2\right).
\end{aligned}$$

4. Simulation Study

Since the evaluation methodology has taken it's place, simulation has become very important in statistical researches. In present study, a simulation study has been performed for the proposed model. The C-AR(1) time series have been generated by taking the covariate from the model

$$w_t=\begin{cases}0.1+0.08w_{t-1}+\epsilon_t & for\ t\le T_B\\ 0.5+0.07w_{t-1}+\epsilon_t & for\ t>T_B\end{cases} \tag{11}$$

The C-AR(1) with structural break at $T_B = 60$ considering initial value y_0 =10 with size of the series $T = 100$ is generated from the model (3). To get more generalized idea about the model, the unit root hypothesis and estimation for the model under study is carry out for different value of $\rho, \lambda_1, \lambda_2, \mu_1, \mu_2, \sigma_1$ and σ_2 which are ρ=(0.90, 0.92, 0.94, 0.97, 0.98, 0.99), λ_1 = (0.5,1), λ_2 = (0.5, 1), μ_1 = (50,100), μ_2 = (100,200), $\sigma_1 = 1$ and $\sigma_2 = 2$. Hyper parameters of the inverse gamma distribution are $a_1 = 0.5$, $a_2 = 1$, $b_1 = 1.5$ and $b_2 = 2$. The normal prior distribution need the mean of the series therefore, hyper mean is assumed before and after break point is y_0 and $\bar{y}$ respectively. A single series cannot give the overall idea, therefore the process is repeated for 5000 times. The posterior odds ratio needs "a" which is obtained by the method discussed by Schotman and VanDijk (1991).

For generated series, first to test the unit root hypothesis using the derived posterior odds ratio. The values of POR are reported in Table 1 for small values of ρ=(0.90, 0.92, 0.94) and Table 2 for large values of ρ=(0.97, 0.98, 0.99) for all the combinations of assumed coefficients of the model. All estimated posterior odds ratio are less than one under assumed equal prior odds ratio. It shows that posterior probability of alternative hypothesis is more than the null hypothesis. Therefore the study conclude that the series are trend stationary. This is correctly following the C-AR(1) model.

Table 1. Bayesian unit root test with lower value of ρ

P			λ_1=0.50, λ_2=0.50		λ_1=0.50, λ_2=1		λ_1=1, λ_2=0.50		λ_1=1, λ_2=1	
μ_1	μ_2	ρ	POR	$\hat{\rho}$	POR	$\hat{\rho}$	POR	$\hat{\rho}$	POR	$\hat{\rho}$
		0.90	4.99E-12	0.9004	7.83E-17	0.8867	4.32E-18	0.8936	3.41E-20	0.9032
50	100	0.92	9.52E-16	0.9214	7.43E-21	0.9184	1.71E-23	0.9175	1.05E-22	0.9152
		0.94	2.18E-22	0.9398	1.68E-17	0.9392	2.67E-20	0.9327	1.93E-21	0.9409
		0.90	1.95E-05	0.8986	1.64E-09	0.8962	1.30E-06	0.8937	1.44E-07	0.8925
100	150	0.92	3.68E-06	0.9142	2.08E-13	0.9129	2.91E-11	0.9198	4.26E-14	0.9136
		0.94	6.00E-10	0.9379	3.89E-11	0.936	7.11E-12	0.9402	2.63E-15	0.9406
		0.90	3.09E-02	0.8931	1.44E-05	0.8916	5.97E-06	0.9006	4.88E-07	0.9011
150	200	0.92	2.28E-04	0.9177	1.42E-06	0.9153	2.39E-05	0.9178	3.74E-08	0.9191
		0.94	1.10E-06	0.9394	4.08E-07	0.9413	5.28E-09	0.9365	3.42E-12	0.9410

After testing the unit root hypothesis, the Bayes estimate of model parameters (P) are obtained under squared error loss function by using the Gibbs sampling algorithm. The performance of Bayes estimate is compared with the classical estimate by using of credible interval. The obtained expression needs the numerical integration for which we have used Chen and Shao (1994) method. Table 3 provides the average biases and mean square error (MSE) with respect to MLE and Bayes estimate over 10000 replications. We have also obtained 95% confidence interval based on highest posterior density (HD) interval. From Table 3, it is observed that as the ρ increases the MSE and Bias is also increases for MLE estimate, but in case of Bayes estimates, it decreases. The approximate confidence intervals obtained from Bayes estimate also decreases compared to MLE. Therefore, it is seen that Bayesian approach provides better results with respect to MLE.

