L^p-MAXIMAL REGULARITY OF DEGENERATE DELAY EQUATIONS WITH PERIODIC CONDITIONS

SHANGQUAN BU

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ABSTRACT. Under suitable assumptions on the delay operator F, we give necessary and sufficient conditions for the inhomogeneous abstract degenerate delay equations: $(Mu)'(t) = Au(t) + Fu_t + f(t)$, $(t \in \mathbb{T})$ to have L^p-maximal regularity.

1. INTRODUCTION

The aim of this paper is to study maximal regularity of the following inhomogeneous abstract degenerate delay equations:

$$(Mu)'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{1.1}$$

where A and M are closed linear operators in a complex Banach space X satisfying $D(A) \subset D(M)$, $u_t(\cdot) = u(t + \cdot)$ is defined on $[-r, 0]$ for some fixed $r > 0$, $f \in L^p(\mathbb{T}, X)$ for some $1 \le p < \infty$, and the delay operator $F : L^p([-r, 0], X) \to X$ is a fixed bounded linear operator. We say that (1.1) has L^p-maximal regularity, if for each $f \in L^p(\mathbb{T}, X)$, there exists a unique function u, such that $u \in L^p(\mathbb{T}, D(A))$, Mu is differentiable a.e. on $\mathbb{T}$, $Mu, (Mu)' \in L^p(\mathbb{T}, X)$ and (1.1) is satisfied a.e on $\mathbb{T}$. In this paper, under suitable assumption on the delay operator F, we are able to give a necessary and sufficient condition for (1.1) to have L^p-maximal regularity when X is a UMD Banach space and $1 < p < \infty$.

The equation (1.1) corresponds to problems related with viscoelastic materials, that is, materials whose stresses at any instant depend on the complete history of strains that the material has undergone or teat conduction with memory. The

most typical examples of (1.1) are the cases when M is the multiplication operator by some fixed non negative function m and A is the Laplacian Δ [8]. We refer the readers to [8] for more concrete examples of degenerate differential equations in Banach spaces.

When M is the identity operator on X, the equation (1.1) was extensively studied. For instance, (1.1) for $t \in \mathbb{R}$ was firstly studied by Hale [9] and Webb [18], necessary and sufficient condition for (1.1) to have L^p-maximal regularity were obtained by Lizama [13], C^α-maximal regularity of the corresponding equation on the real line has been studied by Lizama and Poblete [15], while $B_{p,q}^s$-maximal regularity and $F_{p,q}^s$-maximal regularity of (1.1) have been studied by Bu and Fang [5]. We note that in the special case when M is the identity operator on X and $F = 0$, maximal regularity of (1.1) has been studied by Arendt and Bu in L^p-spaces case and Besov spaces case [1, 2], Bu and Kim in Triebel–Lizorkin spaces case [7]. The corresponding integro-differential equations were treated by Keyantuo and Lizama [10, 11], Bu and Fang [6].

Characterizations of L^p-maximal regularity, $B_{p,q}^s$-maximal regularity and $F_{p,q}^s$-maximal regularity of a similar degenerate differential equation with infinite delay

$$(Mu)'(t) = \alpha Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad t \in \mathbb{T}$$

have been obtained by Lizama and Ponce [16], while a second order degenerate equations

$$(Mu')'(t) = Au(t) + f(t), \ t \in \mathbb{T}$$

has been recently studied by Bu [4]. See also [17] for the study of maximal regularity of second order equations with delay, and [14] for existence and uniqueness of periodic solutions for fractional differential equations with delay.

In this paper, we are interested in L^p-maximal regularity of (1.1). The main results are necessary or sufficient conditions for this problem to have L^p-maximal regularity when $1 < p < \infty$ and X is a UMD Banach space, which generalizes the previous known results by Arendt and Bu [1] when $M = I_X$ and $F = 0$, and Lizama [13] when $M = I_X$. The main tools are operator-valued Fourier multiplier results on $L^p(\mathbb{T}, X)$ established in [1]. We remark that the sufficient condition for a sequence $M = (M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$ to be an L^p-multiplier is a Marcinkiewicz type condition of order 1 involved Rademacher boundedness condition valid in UMD Banach spaces [1].

