Banach J. Math. Anal. 8 (2014), no. 2, 139–145

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

GEOMETRIC PROPERTIES OF THE LUPAŞ q-TRANSFORM

SOFIYA OSTROVSKA

Communicated by L.-E. Persson

ABSTRACT. The *Lupaş q-transform* emerges in the study of the limit q-Lupaş operator. This transform is closely connected to the theory of positive linear operators of approximation theory, the q-boson operator calculus, the methods of summation of divergent series, and other areas.

Given $q \in (0,1),\ f \in C[0,1]$, the *Lupaş q-transform* of f is defined by:

$$(\Lambda_q f)(z) := \frac{1}{(-z;q)_\infty} \cdot \sum_{k=0}^{\infty} \frac{f(1-q^k)q^{k(k-1)/2}}{(q;q)_k} z^k,$$

where

$$(a;q)_k := \prod_{j=0}^{k-1}\left(1-aq^j\right),\ (a;q)_\infty := \prod_{j=0}^{\infty}\left(1-aq^j\right),\quad k \in \mathbb{N}_0,\ a \in \mathbb{C}.$$

The analytical and approximation properties of Λ_q have already been examined. In this paper, some properties of the Lupaş q-transform related to continuous linear operators in normed linear spaces are investigated.

1. INTRODUCTION

The *Lupaş q-transform* emerges in the study of the *limit q-Lupaş operator*. The latter comes out naturally as a limit for a sequence of the Lupaş q-analogues of the Bernstein operator (cf. [8] and [11]). The various properties of this operator have been discussed in [1], [9], and [16].

Date: Received: Sep. 26, 2013; Accepted: Nov. 25, 2013.

2010 *Mathematics Subject Classification.* Primary 47A05; Secondary 46B20, 47B65, 47B38.

Key words and phrases. Lupaş q-transform, Bernstein operator, continuous linear operator, isomorphic embedding.

Definition 1.1. Given $q \in (0,1)$, $f \in C[0,1]$, the *q-Lupaş transform* of f is defined by:

$$(\Lambda_q f)(z) := \frac{1}{(-z;q)_\infty} \cdot \sum_{k=0}^{\infty} \frac{f(1-q^k)q^{k(k-1)/2}}{(q;q)_k} z^k,$$

where

$$(a;q)_k := \prod_{j=0}^{k-1} \left(1-aq^j\right), \ (a;q)_\infty := \prod_{j=0}^{\infty} \left(1-aq^j\right), \quad k \in \mathbb{N}_0, \ a \in \mathbb{C}.$$

As it turns out, the *Lupaş q-transform* is closely connected to various subjects, including the theory of positive operators, q-deformed probability distributions, the q-boson operator calculus, the methods of summation of divergent series, and the theory of analytic functions. See [4], [5], [10], and [12].

In general, $\Lambda_q f$ is a meromorphic function, whose simple poles are contained in the set $J_q := \{-q^{-j}\}_{j=0}^{\infty}$. The function $(-z;q)_\infty$ is an entire function, Taylor's expansion of which is given by Euler's identity (cf., e.g. [2], Ch.10, §10.2):

$$\forall z \in \mathbb{C} \ \forall |q| < 1 \quad (-z;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k} z^k. \tag{1.1}$$

It implies immediately that $\Lambda_q(\mathbf{1}_{[0,1]}) = \mathbf{1}_{[0,\infty)}$ for the indicator functions, and that

$$|\Lambda_q f(x)| \le \|f\|_{C[0,1]} \text{ for all } x \ge 0.$$

Therefore, Λ_q can be viewed as a positive linear operator $C[0,1] \to C_B[0,\infty)$, where by $C_B[0,\infty)$ we denote the space of bounded continuous functions on $[0,\infty)$ equipped with the norm $||f|| = \sup_{x\in[0,\infty)} |f(x)|$. In this context, Λ_q is a positive bounded linear operator with $\|\Lambda_q\| = 1$.

Some of the analytical and approximation properties of Λ_q have been examined in [12] and [16]. In distinction, the present work is focused on the geometric properties of the Lupaş q-transform.