Table 2. Bayesian unit root test with higher value of ρ

P			λ_1=0.50, λ_2=0.50		λ_1=0.50, λ_2=1		λ_1=1, λ_2=0.50		λ_1=1, λ_2=1	
μ_1	μ_2	ρ	POR	$\hat{\rho}$	POR	$\hat{\rho}$	POR	$\hat{\rho}$	POR	$\hat{\rho}$
		0.97	2.58E-20	0.9614	2.09E-21	0.9648	7.89E-25	0.9671	1.04E-25	0.9652
50	100	0.98	2.29E-20	0.9810	2.27E-23	0.9783	8.77E-24	0.9726	3.71E-27	0.9774
		0.99	7.20E-26	0.9901	8.44E-21	0.9826	9.05E-26	0.9936	2.26E-24	0.9833
		0.97	2.19E-18	0.9724	6.30E-14	0.9688	1.28E-22	0.9679	2.16E-15	0.9659
100	150	0.98	5.78E-18	0.9757	3.90E-25	0.9755	1.44E-21	0.9819	1.37E-21	0.9789
		0.99	7.95E-19	0.9871	6.90E-25	0.9849	1.66E-21	0.9896	2.57E-30	0.9867
		0.97	1.37E-11	0.9678	6.71E-12	0.9688	8.40E-12	0.9692	6.54E-18	0.9708
150	200	0.98	4.47E-13	0.9819	6.31E-14	0.9786	7.28E-17	0.9801	1.37E-16	0.9770
		0.99	1.49E-22	0.9889	1.47E-19	0.9850	8.98E-23	0.9870	6.05E-23	0.9923

Table 3. Biases, MSEs and Confidence Intervals of C-AR(1) Model Parameters

		MLE			Bayes		
	P	MSE	Bias	CI	MSE	Bias	CI
ρ = 0.90	ρ	6.21E-06	-1.04E-04	[8.95E-01, 9.04E-01]	4.59E-06	-6.60E-05	[8.96E-01, 9.04E-01]
	μ_1	2.88E-01	-2.76E-03	[9.88E+01, 1.01E+02]	2.07E-02	-3.94E-04	[9.97E+01, 1.00E+02]
	μ_2	7.34E-01	1.75E-02	[1.98E+02, 2.02E+02]	3.37E-02	3.20E-03	[2.00E+02, 2.00E+02]
	λ_1	1.80E-03	-1.75E-03	[1.91E+00, 2.08E+00]	1.44E-03	-1.73E-03	[1.93E+00, 2.08E+00]
	λ_2	5.79E-03	-2.02E-03	[2.85E+00, 3.15E+00]	4.43E-03	-1.19E-03	[2.87E+00, 3.12E+00]
	σ_1	3.43E-04	1.96E-03	[6.57E-02, 1.38E-01]	3.06E-04	-1.65E-03	[6.37E-02, 1.32E-01]
	σ_2	2.27E-03	2.15E-03	[1.13E-01, 2.93E-01]	2.05E-03	-6.54E-03	[1.15E-01, 2.84E-01]
ρ = 0.95	ρ	5.74E-06	-5.32E-06	[9.45E-01, 9.54E-01]	3.01E-06	3.90E-05	[9.47E-01, 9.53E-01]
	μ_1	1.49E+00	-1.81E-02	[9.77E+01, 1.03E+02]	6.77E-03	2.10E-03	[9.98E+01, 1.00E+02]
	μ_2	3.28E+00	3.48E-02	[1.97E+02, 2.04E+02]	1.05E-02	1.14E-03	[2.00E+02, 2.00E+02]
	λ_1	1.85E-03	1.17E-03	[1.92E+00, 2.09E+00]	1.50E-03	1.12E-03	[1.92E+00, 2.07E+00]
	λ_2	5.36E-03	7.26E-04	[2.85E+00, 3.13E+00]	4.18E-03	4.71E-04	[2.88E+00, 3.13E+00]
	σ_1	3.58E-04	1.49E-03	[6.27E-02, 1.35E-01]	3.26E-04	-2.50E-03	[6.61E-02, 1.33E-01]
	σ_2	2.30E-03	1.32E-03	[9.91E-02, 2.88E-01]	2.07E-03	-8.00E-03	[1.02E-01, 2.77E-01]
ρ = 0.97	ρ	6.23E-06	-1.02E-04	[9.65E-01, 9.75E-01]	2.37E-06	-1.97E-05	[9.67E-01, 9.73E-01]
	μ_1	5.38E+00	-6.29E-02	[9.57E+01, 1.05E+02]	2.55E-03	2.18E-03	[9.99E+01, 1.00E+02]
	μ_2	1.07E+01	1.54E-01	[1.94E+02, 2.07E+02]	4.18E-03	-1.48E-04	[2.00E+02, 2.00E+02]
	λ_1	1.68E-03	1.66E-03	[1.92E+00, 2.07E+00]	1.46E-03	1.63E-03	[1.93E+00, 2.07E+00]
	λ_2	5.49E-03	5.37E-04	[2.84E+00, 3.13E+00]	4.07E-03	1.50E-03	[2.87E+00, 3.13E+00]
	σ_1	3.65E-04	1.63E-03	[6.72E-02, 1.40E-01]	3.33E-04	-2.33E-03	[6.31E-02, 1.30E-01]
	σ_2	2.06E-03	-1.41E-03	[1.13E-01, 2.85E-01]	1.92E-03	-1.10E-02	[1.15E-01, 2.79E-01]
ρ = 0.99	ρ	5.41E-06	-1.81E-04	[9.86E-01, 9.95E-01]	1.59E-06	-4.81E-05	[9.88E-01, 9.92E-01]
	μ_1	1.05E+02	-1.12E+00	[8.21E+01, 1.15E+02]	3.48E-04	2.64E-03	[1.00E+02, 1.00E+02]
	μ_2	4.57E+02	-1.04E-01	[1.82E+02, 2.20E+02]	7.83E-04	-2.00E-03	[2.00E+02, 2.00E+02]
	λ_1	1.94E-03	-8.32E-05	[1.92E+00, 2.09E+00]	1.62E-03	1.16E-05	[1.92E+00, 2.08E+00]
	λ_2	5.44E-03	3.11E-04	[2.86E+00, 3.15E+00]	4.20E-03	-1.00E-03	[2.86E+00, 3.11E+00]
	σ_1	3.72E-04	1.86E-03	[6.68E-02, 1.39E-01]	3.43E-04	-1.89E-03	[6.44E-02, 1.35E-01]
	σ_2	2.04E-03	1.29E-03	[1.15E-01, 2.84E-01]	1.83E-03	-9.16E-03	[1.13E-01, 2.72E-01]