We notice that the presence of the degenerate operator M and the delay operator F makes our study of L^p-maximal regularity of (1.1) particularly difficult: we have no assumptions on the commutativity of the operators A, M and F, thus $(ikM - A - B_k)^{-1}$ (the M-resolvent of A) is no longer commutative, where $B_k \in \mathcal{L}(X)$ is defined by $B_k x = F(e_k \otimes x)$. Some careful computation in the estimation of the Marcinkiewicz type condition of order 1 is needed. This is also the reason we have to impose an extra condition $(\mathbf{H}_1)$ on the delay operator F in our main result which gives a characterization of L^p-maximal regularity of (1.1) when X is a UMD Banach space and $1 < p < \infty$.

2. Main Results

We begin by giving some notations and notions. Let X be a Banach space and let $f \in L^1(\mathbb{T}, X)$, we denote by $\hat{f}(k)$ the k-th Fourier coefficient of f defined by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t)f(t)dt,$$

when $k \in \mathbb{Z}$, and $e_k(t) = e^{ikt}$. For $k \in \mathbb{Z}$ and $x \in X$, we denote by $e_k \otimes x$ the X-valued function on $\mathbb{T}$ defined by $(e_k \otimes x)(t) = e_k(t)x$.

For $1 \le p < \infty$, the first order periodic Sobolev spaces [1] is defined by:

$$W_{\text{per}}^{1,p}(\mathbb{T}, X) := \big\{ u \in L^p(\mathbb{T}, X) : \text{ there exists } v \in L^p(\mathbb{T}, X),$$

$$\text{such that } \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\big\}.$$

Let $u \in L^p(\mathbb{T}, X)$, then $u \in W_{\text{per}}^{1,p}(\mathbb{T}, X)$ if and only if u is differentiable a.e. on $\mathbb{T}$ and $u' \in L^p(\mathbb{T}, X)$, in this case u is actually continuous and $u(0) = u(2\pi)$ [1, Lemma 2.1].

Let $1 \le p < \infty$. We consider the following degenerate delay equations

$$(Mu)'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{T} \tag{2.1}$$

where A and M are closed linear operators in X satisfying $D(A) \subset D(M)$, $f \in L^p(\mathbb{T}, X)$ is given,

$$F : L^p([-r, 0], X) \to X$$

is a bounded linear operator for some fixed $r > 0$. Moreover, for fixed $t \in \mathbb{T}$, u_t is considered as an element of $L^p([-r, 0], X)$ defined by $u_t(s) = u(t + s)$ for $-r \le s \le 0$.

Definition 2.1. Let X be a Banach space, $1 \le p < \infty$ and $f \in L^p(\mathbb{T}, X)$ be given. $u \in L^p(\mathbb{T}, D(A))$ is called a strong L^p-solution of (2.1), if $Mu \in W_{\text{per}}^{1,p}(\mathbb{T}; X)$ and (2.1) holds a.e. on $\mathbb{T}$. We say that (2.1) has L^p-maximal regularity, if for each $f \in L^p(\mathbb{T}, X)$, (2.1) has a unique strong L^p-solution.

We let

$$S_p(A, M) := \big\{ u \in L^p(\mathbb{T}, D(A)) :$$

$$Mu \in W_{\text{per}}^{1,p}(\mathbb{T}; X) \text{ and } (2.1) \text{ holds a.e. on } \mathbb{T}\big\}$$

be the solution space of (2.1). $S_p(A, M)$ equipped with the norm

$$\big\|u\big\|_{S_p(A,M)} := \big\|(Mu)'\big\|_{L^p} + \big\|Mu\big\|_{L^p} + \big\|u\big\|_{L^p} + \big\|Au\big\|_{L^p} + \big\|Fu.\big\|_{L^p}$$

is a Banach space. It follows from the Closed Graph Theorem that when (2.1) has L^p-maximal regularity, then there exists a constant $C \ge 0$, such that for all $f \in L^p(\mathbb{T}, X)$, if $u \in S_p(A, M)$ is the unique strong L^p-solution of (2.1), then

$$\big\|u\big\|_{S_p(A,M)} \le C\big\|f\big\|_{L^p}. \tag{2.2}$$

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. If $X = Y$, we will simply denote it by $\mathcal{L}(X)$. The main tool in the study of L^p-maximal regularity of (2.1) is operator-valued L^p-multipliers studied in [1].