The following terminology is adopted in the text. The word *operator* is used for a continuous linear operator between normed linear spaces. An operator $T : X \to Y$ is called an *isomorphic embedding* if there exists a constant $m > 0$ such that $||Tx|| \ge m||x||$ for each $x \in X$. The *range* of an operator $T : X \to Y$ is the set $\{y \in Y : \ \exists x \in X \ Tx = y\}$. The space of all bounded sequences of real numbers with the supremum modulus norm is denoted by ℓ_∞, while the subspace of the convergent sequences is denoted by c and the subspace of those sequences converging to 0 is denoted by c_0. The other related terminology can be found in [7] or [14].

2. The geometric properties of the Lupaş q-transform

Our first goal is to prove the following theorem showing that there is a subspace L of $C[0,1]$ isomorphic to c, such that the restriction of Λ_q on L is an isomorphic embedding.

Theorem 2.1. *Let L be the subspace of $C[0,1]$ consisting of functions, which are linear on the intervals $[1-q^{k-1}, 1-q^k]$ for $k \in \mathbb{N}$. If $q \in (0,1)$ is sufficiently close to 0, then the restriction of the Lupaş q-transform $\Lambda_q : C[0,1] \to C_B[0,\infty)$ to L is an isomorphic embedding.*

Proof. It is easy to see that the map $T : L \to c$ given by $Tf = \{f(1-q^k)\}_{k=0}^{\infty}$ is an isomorphism.

Since the functions in L satisfying the conditions $f(1-q^n) = 1$ for some $n \in \mathbb{N}$ and $|f(1-q^k)| \leq 1$ for all $k \in \mathbb{N}$ are dense in the unit sphere of L, it suffices to show that, for q sufficiently close to 0, there exists $m_q > 0$ such that for each sequence $\{f(1-q^k)\}$ satisfying $f(1-q^n) = 1$ and $|f(1-q^k)| \leq 1$ for all $k \in \mathbb{N}$, there holds:

$$||\Lambda_q(f)||_{C_B[0,\infty)} \geq m_q.$$

By virtue of Euler's identity (1.1), to achieve this goal it suffices, for each number $n \in \mathbb{N}$ to pick a real number $x_n \in [0,\infty)$ in such a way that:

$$\frac{q^{n(n-1)/2}}{(q;q)_n} x_n^n - \sum_{k \neq n} \frac{q^{k(k-1)/2}}{(q;q)_k} x_n^k \geq m_q \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k} x_n^k,$$

or, equivalently,

$$(1-m_q)\frac{q^{n(n-1)/2}}{(q;q)_n} x_n^n \geq (1+m_q) \sum_{k \neq n} \frac{q^{k(k-1)/2}}{(q;q)_k} x_n^k. \tag{2.1}$$

Indeed, in this case, for any f satisfying $f(1-q^n) = 1$ and $|f(1-q^k)| \leq 1$ for all $k \in \mathbb{N}$, one has:

$$\begin{aligned}
||\Lambda_q(f)||_{C_B[0,\infty)} &\geq |(\Lambda_d(f))(x_n)| \\
&= \left| \frac{1}{(-x_n;q)_\infty} \sum_{k=0}^{\infty} \frac{f(1-q^k)\, q^{k(k-1)/2}}{(q;q)_k} x_n^k \right| \\
&\geq \frac{1}{(-x_n;q)_\infty} \left(\frac{q^{n(n-1)/2}}{(q;q)_n} x_n^n - \sum_{k \neq n} \frac{q^{k(k-1)/2}}{(q;q)_k} x_n^k \right) \\
&\overset{(2)}{\geq} \frac{1}{(-x_n;q)_\infty} \left(m_q \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k} x_n^k \right) = m_q.
\end{aligned}$$

To satisfy (2.1), x_n has to be chosen in a suitable way. When $x_n = q^{-n+\frac{1}{2}}$, the left-hand side of (2.1) becomes:

$$(1-m_q)\frac{q^{n(n-1)/2}}{(q;q)_n} q^{n(-n+\frac{1}{2})} = \frac{(1-m_q)q^{-\frac{n^2}{2}}}{(q;q)_n}.$$