5. Empirical Analysis

Statistics is the branch of science that provides various tools and techniques to analysis data and derived conclusions. Time series is one such technique which gives a better modelling of economic series for the purpose of explanation and forecasting. Therefore we have modelled the real effective exchange rate (REER) series of South Asian Association for Regional Cooperation (SAARC) countries under proposed framework. The monthly time series of REER of SAARC countries from January 2009 to May 2017 is taken. REER measures the development of the real value of a country's currency against the basket of the trading partners of the country. It is a frequently used variable in both theoretical and applied economic researches. SAARC is the regional organization of South Asia countries, namely Afghanistan, Bangladesh, Bhutan, India, Nepal, Maldives, Pakistan and Sri Lanka. It brings all the associated countries together for strengthen the economical, technological, social and cultural development. It is also provides support and assistance for establishing of relations with developed nations.

India is one of the world fastest growing economies and may therefore affect the associated countries. Considering this, modelling of REER of SAARC countries has been done with India's REER as covariate assuming single break point. In February, 2014 economic conditions changed due to slumping oil prices and volatility in Indian market. This was also the time when changes in ruling party were initiated. The new government was formed in May, 2014. All SAARC countries are having break at T_B=63 except Bhutan country and Maldives. The most preferred break point and its position are summarized in Table 4 and estimates of C-AR(1) parameters (P) are reported in Table 5. Table 6 recorded the confidence interval using HD interval. Estimation has been carried out using MLE and Bayes method of SAARC countries. The value of POR for SAARC countries strongly favours the rejection of unit root hypothesis. It may be due to the participation of India among the SAARC countries in their economic growth as well as their political, diplomatic and security concerns. Among all the countries, Sir Lanka's economic series contain minimum AIC and BIC values due to assumed covariate of Indian economy. This may have happened because of the India–Sir Lanka free trade agreement, the India-Sir Lanka intergovernmental initiate and the comprehensive economic partnership agreement.