Definition 2.2. Let X, Y be Banach spaces, $1 \leq p < \infty$ and let $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$. We say that $(M_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier, if for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$, such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

When $(M_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier, then there exists a constant $C \geq 0$, such that for all $f \in L^p(\mathbb{T}, X)$,

$$\Big\| \sum_{n \in \mathbb{Z}} e_k \otimes M_k \hat{f}(k) \Big\|_{L^p} \leq C \|f\|_{L^p}.$$

This follows easily from the Closed Graph Theorem.

Let γ_j be the j-th Rademacher function on $[0,1]$ given by $\gamma_j(t) = sgn(\sin(2^j t))$ when $j \geq 1$. For $x \in X$, we denote by $\gamma_j \otimes x$ the X-valued function $t \to r_j(t)x$ on $[0, 1]$.

Definition 2.3. Let X and Y be Banach spaces. A set $\mathbf{T} \subset \mathcal{L}(X, Y)$ is said to be Rademacher bounded (R-bounded, in short), if there exists $C > 0$ such that

$$\Big\| \sum_{j=1}^{n} \gamma_j \otimes T_j x_j \Big\|_{L^1([0,1],Y)} \leq C \Big\| \sum_{j=1}^{n} \gamma_j \otimes x_j \Big\|_{L^1([0,1],X)}$$

for all $T_1, \ldots, T_n \in \mathbf{T}, x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$.

We may replace the L^1-norm in the definition of R-boundedness by any L^p-norm when $1 \leq p < \infty$. This follows immediately from Kahane's inequality [12, Theorem 1.e.13]. It is clear from the definition that when $\mathbf{T}$ is R-bounded, then it is norm bounded. The converse implication is not true in general, it is known that every norm bounded subset $\mathbf{T} \subset \mathcal{L}(X, Y)$ is R-bounded if and only if X is of cotype 2 and Y is of type 2 [1, Proposition 1.13].

Remarks 1. (i) Let $\mathbf{S}, \mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then it can be seen easily from the definition that $\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$ and $\mathbf{S} + \mathbf{T} := \{S + T : S \in \mathbf{S}, T \in \mathbf{T}\}$ are still R-bounded.

(ii) Let X be a UMD Banach space and let $M_k = m_k I_X$ with $m_k \in \mathbb{C}$, where I_X is the identity operator on X, if

$$\sup_{k \in \mathbb{Z}} |m_k| < \infty, \quad \sup_{k \in \mathbb{Z}} |k(m_{k+1} - m_k)| < \infty,$$

then $(M_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier whenever $1 < p < \infty$ [3].

The following results will be fundamental in our study of L^p-maximal regularity for (2.1).

Proposition 2.4. ([1, Proposition 1.11]) *Let X, Y be Banach spaces, $1 \leq p < \infty$, and let $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$ be an L^p-multiplier. Then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

Theorem 2.5. ([1, Theorem 1.3]) *Let X, Y be UMD Banach spaces and $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X, Y)$. If the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier whenever $1 < p < \infty$.*

Let $r > 0$ be fixed, $F \in \mathcal{L}(L^p([-r, 0], X), X)$ and $k \in \mathbb{Z}$, we define the operator B_k on X by $B_k x = F(e_k \otimes x)$ for all $x \in X$. An easy calculation shows that if $u \in L^p(\mathbb{T}, X)$, then

$$\widehat{Fu\cdot}(k) = B_k \hat{u}(k) \tag{2.3}$$

when $k \in \mathbb{Z}$. It is clear that $B_k \in \mathcal{L}(X)$ and $\|B_k\| \leq r^{1/p} \|F\|$. We define the M-resolvent of A for (2.1) by

$$\rho_M(A) := \big\{ k \in \mathbb{Z} : \quad ikM - A - B_k \text{ is a bijection from}$$
$$D(A) \text{ onto } X \text{ and } (ikM - A - B_k)^{-1} \in \mathcal{L}(X) \big\}.$$

It follows easily from the assumption $D(A) \subset D(M)$ and the closedness of M and A that when $k \in \rho_M(A)$, then

$$M(ikM - B_k - A)^{-1} \in \mathcal{L}(X), \ A(ikM - A - B_k)^{-1} \in \mathcal{L}(X).$$

We begin by giving a necessary condition for the problem (2.1) to have L^p-maximal regularity.