Meanwhile, the right-hand side is:

$$\begin{aligned}(1+m_q)\sum_{k\neq n}\frac{q^{k(k-1)/2}}{(q;q)_k}(q^{-n+\frac{1}{2}})^k &= q^{-\frac{n^2}{2}}(1+m_q)\sum_{k\neq n}\frac{1}{(q;q)_k}q^{\frac{1}{2}(k-n)^2}\\ &\leq q^{-\frac{n^2}{2}}(1+m_q)\sum_{k\neq n}\frac{1}{(q;q)_k}q^{\frac{1}{2}|k-n|}\\ &\leq q^{-\frac{n^2}{2}}(1+m_q)\frac{2\sqrt{q}}{(q;q)_\infty(1-\sqrt{q})}.\end{aligned}$$

The desired inequality would follow from:

$$\frac{1-m_q}{(q;q)_n}\geq(1+m_q)\frac{2\sqrt{q}}{(q;q)_\infty(1-\sqrt{q})}. \tag{2.2}$$

It is clear that picking $q>0$ small enough to yield $(q;q)_\infty>\frac{2\sqrt{q}}{(1-\sqrt{q})}$ and, then, taking m_q sufficiently close to 0, it can be achieved that inequality (2.2) is satisfied for all $n\in\mathbb{N}$. □

Remark 2.2. The restriction of Λ_q to any subspace of $C[0,1]$ which does not contain a subspace isomorphic to c_0 is strictly singular, and as such, is not an isomorphic embedding. To see this, observe that Λ_q factors through L, which itself is isomorphic to c_0. Applying the well-known results on the Banach space geometry (see [7, Ch. 2] or [14]), we derive the statement as in the previous sentence.

Combining Remark 2.2 with the classical Banach-Mazur theorem [3, Ch. XI, §8] on the universality of $C[0,1]$, it is concluded that there are many different subspaces of $C[0,1]$ on which the operator Λ_q is not an isomorphic embedding.

The following simple property of the Lupaş q-transform related to the regular method of summation holds.

Lemma 2.3. *The following equality is valid:*

$$\lim_{x\to+\infty}\Lambda_q f(x)=f(1).$$

Proof. Let $f(1-q^k)=f(1)+a_k$, where $\{a_k\}\to 0$ and $|a_k|\leq M$, $M>0$. Given $\varepsilon>0$, choose $N=N(\varepsilon)\in\mathbb{N}$ so that $|a_k|<\varepsilon$ for all $k\geq N$. Further, select $x_0>0$ in such a way that:

$$\frac{1}{(-x;q)_\infty}\cdot\sum_{k=0}^{N(\varepsilon)}\frac{q^{k(k-1)/2}}{(q;q)_k}x^k<\frac{\varepsilon}{M}$$

for all $x>x_0$. Then, for all $x>x_0$, one has:

$$|\Lambda_q f(x)-f(1)|\leq\frac{M}{(-x;q)_\infty}\cdot\sum_{k=0}^{N}\frac{q^{k(k-1)/2}}{(q;q)_k}x^k+\frac{\varepsilon}{(-x;q)_\infty}\cdot\sum_{k=N+1}^{\infty}\frac{q^{k(k-1)/2}}{(q;q)_k}x^k<2\varepsilon.$$

□

Let us denote by $C_L[0,\infty)$ the subspace of $C_B[0,\infty)$ having a finite limit at ∞. By Lemma 2.3, the image $\Lambda_q(C[0,1])$ is contained in $C_L[0,\infty)$. It is easy to observe that the space $C_L[0,\infty)$ is separable.

Corollary 2.4. *If q is sufficiently close to 0, the range of Λ_q is a closed complemented subspace of $C_L[0,\infty)$. There exists an operator $\Theta_q : C_L[0,\infty) \to C[0,1]$, such that the composition $\Theta_q\Lambda_q$ is the identity operator on L.*

Proof. The point that the range of Λ_q is closed follows from the general fact: the image of a Banach space under an isomorphic embedding is closed. The range of Λ_q is complemented because it is, in essence, the range of an isomorphic embedding of c, which is isomorphic to c_0 and, by the Sobczyk theorem [15] - see also [7, p. 106] and [13] - the range of an isomorphic embedding of c_0 in a separable Banach space is always complemented. Let P_q be a continuous linear projection of $C_L[0,\infty)$ onto $\Lambda_q(L)$. We define Θ_q as the composition $\Lambda_q^{-1}P_q$, where $\Lambda_q^{-1} : \Lambda_q(L) \to L$ is the inverse defined in a natural way on the range of Λ_q. □

For the sequel, the following lemmas are needed.