Table 4. Summary of monthly series of SAARC Countries

Country	Number of Breaks	TB
India	1	63
Nepal	1	63
Pakistan	1	63
Bangladesh	1	63
Afghanistan	1	63
Sri Lanka	1	63
Bhutan	1	38
Maldives	NA	NA

Table 5. Estimates of Parameters, POR, AIC and BIC of C-AR(1) Model

P	Nepal		Pakistan		Bangladesh		Afghanistan		Sri Lanka	
	MLE	Bayes	MLE	Bayes	MLE	Bayes	MLE	Bayes	MLE	Bayes
ρ	0.7955	0.8971	0.9344	0.938	0.9037	0.9144	0.9055	0.9148	0.9275	0.9339
μ_1	0.3277	0.9334	3.3933	0.9777	1.4267	0.935	2.007	0.9344	1.9073	0.9748
μ_2	0.6241	1.0811	0.9513	1.0776	0.9277	1.0798	0.9839	1.0804	1.8359	1.093
λ_1	0.1402	0.0149	-0.1502	0.0023	-0.0323	0.0104	-0.0846	0.0109	-0.0539	0.003
λ_2	0.094	0.0108	0.0143	0.0077	0.0174	0.0075	0.0163	0.009	-0.0377	0.0075
σ_1	0.0003	0.0016	0.0003	0.0022	0.0003	0.001	0.0003	0.001	0.0003	0.0013
σ_2	0.0002	0.0009	0.0002	0.0007	0.0002	0.0008	0.0002	0.0008	0.0002	0.0011
POR	2.14E-03		6.90E-04		7.89E-04		7.63E-04		6.89E-04	
AIC	155.9351		114.4217		114.3346		114.585		109.7219	
BIC	169.0107		127.4973		127.4102		127.6606		122.7975	

Table 6. Bayesian Confidence Interval

P	Nepal	Pakistan	Bangladesh	Afghanistan	Sri Lanka
ρ	[0.7328, 0.9998]	[0.8411, 0.9922]	[0.8140, 0.9931]	[0.8151, 0.9926]	[0.8347, 0.9925]
μ_1	[0.7922, 1.0485]	[0.8539, 1.0148]	[0.8457, 1.0283]	[0.8436, 1.0261]	[0.8471, 1.0172]
μ_2	[1.0084, 1.1467]	[1.0208, 1.1402]	[1.0227, 1.1432]	[1.0201, 1.1398]	[1.0225, 1.1402]
λ_1	[-0.0086, 0.0559]	[-0.0038,0.0275]	[-0.0041, 0.0301]	[-0.0048, 0.0311]	[-0.0038, 0.0254]
λ_2	[-0.0038, 0.0293]	[-0.0012, 0.0195]	[-0.0011, 0.0179]	[-0.0021, 0.0209]	[-0.0006, 0.0203]
σ_1	[0.0003, 0.0047]	[0.0004, 0.0014]	[0.0004, 0.0018]	[0.0004, 0.0018]	[0.0004, 0.0015]
σ_2	[0.0004, 0.0018]	[0.0004, 0.0012]	[0.0004, 0.0012]	[0.0004, 0.0013]	[0.0004, 0.0012]

6. Conclusion

Structural break may occur in economic time series due to various reasons such as changes in financial and political polices, environmental factors etc. The autoregressive time series model with covariate has been explored considering break in mean and variance. We have noticed similar break point in six other countries in February, 2014. This was the time when political shift in central government was under discussion and a new government was formed in May, 2014. It was also the time when economic conditions changed due to slumping oil prices and volatility in Indian market. The posterior odd ratio shows that proposed C-AR(1) model satisfied the stationary condition of the REER series. The proposed model may be extended to other non-normal priors and non-normal errors as well as for accommodating multiple breaks and temporary shifts.

Data Source: The monthly REER data collected from January 2009 to May 2017, http://bruegel.org/publications/datasets/real-effective-exchange-rates-for-178-countries-a-new-database/.