Proposition 2.6. *Let $1 \leq p < \infty$ and assume that (2.1) has L^p-maximal regularity. Then $\rho_M(A) = \mathbb{Z}$, $(ikM(ikM - B_k - A)^{-1})_{k \in \mathbb{Z}}$ and $((ikM - B_k - A)^{-1})_{k \in \mathbb{Z}}$ are L^p-multipliers. In particular, the sets $\{ikM(ikM - B_k - A)^{-1} : k \in \mathbb{Z}\}$ and $\{(ikM - B_k - A)^{-1} : k \in \mathbb{Z}\}$ are R-bounded.*

Proof. Let $k \in \mathbb{Z}$ and $y \in X$ be fixed, and let $f = e_k \otimes y$. Then $f \in L^p(\mathbb{T}, X)$, $\hat{f}(k) = y$ and $\hat{f}(n) = 0$ when $n \neq k$. Since (2.1) has L^p-maximal regularity, there exists a unique $u \in S_p(A, M)$ satisfying

$$(Mu)'(t) = Au(t) + Fu_t + f(t)$$

a.e. on $\mathbb{T}$. We have $\hat{u}(n) \in D(A) \subset D(M)$ when $n \in \mathbb{Z}$ by [1, Lemma 3.1] as $u \in L^p(\mathbb{T}, D(A))$, and $\widehat{(Mu)'}(k) = ikM\hat{u}(k)$ when $k \in \mathbb{Z}$. Taking Fourier transforms on both sides, we obtain

$$(ikM - A - B_k)\hat{u}(k) = y$$

and $(inM - A - B_n)\hat{u}(n) = 0$ when $n \neq k$ by (2.3). This shows in particular that $ikM - A - B_k$ is surjective. To show that $ikM - A - B_k$ is injective, we take $x \in D(A)$ such that

$$(ikM - A - B_k)x = 0.$$

Let $u = e_k \otimes x$, then clearly $u \in S_p(A, M)$ and (2.1) holds a.e. on $\mathbb{T}$ when taking $f = 0$. We obtain $x = 0$ by the uniqueness assumption. We have shown that $ikM - A - B_k$ is also injective. Therefore $ikM - A - B_k$ is bijective from $D(A)$ onto X.

Next we show that $(ikM - A - B_k)^{-1} \in \mathcal{L}(X)$. For $f = e_k \otimes y$, we let $u \in S_p(A, M)$ be the unique strong L^p-solution of (2.1). Then

$$\hat{u}(n) = \begin{cases} 0, & n \neq k, \\ (ikM - A - B_k)^{-1}y, & n = k. \end{cases}$$

This implies that $u = e_k \otimes (ikM - A - B_k)^{-1}y$. By (2.2), there exists a constant $C \geq 0$ independent from y and k, such that

$$\|Mu\|_{L^p} + \|(Mu)'\|_{L^p} + \|u\|_{L^p} + \|Au\|_{L^p} + \|Fu.\|_{L^p} \leq C \|f\|_{L^p}.$$

This implies that $\|(ikM - A - B_k)^{-1}y\| \leq C \|y\|$ for all $y \in X$. Therefore $(ikM - A - B_k)^{-1} \in \mathcal{L}(X)$. We have shown that $k \in \rho_M(A)$ for all $k \in \mathbb{Z}$. Thus $\mathbb{Z} = \rho_M(A)$.