Lemma 2.5. *For each $\varepsilon > 0$ and $M > 0$, there exists $n \in \mathbb{N}$, such that if $||f||_{C[0,1]} \leq 1$ and $f(1-q^k) = 0$ for $k = 0, \dots, n$, then $|(\Lambda_q f)(x)| < \varepsilon$ for $x \in [0, M]$.*

Proof. Indeed, since $\frac{1}{(-x;q)_\infty} \leq 1$ on $[0, +\infty)$, it follows for all $x \in [0, M]$ that:

$$|\Lambda_q f(x)| \leq \frac{1}{(q;q)_\infty} \sum_{k=n+1}^{\infty} q^{k(k-1)/2} M^k < \varepsilon$$

for sufficiently large n's as the series converges for all M. □

Lemma 2.6. *Given $n \in \mathbb{N}$, consider the set of functions $A_n = \{f \in C[0,1] : ||f|| = 1$ and $f(1-q^k) = 0$ for $k = 0, \dots, n\}$. Then, for any $\alpha > 0$ and any $n \in \mathbb{N}$, there exists a function $\widetilde{f} \in A_n$ such that $||\Lambda_q \widetilde{f}|| \geq \alpha$ and $\widetilde{f}(1-q^k) = 0$ also for sufficiently large k.*

Proof. Opt for any $\alpha \in (0,1)$. As it has already been mentioned, the transform Λ_q maps $\mathbf{1}_{[0,1]}$ to $\mathbf{1}_{[0,\infty)}$. Combining this fact with Lemma 2.5, one can conclude that any function f satisfying

$$f(1-q^k) = \begin{cases} 0 & \text{if } k = 0, \dots, n \\ 1 & \text{if } k > n \end{cases}$$

also fulfills $(\Lambda_q f)(x) \to 1$ as $x \to +\infty$, whence there is $x_0 \in (0,\infty)$ such that $(\Lambda_q f)(x_0) > \alpha$. Now, note that there exists $N \in \mathbb{N}$ such that

$$\frac{1}{(-x_0,q)_\infty} \sum_{k=m+1}^{N} \frac{q^{k(k-1)/2}}{(q;q)_k} x_0^k > \alpha.$$

The function $\widetilde{f} \in C[0,1]$ given by

$$\widetilde{f}(1-q^k) = \begin{cases} 0 & \text{if } k = 0, \dots, m; N+1, N+2, \dots \\ 1 & \text{if } k = m+1, \dots, N \end{cases}$$

has the desired properties. □

Finally, the following assertion can be reached.

Theorem 2.7. *If q is sufficiently close to 0, the range of Λ_q is a closed, uncomplemented subspace of $C_B[0,\infty)$.*

Proof. This proof is based on another theorem of Sobczyk [15]: c_0 is uncomplemented in ℓ_∞. Here, we consider c_0 as a subspace of ℓ_∞ which is embedded in a natural way. It is worth mentioning that a much stronger result than this theorem of Sobczyk is known: Lindenstrauss [6] (see also [7, Theorem 2.a.7]) proved that each infinite-dimensional complemented subspace of ℓ_∞ is isomorphic to ℓ_∞; hence it cannot be separable and, for this reason, cannot be isomorphic to c_0 either.

In what proceeds, Sobczyk's theorem is used as follows. Construct a sequence of norm-one functions $\{f_n\}_{n=1}^{\infty}$ in L and a sequence $\{[a_n,b_n]\}_{n=1}^{\infty}$, $a_n < b_n$, of disjoint intervals in $[0,\infty)$ in a way that, for some $\alpha > 0$, the conditions below hold:

(1) $||\Lambda_q(f_n)|| \geq \alpha$.
(2) $||\Lambda_q(f_n) - g_n|| \leq \frac{\alpha}{2^n}$, where $\{g_n\}$ is a sequence of functions in $C_B[0,\infty)$ with $\mathrm{supp}(g_n) \subset [a_n,b_n]$.
(3) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$.