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Appendix

Likelihood function under unit root hypothesis

$$P\left(y|\Lambda_1,\Lambda_2,\sigma_1,\sigma_2\right)=\left(\frac{1}{2\pi}\right)^{\frac{T}{2}}\left(\frac{1}{\sigma_1^2}\right)^{\frac{T_B}{2}}\exp\left[-\frac{1}{2\sigma_1^2}\left(\Delta y_{T_B}-W_{T_B}\Lambda_1\right)'\left(\Delta y_{T_B}-W_{T_B}\Lambda_1\right)\right]$$
$$\left(\frac{1}{\sigma_2^2}\right)^{\frac{T-T_B}{2}}\exp\left[-\frac{1}{2\sigma_2^2}\left(\Delta y_{T-T_B}-W_{T-T_B}\Lambda_2\right)'\left(\Delta y_{T-T_B}-W_{T-T_B}\Lambda_2\right)\right] \tag{12}$$

Posterior distribution under H_0

$$\begin{aligned}P\left(y|H_0\right)=&\int_0^{\infty}\int_0^{\infty}\int_{R^{p_1+r_1}}\int_{R^{p_2+r_2}}\left(\frac{b_1^{a_1}b_2^{a_2}\left(\sigma_1^2\right)^{-\frac{T_B}{2}-a_1-1}\left(\sigma_2^2\right)^{-\frac{T-T_B}{2}-a_2-1}}{(2\pi)^{\frac{T}{2}}\Gamma(a_1)\Gamma(a_2)}\right)\\&\exp\left[-\frac{1}{2\sigma_1^2}\left\{\left(\Delta y_{T_B}-W_{T_B}\Lambda_1\right)'\left(\Delta y_{T_B}-W_{T_B}\Lambda_1\right)-2b_1\right\}\right]d\Lambda_1 d\sigma_1^2\\&\exp\left[-\frac{1}{2\sigma_2^2}\left\{\left(\Delta y_{T-T_B}-W_{T-T_B}\Lambda_2\right)'\left(\Delta y_{T-T_B}-W_{T-T_B}\Lambda_2\right)-2b_2\right\}\right]d\Lambda_2 d\sigma_2^2\end{aligned} \tag{13}$$

Solving equation (13) for $\Lambda_1, \Lambda_2, \sigma_1$and σ_2 by supposing $\hat{\Lambda}_1 = W'_{T_B}\left(W'_{T_B}W_{T_B}\right)^{-1}\Delta y_{T_B}$ and $\hat{\Lambda}_2 = W'_{T-T_B}\left(W'_{T-T_B}W_{T-T_B}\right)^{-1}\Delta y_{T-T_B}$. We will get required solution

$$P(y|H_0) = \frac{2^{a_1+a_2}b_1^{a_1}b_2^{a_2}\Gamma\left(\frac{T_B}{2}+a_1-\frac{p_1+r_1}{2}\right)\Gamma\left(\frac{T-T_B}{2}+a_2-\frac{p_2+r_2}{2}\right)}{\Gamma(a_1)\Gamma(a_2)\pi^{\frac{T-(p_1+r_1+p_2+r_2)}{2}}} \left[\frac{|W'_{T_B}W_{T_B}|^{-\frac{1}{2}}|W'_{T-T_B}W_{T-T_B}|^{-\frac{1}{2}}}{\left(\Delta y'_{T_B}\Sigma_1\Delta y_{T_B}+2b_1\right)^{\frac{T_B}{2}+a_1-\frac{p_1+r_1}{2}}\left(\Delta y'_{T-T_B}\Sigma_2\Delta y_{T-T_B}+2b_2\right)^{\frac{T-T_B}{2}+a_2-\frac{p_2+r_2}{2}}}\right] \tag{14}$$

Likelihood function under alternative hypothesis

$$\begin{aligned}
&P\left(y|\rho,\mu_1,\mu_2,\Lambda_1,\Lambda_2,\sigma_1,\sigma_2\right)\\
&=\left(\frac{1}{2\pi}\right)^{\frac{T}{2}}\left(\frac{1}{\sigma_1^2}\right)^{\frac{T_B}{2}}\left(\frac{1}{\sigma_2^2}\right)^{\frac{T-T_B}{2}}\exp\left[-\frac{1}{2\sigma_1^2}\left(y_{T_B}-\rho y_{-1}^{T_B}-(1-\rho)\mu_1 l_{T_B}-W_{T_B}\Lambda_1\right)'\right.\\
&\left.\left(y_{T_B}-\rho y_{-1}^{T_B}-(1-\rho)\mu_1 l_{T_B}-W_{T_B}\Lambda_1\right)\right]\exp\left[-\frac{1}{2\sigma_2^2}\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}-(1-\rho)\mu_2 l_{T-T_B}-W_{T-T_B}\Lambda_2\right)'\right.\\
&\left.\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}-(1-\rho)\mu_2 l_{T-T_B}-W_{T-T_B}\Lambda_2\right)\right]
\end{aligned} \tag{15}$$

