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EXPONENTIAL MONOMIALS ON STURM–LIOUVILLE HYPERGROUPS

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Abstract. Using the concept of exponential monomial on Sturm–Liouville hypergroups we show that an important subclass of exponential monomials, the class of special exponential monomials has a linear independence property. The result can be reformulated as the linear independence of the derivatives with respect to the parameter of the solutions of eigenvalue problems for second order linear differential equations.

1. Introduction

In this paper $\mathbb{R}_0$, $\mathbb{R}_+$ and $\mathbb{C}$ denotes the set of nonnegative real numbers, the set of positive real numbers and the set of complex numbers, respectively. Concerning hypergroups and different concepts on hypergroups the reader should consult with [1].

The study of spectral analysis and spectral synthesis problems is based on the concept of exponential monomials. Unfortunately at this moment we do not have a general definition of this concept on arbitrary (commutative) hypergroups hence on each special type of hypergroups we need to introduce the most appropriate form. In the papers [2] and [3] the corresponding definition on polynomial hypergroups in one variable and in several variables is given. Using these concepts it is possible to prove spectral synthesis on these types of hypergroups. Here we

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define exponential monomials on Sturm–Liouville hypergroups and we prove a special linear independence property of them.

Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup. We recall that the continuous function $m : \mathbb{R}_0 \to \mathbb{C}$ is an exponential on K if and only if it is C^∞ on the positive reals and there exists a complex number λ such that

$$m''(x) + \frac{A'(x)}{A(x)} m'(x) = \lambda m(x), \qquad m(0) = 1, \qquad m'(0) = 0$$

holds for any positive x. Exponential functions satisfy Cauchy's functional equation

$$m(x * y) = m(x)m(y)$$

for all x, y in$K$.

It is obvious that we can define an exponential family $\varphi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ with the property that the function $x \mapsto \varphi(x, \lambda)$ is an exponential of K for each complex λ, and for each exponential m of K there exists a unique complex λ such that $m(x) = \varphi(x, \lambda)$ holds for every x in $\mathbb{R}_0$. Hence the exponential family satisfies

$$\partial_1^2 \varphi(x, \lambda) + p(x)\partial_1 \varphi(x, \lambda) = \lambda \varphi(x, \lambda), \qquad \varphi(0, \lambda) = 1, \qquad \partial_1 \varphi(0, \lambda) = 0$$

for each x in $\mathbb{R}_+$ and complex number λ, where $p(x) = \frac{A'(x)}{A(x)}$. Here ∂_2 denotes the partial differential operator with respect to the second variable. Actually, (2.5) characterizes the exponential family. Clearly φ is C^∞ on $\mathbb{R}_0$ in x and entire in λ.

Using the exponential family we define *exponential monomials* on K as functions of the form $x \mapsto P(\partial_2)\varphi(x, \lambda)$, where P is a complex polynomial and λ is a complex number. The meaning of $P(\partial_2)$ is obvious. In particular, if $P \equiv 1$, then we have that any exponential function is an exponential monomial. Observe, that this is an analogous concept to the "exponential monomial" on polynomial hypergroups in several variables in [2] and [3]. Sums of exponential monomials are called *exponential polynomials*.

A particular subclass of exponential monomials is formed by the functions of the type $x \mapsto \partial_2^k \varphi(x, \lambda)$, where k is a nonnegative integer and λ is a complex number. Here we note that if $\lambda = 0$, then $\varphi(x, 0) = 1$ for each x in $\mathbb{R}_0$, hence the corresponding function $x \mapsto \partial_2^k \varphi(x, 0)$ is identically 1 for $k = 0$, and it is identically 0 for $k > 0$. For the sake of simplicity we will call the functions $x \mapsto \partial_2^k \varphi(x, \lambda)$ *special exponential monomials* if k is a nonnegative integer and λ is a complex number, supposing that if $\lambda = 0$, then $k = 0$. Our aim is to show that different special exponential monomials are linearly independent.

2. Main results

First we show that different exponential functions are linearly independent.