The well-known perturbation argument (see, for example, [7, p. 6]) implies the existence of a continuous automorphism $A : C_B[0,\infty) \to C_B[0,\infty)$ satisfying $A(\Lambda_q(f_n)) = g_n$.

Now, consider the subspace $N \subset C_B[0,\infty)$ consisting of all the bounded functions which on $[a_n,b_n]$ coincide with a multiple of g_n, and are equal to 0 on the complement of $\cup_{n=1}^{\infty}[a_n,b_n]$. Also, let D be the closed linear span of the functions $\{g_n\}_{n=1}^{\infty}$. It is clear that N is isometric to ℓ_∞ while D is isometric to c_0. Furthermore, the diagram

$$\begin{array}{ccc} D & \xrightarrow{I_D} & c_0 \\ \downarrow & & \downarrow \\ N & \xrightarrow{I_N} & \ell_\infty \end{array}$$

commutes, where I_D and I_N are the above-mentioned isometries, and vertical arrows correspond to canonical embeddings. Therefore, D is uncomplemented in N, and, consequently, D is uncomplemented in $C_B[0,\infty)$. On the other hand, D is complemented in $\Lambda_q(L)$ by the first theorem of Sobczyk, which claims that a subspace isomorphic to c_0 is complemented in any separable Banach space. Therefore $\Lambda_q(L)$ is uncomplemented in $C_B[0,\infty)$. □

Acknowledgement. The author expresses her sincere gratitude to Mr. P. Danesh from Atilim University Academic Writing and Advisory Centre for his help in the preparation of the manuscript.

References

1. O. Agratini, *On certain q-analogues of the Bernstein operators*, Carpathian J. Math. **24** (2009), no. 3, 281–286.

2. G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia of Mathematics and Its Applications, Cambridge Univ. Press., Cambridge, 1999.
3. S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne, Warszawa, 1932.
4. A. Guessab and G. Schmeisser, *Two Korovkin-type theorems in multivariate approximation*, Banach J. Math. Anal. **2** (2008), no. 2, 121–128.
5. S. Jing, *The q-deformed binomial distribution and its asymptotic behaviour*, J. Phys. A: Math. Gen. **27** (1994), 493–499.
6. J. Lindenstrauss, *On complemented subspaces of m*, Israel J. Math. **5** (1967), 153–156.
7. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I Sequence spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. Springer-Verlag, Berlin-New York, 1977.
8. A. Lupaş, *A q-analogue of the Bernstein operator*, University of Cluj-Napoca, Seminar on numerical and statistical calculus (1987), no. 9, 85–92.
9. N. Mahmudov, P. Sabancigil, *Voronovskaja type theorem for the Lupas q-analogue of the Bernstein operators*, Math. Comm. **17** (2012), no. 1, 83–91.
10. I.Y. Novikov, V.Yu. Protasov and M.A. Skopina, *Wavelet theory*, AMS Translations of Mathematical Monographs, Vol. 239, 2011.
11. S. Ostrovska, *On the Lupas q-analogue of the Bernstein operator*, Rocky Mountain J. Math. **36** (2006), no. 5, 1615–1629.
12. S. Ostrovska, *Analytical properties of the Lupas q-transform*, J. Math. Anal. Appl. **394** (2012), no. 1, 177–185.
13. M.I. Ostrovskii, *Separably injective Banach spaces*, Funct. Anal. Appl. **20** (1986), no. 2, 154–155.
14. M.M. Popov, *Complemented subspaces and some problems of the modern geometry of Banach spaces* (Ukrainian), Mathematics today **13** (2007), 78–116.
15. A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. **47** (1941), 938–947.
16. V.S. Videnskii, *A remark on the rational linear operators considered by A. Lupas*, Some current problems in modern mathematics and education in mathematics (Russian), 134–146, St. Petersburg, 2008.

Department of Mathematics, Atilim University, Incek 06836 Ankara, Turkey
E-mail address: ostrovsk@atilim.edu.tr; ostrovskasofiya@yahoo.com