Now let $f \in L^p(\mathbb{T}, X)$ be given. There exists a unique $u \in S_p(A, M)$ such that

$$(Mu)'(t) = Au(t) + Fu_t + f(t)$$

a.e. on $\mathbb{T}$ by assumption. Taking Fourier transform on both sides, we obtain $\hat{u}(k) \in D(A) \subset D(M)$ by [1, Lemma 3.1], and

$$ikM\hat{u}(k) = A\hat{u}(k) + B_k\hat{u}(k) + \hat{f}(k)$$

for all $k \in \mathbb{Z}$ by (2.3). Equivalently $\hat{u}(k) = (ikM - A - B_k)^{-1}\hat{f}(k)$ for all $k \in \mathbb{Z}$. This implies that

$$\widehat{(Mu)'}(k) = ikM(ikM - A - B_k)^{-1}\hat{f}(k)$$

whenever $k \in \mathbb{Z}$. We have shown that $(ikM(ikM - A - B_k)^{-1})_{k \in \mathbb{Z}}$ is an L^p-multiplier as $(Mu)' \in L^p(\mathbb{T}, X)$. We deduce that the set

$$\left\{ ikM(ikM - A - B_k)^{-1} : k \in \mathbb{Z} \right\}$$

is R-bounded by Proposition 2.4. Since $u \in L^p(\mathbb{T}, D(A))$, then $u \in L^p(\mathbb{T}, X)$ as $D(A)$ is equipped with the graph norm $\|x\|_{D(A)} := \|x\| + \|Ax\|$. Therefore $((ikM - A - B_k)^{-1})_{k \in \mathbb{Z}}$ is an L^p-multiplier. It follows that the set $\{(ikM - A - B_k)^{-1} : k \in \mathbb{Z}\}$ is also R-bounded by Proposition 2.4. This finishes the proof. $\square$

The set $\{B_k : k \in \mathbb{Z}\}$ is always R-bounded. Indeed, let $n \geq 1$ and $x_1, x_2, \cdots, x_n \in X$, then by Fubini's Theorem and Kahane's concentration principle [12]

$$\left\| \sum_{k=1}^{n} \gamma_k \otimes B_k x_k \right\|_{L^p([0,1],X)} = \left\| \sum_{k=1}^{n} \gamma_k \otimes F(e_k \otimes x_k) \right\|_{L^p([0,1],X)}$$

$$\leq \|F\| \left(\int_0^1 \int_0^r \left\| \sum_{k=1}^{n} \gamma_k(t)e_k(s)x_k \right\|^p ds dt \right)^{1/p}$$

$$\leq \|F\| \left(\int_0^1 \int_0^r \left\| \sum_{k=1}^{n} \gamma_k(t)e_k(s)x_k \right\|^p dt ds \right)^{1/p}$$

$$\leq 2\|F\| \left(\int_0^1 \left\| \sum_{k=1}^{n} \gamma_k(t)x_k \right\|^p dt \right)^{1/p}$$

$$= 2\|F\| \left\| \sum_{k=1}^{n} \gamma_k \otimes x_k \right\|_{L^p([0,1],X)}.$$

We will use the following condition on the delay operator F: we say that F satisfies the condition $(\mathbf{H}_1)$, if the set $\{k(B_{k+1} - B_k) : k \in \mathbb{Z}\}$ is R-bounded. Now we are ready to give a necessary and sufficient condition for (2.1) to have L^p-maximal regularity when X is a UMD Banach space and $1 < p < \infty$.

Theorem 2.7. *Let X be a UMD Banach space and $1 < p < \infty$. We assume that the delay operator F satisfies the condition $(\mathbf{H_1})$. Then (2.1) has L^p-maximal regularity if and only if $\rho_M(A) = \mathbb{Z}$ and the sets*

$$\{ikM(ikM - A - B_k)^{-1} : k \in \mathbb{Z}\}, \ \{(ikM - A - B_k)^{-1} : k \in \mathbb{Z}\}$$

are R-bounded.