Theorem 2.1. *On any hypergroup different exponentials are linearly independent.*

Proof. Let $m_1, m_2, \dots, m_n$ be different exponentials on the hypergroup K. We prove by induction on n. For $n = 1$ the statement is trivial. Suppose that $n > 1$ and

$$c_1 m_1(t) + c_2 m_2(t) + \cdots + c_{n-1} m_{n-1}(t) + c_n m_n(t) = 0 \tag{2.1}$$

holds for each t in K. Let x, y be arbitrary in K and we integrate both sides of equation (2.1) with respect to the measure $\delta_x * \delta_y$:

$$c_1 m_1(x * y) + c_2 m_2(x * y) + \cdots + c_{n-1} m_{n-1}(x * y) + c_n m_n(x * y) = 0\,.$$

Using the exponential properties of the m's we have

$$c_1 m_1(x) m_1(y) + \cdots + c_{n-1} m_{n-1}(x) m_{n-1}(y) + c_n m_n(x) m_n(y) = 0\,. \tag{2.2}$$

Now we write $t = x$ in (2.1) and multiply the equation obtained by $m_n(y)$:

$$c_1 m_1(x) m_n(y) + \cdots + c_{n-1} m_{n-1}(x) m_n(y) + c_n m_n(x) m_n(y) = 0\,. \tag{2.3}$$

We subtract (2.3) from (2.2) to get

$$c_1 m_1(x)[m_1(y) - m_n(y)] + \cdots + c_{n-1} m_{n-1}(x)[m_{n-1}(y) - m_n(y)] = 0\,.$$

By assumption the exponentials $m_1, m_2, \dots, m_{n-1}$ are linearly independent, hence

$$c_i[m_i(y) - m_n(y)] = 0$$

for $i = 1, 2, \dots, n-1$. As $m_n \neq m_1$ we can choose a y in K such that $m_n(y) \neq m_1(y)$; it follows that $c_1 = 0$. Continuing this argument we get $c_i = 0$ for $i = 1, 2, \dots, n-1$, which also implies $c_n = 0$. The proof is complete. □

We shall also need the following result in the sequel.

Theorem 2.2. *Let K be a Sturm–Liouville hypergroup with the exponential family $\varphi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$, n a nonnegative integer and $\lambda_0 \neq 0$ a complex number. Then the special exponential monomials*

$$x \mapsto \varphi(x, \lambda_0), x \mapsto \partial_2 \varphi(x, \lambda_0), \dots, x \mapsto \partial_2^n \varphi(x, \lambda_0)$$

are linearly independent.

Proof. We prove the statement by induction on n, which is obviously true for $n = 0$. Suppose that we have proved it for n, and we prove it for $n + 1$, where n is some nonnegative integer. Proving our statement by contradiction we suppose that the function $x \mapsto \partial_2^{n+1} \varphi(x, \lambda_0)$ is a linear combination of the functions

$$x \mapsto \partial_2^k \varphi(x, \lambda_0)$$

for $k = 0, 1, \dots, n$, that is there are complex numbers c_k for $k = 0, 1, \dots, n$ such that

$$\partial_2^{n+1} \varphi(x, \lambda_0) = \sum_{k=0}^{n} c_k \partial_2^k \varphi(x, \lambda_0) \tag{2.4}$$

holds for each x in K. By the definition of the exponential family we have

$$\partial_1^2 \varphi(x, \lambda) + p(x) \partial_1 \varphi(x, \lambda) = \lambda \varphi(x, \lambda) \tag{2.5}$$

for each $x > 0$ and λ in $\mathbb{C}$. We differentiate both sides k times with respect to λ for $k = 0, 1, \ldots, n+1$. We then obtain

$$\partial_1^2 \partial_2^k \varphi(x,\lambda) + p(x)\partial_1 \partial_2^k \varphi(x,\lambda) = \sum_{j=0}^{k} \binom{k}{j} \lambda^{(j)} \cdot \partial_2^{k-j} \varphi(x,\lambda)\,,$$

or, equivalently

$$\partial_1^2 \partial_2^k \varphi(x,\lambda) + p(x)\partial_1 \partial_2^k \varphi(x,\lambda) = \lambda \partial_2^k \varphi(x,\lambda) + k\partial_2^{k-1} \varphi(x,\lambda) \tag{2.6}$$

for each $x > 0$ and λ in$\mathbb{C}$ and for $k = 0, 1, \ldots, n+1$. (Here $\partial_2^{-1}\varphi(x,\lambda) = 0$.) We shall use this equation several times in the sequel.