Posterior distribution under H_1

$$\begin{aligned}
P\left(y|H_1\right) =& \int_a^1\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{R^{p_1+r_1}}\int_{R^{p_2+r_2}}\left(\frac{b_1^{a_1}b_2^{a_2}\left(\sigma_1^2\right)^{-\frac{T_B+1}{2}-a_1}\left(\sigma_2^2\right)^{-\frac{T-T_B+1}{2}-a_2}}{(2\pi)^{\frac{T}{2}+1}\Gamma(a_1)\Gamma(a_2)(1-a)}\right)\\
&\exp\left[-\frac{1}{2\sigma_1^2}\left(y_{T_B}-\rho y_{-1}^{T_B}-(1-\rho)\mu_1 l_{T_B}-W_{T_B}\Lambda_1\right)'\left(y_{T_B}-\rho y_{-1}^{T_B}-(1-\rho)\mu_1 l_{T_B}-W_{T_B}\Lambda_1\right)\right.\\
&\left.+\left(\mu_1-\mu_1'\right)^2+2b_1\right]\exp\left[-\frac{1}{2\sigma_2^2}\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}-(1-\rho)\mu_2 l_{T-T_B}-W_{T-T_B}\Lambda_2\right)'\right.\\
&\left.\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}-(1-\rho)\mu_2 l_{T-T_B}-W_{T-T_B}\Lambda_2\right)+\left(\mu_2-\mu_2'\right)^2+2b_2\right]d\Lambda_1 d\Lambda_2\\
&d\mu_1 d\mu_2\sigma_1^2 d\sigma_2^2 d\rho
\end{aligned} \tag{16}$$

Solving equation (16) for $\Lambda_1, \Lambda_2, \mu_1, \mu_2, \rho, \sigma_1$and σ_2 by supposing

$$\begin{aligned}
\tilde{\Lambda}_1 =& W'_{T_B}\left(W'_{T_B}W_{T_B}\right)^{-1}\left(y_{T_B}-\rho y_{-1}^{T_B}-(1-\rho)\mu_1 l_{T_B}\right)\\
\tilde{\Lambda}_2 =& W'_{T-T_B}\left(W'_{T-T_B}W_{T-T_B}\right)^{-1}\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}-(1-\rho)\mu_2 l_{T-T_B}\right)\\
\tilde{\mu}_1 =& \left[(1-\rho)l_{T_B}\Sigma_1\left(y_{T_B}-\rho y_{-1}^{T_B}\right)+\mu_1'\right]\left[1+(1-\rho)^2 l'_{T_B}\Sigma_1 l_{T_B}\right]^{-1}\\
\tilde{\mu}_2 =& \left[(1-\rho)l_{T-T_B}\Sigma_2\left(y_{T-T_B}-\rho y_{-1}^{T-T_B}\right)+\mu_2'\right]\left[1+(1-\rho)^2 l'_{T-T_B}\Sigma_2 l_{T-T_B}\right]^{-1}
\end{aligned}$$

We will get required solution

$$P(y|H_1) = \frac{2^{a_1+a_2} b_1^{a_1} b_2^{a_2} \Gamma\left(\frac{T_B}{2} + a_1 - \frac{p_1+r_1}{2}\right) \Gamma\left(\frac{T-T_B}{2} + a_2 - \frac{p_2+r_2}{2}\right) |W'_{T_B} W_{T_B}|^{-\frac{1}{2}} |W'_{T-T_B} W_{T-T_B}|^{-\frac{1}{2}}}{\Gamma(a_1)\Gamma(a_2)\pi^{\frac{T-(p_1+r_1+p_2+r_2)}{2}}}$$
$$\int_a^1 \left[\frac{|1+(1-\rho)^2 l'_{T_B} \Sigma_1 l_{T_B}|^{-\frac{1}{2}} |1+(1-\rho)^2 l'_{T-T_B} \Sigma_2 l_{T-T_B}|^{-\frac{1}{2}}}{(1-a) A(\rho)^{\frac{T_B}{2}+a_1-\frac{p_1+r_1}{2}} B(\rho)^{\frac{T-T_B}{2}+a_2-\frac{p_2+r_2}{2}}} \right]^{-1} d\rho \tag{17}$$