Proof. The condition is clearly necessary by Proposition 2.6. We are going to show that the condition is also sufficient. Assume that $\rho_M(A) = \mathbb{Z}$ and the sets

$$\{ikM(ikM - A - B_k)^{-1} : k \in \mathbb{Z}\}, \ \{(ikM - A - B_k)^{-1} : k \in \mathbb{Z}\}$$

are R-bounded. Let

$$C_k = ikM - B_k, \ N_k = (C_k - A)^{-1},$$
$$S_k = ikM(C_k - A)^{-1}, \ T_k = B_k(C_k - A)^{-1}$$

when $k \in \mathbb{Z}$. Then $\{N_k : k \in \mathbb{Z}\}$, $\{S_k : k \in \mathbb{Z}\}$ and $\{T_k : k \in \mathbb{Z}\}$ are R-bounded. Here we have used the fact that the set $\{B_k : k \in \mathbb{Z}\}$ is R-bounded and Remarks 1. We are going to show that $(N_k)_{k \in \mathbb{Z}}$, $(S_k)_{k \in \mathbb{Z}}$ and $(T_k)_{k \in \mathbb{Z}}$ are L^p-multipliers.

We notice that for $k \in \mathbb{Z}$

$$\begin{aligned} N_{k+1} - N_k &= N_{k+1}[(C_k - A) - (C_{k+1} - A)]N_k \qquad (2.4)\\ &= N_{k+1}[-iM + (B_{k+1} - B_k)]N_k. \end{aligned}$$

Consequently

$$k(N_{k+1} - N_k) = -N_{k+1}S_k + N_{k+1}k(B_{k+1} - B_k)N_k.$$

This implies that the set $\{k(N_{k+1} - N_k) : k \in \mathbb{Z}\}$ is R-bounded by Remarks 1. Hence $(N_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier by Theorem 2.5.

For $k \in \mathbb{Z}$, by (2.4)

$$\begin{aligned} k(S_{k+1} - S_k) &= ikM[(k+1)N_{k+1} - kN_k]\\ &= ikM[(k+1)(N_{k+1} - N_k) + N_k]\\ &= ikM[(k+1)N_{k+1}(-iM + (B_{k+1} - B_k))N_k + N_k]\\ &= -S_{k+1}S_k + S_{k+1}k(B_{k+1} - B_k)N_k + S_k. \end{aligned}$$

It follows that the set $\{k(S_{k+1} - S_k) : k \in \mathbb{Z}\}$ is R-bounded by assumption. Here we have used Remarks 1. Thus $(S_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier by Theorem 2.5.

For $k \in \mathbb{Z}$,

$$k(T_{k+1} - T_k) = k(B_{k+1} - B_k)N_{k+1} - kB_k(N_{k+1} - N_k).$$

To show that $(T_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier, it will suffice to show that the set $\{k(T_{k+1} - T_k) : k \in \mathbb{Z}\}$ is R-bounded by assumption and Theorem 2.5. The set $\{k(B_{k+1} - B_k)N_{k+1} : k \in \mathbb{Z}\}$ is clearly R-bounded by assumption and Remarks 1. On the other hand, by (2.4)

$$\begin{aligned} kB_k(N_{k+1} - N_k) &= kB_kN_{k+1}[-iM + (B_{k+1} - B_k)]N_k\\ &= -B_kN_{k+1}S_k + B_kN_{k+1}k(B_{k+1} - B_k)N_k. \end{aligned}$$

Thus the set $\{kB_k(N_{k+1} - N_k) : k \in \mathbb{Z}\}$ is R-bounded. Here we have used Remarks 1 and the fact that the set $\{B_k : k \in \mathbb{Z}\}$ is R-bounded. We have shown that

$(N_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are L^p-multipliers. Since $ikMN_k - AN_k - B_kN_k = I_X$, We deduce that $(AN_k)_{k\in\mathbb{Z}}$ is also an L^p-multiplier.

Now let $f \in L^p(\mathbb{T}, X)$ be fixed. Then there exist u, v, w, $q \in L^p(\mathbb{T}, X)$, such that

$$\hat{u}(k) = N_k\hat{f}(k), \ \ \hat{v}(k) = S_k\hat{f}(k), \ \ \hat{w}(k) = T_k\hat{f}(k), \ \ \hat{q}(k) = AN_k\hat{f}(k)$$

when $k \in \mathbb{Z}$. We have $\hat{u}(k) \in D(A)$ and $A\hat{u}(k) = \hat{q}(k)$ when $k \in \mathbb{Z}$. We deduce that $u(t) \in D(A)$ a.e. on $\mathbb{T}$, $q = Au \in L^p(\mathbb{T}, X)$ by [1, Lemma 3.1]. Hence $u \in L^p(\mathbb{T}, D(A))$.