Differentiating equation (2.4) two times with respect to x we have the equations

$$\partial_1 \partial_2^{n+1} \varphi(x,\lambda_0) = \sum_{k=0}^{n} c_k \partial_1 \partial_2^k \varphi(x,\lambda_0)$$

and

$$\partial_1^2 \partial_2^{n+1} \varphi(x,\lambda_0) = \sum_{k=0}^{n} c_k \partial_1^2 \partial_2^k \varphi(x,\lambda_0)$$

for each $x > 0$. From these equations by (2.6) we have

$$\sum_{k=0}^{n} c_k \partial_1^2 \partial_2^k \varphi(x,\lambda_0) + \sum_{k=0}^{n} c_k p(x) \partial_1 \partial_2^k \varphi(x,\lambda_0) =$$

$$= \lambda_0 \partial_2^{n+1} \varphi(x,\lambda_0) + (n+1)\partial_2^n \varphi(x,\lambda_0) = \sum_{k=0}^{n} \lambda_0 c_k \partial_2^k \varphi(x,\lambda_0) + (n+1)\partial_2^n \varphi(x,\lambda_0)\,.$$

We can reorder the terms in this equation to obtain

$$\sum_{k=0}^{n} c_k [\partial_1^2 \partial_2^k \varphi(x,\lambda_0) + p(x)\partial_1 \partial_2^k \varphi(x,\lambda_0) - \lambda_0 \partial_2^k \varphi(x,\lambda_0)] =$$

$$= (n+1)\partial_2^n \varphi(x,\lambda_0)\,,$$

or, equivalently, using again (2.6)

$$\sum_{k=1}^{n} k c_k \partial_2^{k-1} \varphi(x,\lambda_0) - (n+1)\partial_2^n \varphi(x,\lambda_0) = 0\,. \tag{2.7}$$

But this is a contradiction, because equation (2.7) presents a nontrivial linear combination of linearly independent functions, which has the value zero. Hence the proof is complete. □

Now we are in the position to prove linear independence of the special exponential monomials.

Theorem 2.3. *On any Sturm–Liouville hypergroup different special exponential monomials are linearly independent.*

Proof. We have to show that any finite set of special exponential monomials is linearly independent. First we suppose that this set does not include the special exponential monomial 1. We may suppose that this set consists of special exponential monomials of the form

$$x \mapsto \partial_2^l \varphi(x, \lambda_j)$$

for $l = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, k$ with some restrictions on the nonnegative integer n and the positive integer k. Actually, we shall consider two cases: in the first case we suppose that we have proved the linear independence of the functions

$$x \mapsto \partial_2^l \varphi(x, \lambda_j)$$

for $l = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, k$, where n is a nonnegative integer and k is a positive integer, and we show that the function $x \mapsto \partial_2^{n+1} \varphi(x, \lambda_1)$ is not a linear combination of them, and in the second case we suppose that we have proved the linear independence of the functions

$$x \mapsto \partial_2^l \varphi(x, \lambda_s), x \mapsto \partial_2^{n+1} \varphi(x, \lambda_t)$$

for $l = 0, 1, \ldots, n$, $s = 1, 2, \ldots, k$ and $t = 1, 2, \ldots, j$, where n is a nonnegative integer, $k \geq 2$ is a positive integer and j is a positive integer with $j \leq k - 1$, and we show that the function $x \mapsto \partial_2^{n+1} \varphi(x, \lambda_{j+1})$ is not a linear combination of them. It is easy to see that any other case can be reduced to these two cases (eventually, by renumbering the λ's). We apply induction again: in the first case the statement is clearly true for $n = 0$ and $k = 1$. Also, if $n = 0$ and k is arbitrary, then the statement follows from Theorem 2.1, and if $k = 1$ and n is arbitrary, then the statement follows from Theorem 2.2. Hence we can consider the first case and prove by contradiction: suppose that the function $x \mapsto \partial_2^{n+1} \varphi(x, \lambda_1)$ is a linear combination of the functions

$$x \mapsto \partial_2^l \varphi(x, \lambda_j)$$

for $l = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, k$, where n is a nonnegative integer and k is a positive integer. This means that there are complex numbers $c_{l,j}$ for $l = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, k$ such that

$$\partial_2^{n+1} \varphi(x, \lambda_1) = \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} \partial_2^l \varphi(x, \lambda_j)$$