Let $a_k = \frac{1}{k}$ when $k \neq 0$ and $a_0 = 1$. Then $(a_kI_X)_{k\in\mathbb{Z}}$ is an L^p-multiplier by Remarks 1. Consequently $(MN_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier as $(S_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier. Here we have used the easy fact that the product of two L^p-multipliers is still an L^p-multiplier. We deduce that $Mu \in L^p(\mathbb{T}, X)$. Since $(ikMN_k)_{k\in\mathbb{Z}}$ is an L^p-multiplier, we deduce that $Mu \in W^{1,p}_{\text{per}}(\mathbb{T}, X)$. We have

$$\widehat{(Mu)'}(k) = ikMN_k\hat{f}(k), \ \ \widehat{Au}(k) = AN_k\hat{f}(k), \ \ \widehat{Fu.} = B_kN_k\hat{f}(k)$$

for all $k \in \mathbb{Z}$. It follows from the identity $ikMN_k - AN_k - B_kN_k = I_X$ that

$$(Mu)'(t) = Au(t) + Fu_t + f(t)$$

a.e. on $\mathbb{T}$ by the uniqueness of the Fourier coefficients [1, page 314]. This shows the existence of strong L^p-solution.

To show the uniqueness, we assume that $u \in S_p(A, M)$ be such that

$$(Mu)'(t) = Au(t) + Fu_t$$

a.e. on $\mathbb{T}$. Taking Fourier transform on both sides, we obtain $\hat{u}(k) \in D(A) \subset D(M)$ by [1, Lemma 3.1] and $(ikM - A - B_k)\hat{u}(k) = 0$ when $k \in \mathbb{Z}$. This implies that $u = 0$ as $\rho_M(A) = \mathbb{Z}$ by the uniqueness of the Fourier coefficients [1, page 314]. The proof is completed. $\square$

Since the necessary and sufficient condition given in Theorem 2.7 for (2.1) to have L^p-maximal regularity does not depends on the space parameter $1 < p < \infty$. Actually we have the following result.

Corollary 2.8. *Let X be a UMD Banach space and we assume that the delay operator F satisfies the condition ($\mathbf{H}_1$). Then (2.1) has L^p-maximal regularity for some $1 < p < \infty$ if and only if it has (2.1) has L^p-maximal regularity for all $1 < p < \infty$*

Almost the same argument used in the proof of Theorem 2.7 give a sufficient condition for (2.1) to have L^p-maximal regularity without the assumption ($\mathbf{H}_1$).

Theorem 2.9. *Let X be a UMD Banach space and $1 < p < \infty$. We assume that $\rho_M(A) = \mathbb{Z}$ and the sets*

$$\left\{ ikM(ikM - A - B_k)^{-1} : \ k \in \mathbb{Z} \right\}, \ \left\{ k(ikM - A - B_k)^{-1} : \ k \in \mathbb{Z} \right\}$$

are R-bounded. Then (2.1) has L^p-maximal regularity.

It was shown in [13] that when X is a UMD Banach space, $1 < p < \infty$ and $M = I_X$, then (2.1) has L^p-maximal regularity if and only if $\rho_M(A) = \mathbb{Z}$ and the set $\{ik(ik - A - B_k) : k \in \mathbb{Z}\}$ is R-bounded. Theorem 2.9 together with Proposition 2.6 may be considered as a generalization of the previous mentioned result by C. Lizama. Thus our results may be also considered as generalizations of the known results by W. Arendt and S. Bu in the special case when $M = I_X$ and $F = 0$.

Even the underlying Banach space X is a concrete L^p-space, it is not an easy task to verify a given set is R-bounded [19]. But things are different when X is a Hilbert space, it is known that when X is a Hilbert space, then a set $\mathbf{T} \subset \mathcal{L}(X)$ is R-bounded if and only if it is norm bounded [1, Proposition 1.13]. Thus we have the followings corollaries of Theorem 2.7 and Theorem 2.9.