holds for each $x > 0$. Differentiating two times with respect to x we get the equations

$$\partial_1 \partial_2^{n+1} \varphi(x, \lambda_1) = \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} \partial_1 \partial_2^l \varphi(x, \lambda_j)$$

and

$$\partial_1^2 \partial_2^{n+1} \varphi(x, \lambda_1) = \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} \partial_1^2 \partial_2^l \varphi(x, \lambda_j)$$

for each $x > 0$. From these equations by (2.6) we have

$$\sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}\partial_1^2\partial_2^l\varphi(x,\lambda_j) + \sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}p(x)\partial_1\partial_2^l\varphi(x,\lambda_j) =$$

$$= \lambda_1\partial_2^{n+1}\varphi(x,\lambda_1)+(n+1)\partial_2^n\varphi(x,\lambda_1) = \sum_{l=0}^{n}\sum_{j=1}^{k}\lambda_1 c_{l,j}\partial_2^l\varphi(x,\lambda_j)+(n+1)\partial_2^n\varphi(x,\lambda_1)\,.$$

We can reorder the terms in this equation to obtain

$$\sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}[\partial_1^2\partial_2^l\varphi(x,\lambda_j) + p(x)\partial_1\partial_2^l\varphi(x,\lambda_j) - \lambda_1\partial_2^l\varphi(x,\lambda_j)] =$$

$$= (n+1)\partial_2^n\varphi(x,\lambda_1)\,,$$

or, equivalently, using again (2.6)

$$\sum_{l=1}^{n}\sum_{j=1}^{k} l c_{l,j}\partial_2^{l-1}\varphi(x,\lambda_j) - (n+1)\partial_2^n\varphi(x,\lambda_1) = 0\,. \tag{2.8}$$

But this is a contradiction, because equation (2.8) presents a nontrivial linear combination of linearly independent functions, which has the value zero. Hence the proof of our statement in the first case is complete.

Now we consider the second case and we prove again by contradiction: we suppose that we have proved the linear independence of the functions

$$x \mapsto \partial_2^l\varphi(x,\lambda_s), x \mapsto \partial_2^{n+1}\varphi(x,\lambda_t)$$

for $l = 0,1,\ldots,n$, $s = 1,2,\ldots,k$ and $t = 1,2,\ldots,j$, where n is a nonnegative integer, $k \geq 2$ is a positive integer and j is a positive integer with $j \leq k-1$, and we show that the function $x \mapsto \partial_2^{n+1}\varphi(x,\lambda_{j+1})$ is a linear combination of them. This means that there are complex numbers $c_{l,s}, d_t$ for $l = 0,1,\ldots,n$ and $s = 1,2,\ldots,k$, $t = 1,2,\ldots,j$ such that

$$\partial_2^{n+1}\varphi(x,\lambda_{j+1}) = \sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}\partial_2^l\varphi(x,\lambda_s) + \sum_{t=1}^{j} d_t\partial_2^{n+1}\varphi(x,\lambda_t)$$

holds for each $x > 0$. Differentiating two times with respect to x we get the equations

$$\partial_1\partial_2^{n+1}\varphi(x,\lambda_{j+1}) = \sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}\partial_1\partial_2^l\varphi(x,\lambda_s) + \sum_{t=1}^{j} d_t\partial_1\partial_2^{n+1}\varphi(x,\lambda_t)$$

and

$$\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_{j+1}) = \sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}\partial_1^2\partial_2^l\varphi(x,\lambda_s) + \sum_{t=1}^{j} d_t\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_t)$$

for each $x > 0$. From these equations by (2.6) we have

$$\sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}\partial_1^2\partial_2^l\varphi(x,\lambda_s) + \sum_{t=1}^{j} d_t\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_t)+$$

$$+\sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}p(x)\partial_1\partial_2^l\varphi(x,\lambda_s)+\sum_{t=1}^{j} d_t p(x)\partial_1\partial_2^{n+1}\varphi(x,\lambda_t)=$$
$$=\lambda_{j+1}\partial_2^{n+1}\varphi(x,\lambda_{j+1})+(n+1)\partial_2^n\varphi(x,\lambda_{j+1})=$$
$$=\sum_{l=0}^{n}\sum_{s=1}^{k}\lambda_{j+1}c_{l,s}\partial_2^l\varphi(x,\lambda_s)+\sum_{t=1}^{j} d_t\lambda_{j+1}\partial_2^{n+1}\varphi(x,\lambda_t)+(n+1)\partial_2^n\varphi(x,\lambda_{j+1})\,.$$