Corollary 2.10. *Let X be a Hilbert space and $1 < p < \infty$. We assume that the delay operator F satisfies the condition $(\mathbf{H}_1)$. Then (2.1) has L^p-maximal regularity if and only if $\rho_M(A) = \mathbb{Z}$ and*

$$\sup_{k \in \mathbb{Z}} \left\| kM(ikM - A - B_k)^{-1} \right\| < \infty, \quad \sup_{k \in \mathbb{Z}} \left\| (ikM - A - B_k)^{-1} \right\| < \infty.$$

Corollary 2.11. *Let X be a Hilbert space and $1 < p < \infty$. We assume that $\rho_M(A) = \mathbb{Z}$ and the set*

$$\sup_{k \in \mathbb{Z}} \left\| kM(ikM - A - B_k)^{-1} \right\| < \infty, \quad \sup_{k \in \mathbb{Z}} \left\| k(ikM - A - B_k)^{-1} \right\| < \infty.$$

Then (2.1) has L^p-maximal regularity.

Example 2.12. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, m be a non negative bounded measurable function defined on Ω and f be a given function on $[0, 2\pi] \times \Omega$. We let $X = H^{-1}(\Omega)$ and consider the following periodic problem

$$\begin{cases} \dfrac{\partial(m(x)u)}{\partial t} - \Delta u + Fu_t = f(t,x), & (t,x) \in [0, 2\pi] \times \Omega \\[2mm] u(t,x) = 0, \ (t,x) \in [0, 2\pi] \times \partial\Omega \\[2mm] m(x)u(0,x) = m(x)u(2\pi, x), \ x \in \Omega, \end{cases}$$

where $u_t(s,x) = u(t+s,x)$ when $s \in [-r, 0]$ for some fixed $r > 0$, and the delay operator $F : L^p([-r,0], X) \to X$ is a fixed bounded linear operator for some $1 < p < \infty$.

Let M be the multiplication operator by m, then by [8, Section 3.7] (see also references therein), there exists a constant $c > 0$ such that

$$\left\| M(zM - \Delta)^{-1} \right\| \leq \frac{c}{1 + |z|}$$

when $\mathrm{Re}(z) \geq -c(1 + |\mathrm{Im}(z)|)$. This implies in particular that

$$\left\| M(ikM - \Delta)^{-1} \right\| \leq \frac{c}{1 + |k|} \tag{2.5}$$

whenever $k \in \mathbb{Z}$. If we assume furthermore that $m(x) > 0$ a.e. on Ω, and m^{-1} is regular enough so that the multiplication operator by the function m^{-1} is bounded linear on $H^{-1}(\Omega)$, then there exists a constant $c' > 0$ such that

$$\left\| (ikM - \Delta)^{-1} \right\| \leq \frac{c'}{1 + |k|} \tag{2.6}$$

when $k \in \mathbb{Z}$. Now the identity

$$ikM - \Delta - B_k = (I - (ikM - \Delta)^{-1} B_k)(ikM - \Delta)$$

implies that $\rho_M(\Delta) = \mathbb{Z}$ and

$$(ikM - \Delta - B_k)^{-1} = (ikM - \Delta)^{-1} [I - (ikM - \Delta)^{-1} B_k]^{-1}.$$

We have already shown that the set $\{B_k : k \in \mathbb{Z}\}$ is Rademacher bounded, thus it is norm bounded, this together with (2.6) shows that

$$\lim_{|k| \to +\infty} \left\| (ikM - \Delta)^{-1} B_k \right\| = 0.$$

Therefore by (2.6)

$$\sup_{k \in \mathbb{Z}} \left\| k(ikM - \Delta - B_k)^{-1} \right\| < \infty,$$

and by (2.5)

$$\sup_{k \in \mathbb{Z}} \left\| kM(ikM - \Delta - B_k)^{-1} \right\| < \infty.$$

We deduce from Corollary 2.11 (or Theorem 2.9) that the above periodic problem has L^p-maximal regularity when taking $X = H^{-1}(\Omega)$. Here we have used the fact that $H^{-1}(\Omega)$ is a Hilbert space.

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DEPARTMENT OF MATHEMATICAL SCIENCE, UNIVERSITY OF TSINGHUA, BEIJING 100084, CHINA.

E-mail address: sbu@math.tsinghua.edu.cn