We can reorder the terms in this equation to obtain

$$\sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}[\partial_1^2\partial_2^l\varphi(x,\lambda_s)+p(x)\partial_1\partial_2^l\varphi(x,\lambda_s)-\lambda_{j+1}\partial_2^l\varphi(x,\lambda_s)]+$$
$$+\sum_{t=1}^{j} d_t[\partial_1^2\partial_2^{n+1}\varphi(x,\lambda_t)+p(x)\partial_1\partial_2^{n+1}\varphi(x,\lambda_t)-\lambda_{j+1}\partial_2^{n+1}\varphi(x,\lambda_t)]=$$
$$=(n+1)\partial_2^n\varphi(x,\lambda_{j+1})\,,$$

or, equivalently, using again (2.6)

$$\sum_{l=1}^{n}\sum_{s=1}^{k} l c_{l,s}\partial_2^{l-1}\varphi(x,\lambda_s)+\sum_{t=1}^{j} d_t(n+1)\partial_2^n\varphi(x,\lambda_t)$$
$$+\sum_{l=0}^{n}\sum_{s=1}^{k} c_{l,s}(\lambda_s-\lambda_{j+1})\partial_2^l\varphi(x,\lambda_s)$$
$$+\sum_{t=1}^{j} d_t(\lambda_t-\lambda_{j+1})\partial_2^{n+1}\varphi(x,\lambda_t)-(n+1)\partial_2^n\varphi(x,\lambda_{j+1})=0\,.$$

The term containing $\partial_2^n\varphi(x,\lambda_{j+1})$ does not appear in the first two sums, it appears with zero coefficient in the third sum, it does not appear in the fourth sum, hence its coefficient on the left hand side is $-(n+1)\neq 0$. This is a contradiction and the proof of our statement also in the second case is complete.

To finish the proof we have to consider the case where the special exponential monomial 1 is in the set of the exponential monomials. We prove by contradiction again: suppose that there are nonzero complex numbers $\lambda_1,\lambda_2,\dots,\lambda_k$ and there is a nonnegative integer n such that

$$1=\sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}\partial_2^l\varphi(x,\lambda_j) \tag{2.9}$$

holds for each $x>0$. Differentiating equation two times with respect to x we obtain

$$0=\sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}\partial_1\partial_2^l\varphi(x,\lambda_j) \tag{2.10}$$

and

$$0=\sum_{l=0}^{n}\sum_{j=1}^{k} c_{l,j}\partial_1^2\partial_2^l\varphi(x,\lambda_j) \tag{2.11}$$

for each $x > 0$. Adding equations (2.10) and (2.11) we get

$$0 = \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} [\partial_1^2 \partial_2^l \varphi(x, \lambda_j) + p(x) \partial_1 \partial_2^l \varphi(x, \lambda_j)] =$$

$$= \sum_{l=0}^{n} \sum_{j=1}^{k} c_{l,j} [\lambda_j \partial_2^l \varphi(x, \lambda_j) + l \partial_2^{l-1} \varphi(x \lambda_j)]$$

for each $x > 0$. On the right hand side we have a linear combination of linearly independent functions. The coefficient of $\partial_2^n \varphi(x, \lambda_j)$ is $c_{n,j} \lambda_j$, which must be zero, hence $c_{n,j} = 0$ for $j = 1, 2, \ldots, k$. Continuing recursively we get that $c_{n-1,j} = c_{n-2,j} = \cdots = c_{0,j} = 0$ for $j = 1, 2, \ldots, k$, which contradicts to equation (2.9). Now the proof of the theorem is complete. □

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References

1. W.R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyter Studies in Mathematics, de Gruyter, Berlin, New York, 1995.
2. L. Székelyhidi, *Spectral Analysis and Spectral Synthesis on Polynomial Hypergroups*, Monatsh. Math. **141** (2004), no. 1, 33–43.
3. L. Székelyhidi, *Spectral Synthesis on Multivariate Polynomial Hypergroups*, Monatsh. Math. **153** (2008), no. 2, 145–152.

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