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REFINED MULTIDIMENSIONAL HARDY-TYPE INEQUALITIES VIA SUPERQUADRACITY

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Dedicated to Professor Josip Pečarić

Submitted by José M. Isidro

Abstract. Some new refined multidimensional Hardy-type inequalities for $p \geq 2$ and their duals are derived and discussed. Moreover, these inequalities hold in the reversed direction when $1 < p \leq 2$. The results obtained are based mainly on some new results for superquadratic and subquadratic functions. In particular, our results further extend the recent results in [J.A. Oguntuase and L.-E. Persson, Refinement of Hardy's inequalities via superquadratic and subquadratic functions, J. Math. Anal. Appl., 339 (2008), no. 2, 1305– 1312] to a multidimensional setting.

1. **Introduction**

In 1920 G.H. Hardy [4] announced and proved in [5] (see also [6, 9, 10]) the following result: Let $p > 1$ and $f \in L^p(0, \infty)$ be a non-negative function, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)dx \tag{1.1}$$

holds, where the constant $\left(\frac{p}{p-1}\right)^p$ on the right hand side of (1.1) is the best possible. This interesting result is today referred to as the classical Hardy's integral inequality. Inequality (1.1) has an interesting prehistory and history (see e.g. [6, 8, 9] and the references given there). A well-known simple fact is that

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(1.1) can equivalently (via the substitution $f(x) = h(x^{1-\frac{1}{p}})x^{-\frac{1}{p}}$) be rewritten in the form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x h(t)dt\right)^p \frac{dx}{x} \leq \int_0^\infty h^p(x)\frac{dx}{x}$$

and in this form it even holds with equality when $p = 1$ (see [9] and also [7]). In this form we see that Hardy's inequality is a simple consequence of Jensen's inequality but this was not discovered in the dramatic period when Hardy discovered and finally proved his inequality in 1925 (see [6, 8, 9]).

In a recent paper Oguntuase and Persson [11] used mainly the notion of superquadratic and subquadratic functions to obtain a new refinement of the Hardy inequality for $p \geq 2$, which holds in the reversed direction for $1 < p \leq 2$. The result is indeed surprising and in the breaking point $p = 2$ we even get equality (like some new Parseval formula for this operator) and this is completely different from the usual Hardy situation where the breaking point is $p = 1$ and no such equality can appear in this point. In this paper we prove a multidimensional version of this result. The key point is to use the notion of superquadratic and subquadratic functions introduced by Abramovich, Jameson and Sinnamon in [2] (see also [3]).

The paper is organized as follows: In Section 2 we present and prove some multidimensional inequalities involving superquadratic and subquadratic functions of independent interest. In Section 3 our new multidimensional refined Hardy type inequalities and their proofs are presented. Our final Section is devoted to some concluding remarks and examples.

Notations and Conventions Throughout this paper we use bold letters to denote $n-$tuples of real numbers, e.g. $\mathbf{x} = (x_1, ..., x_n)$, or $\mathbf{y} = (y_1, ..., y_n)$. Also, we set $\mathbf{0} = (0, ..., 0) \in \mathbb{R}^n$ and $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^n$. Furthermore, the relations $<, \leq, >$, and $\geq$ are, as usual, defined componentwise. For example, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we write $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$, $i = 1, ..., n$. Moreover, $\mathbf{0} < \boldsymbol{b} \leq \boldsymbol{\infty}$ means that $0 < b_i \leq \infty$, $i = 1, ..., n$. In addition, we introduce a notation for $n-$cells, that is, axis parallel rectangular blocks in $\mathbb{R}^n$. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, $\boldsymbol{a} < \boldsymbol{b}$, let

$$\begin{aligned} (\boldsymbol{a}, \boldsymbol{b}) &= (a_1, b_1) \times ... \times (a_n, b_n) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a} < \boldsymbol{x} < \boldsymbol{b}\}, \\ (\boldsymbol{a}, \boldsymbol{b}] &= (a_1, b_1] \times ... \times (a_n, b_n] = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a} < \boldsymbol{x} \leq \boldsymbol{b}\}, \end{aligned}$$

and analogously also for $[\boldsymbol{a}, \boldsymbol{b})$ and $[\boldsymbol{a}, \boldsymbol{b}]$. In particular, we have $\mathbb{R}^n_+ = (\mathbf{0}, \boldsymbol{\infty})$, $(\mathbf{0},\ \boldsymbol{\infty}] = \{\mathbf{0} < \boldsymbol{x} \leq \boldsymbol{\infty}\}$, and $[\mathbf{0}, \boldsymbol{\infty}) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \mathbf{0}\}$. Furthermore, all functions are assumed to be measurable and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$, and $\frac{0}{0}$ are taken to be equal to zero. Moreover, $u(\boldsymbol{x})$ denotes a weight function, i.e. a nonnegative and measurable function on $\mathbb{R}^n$, and we define a corresponding weight function $v(\boldsymbol{t})$ by

$$v(\boldsymbol{t}) = \begin{cases} t_1...t_n \int_{t_1}^{b_1} ... \int_{t_n}^{b_n} \frac{u(\boldsymbol{x})}{x_1^2...x_n^2} d\boldsymbol{x}, & \boldsymbol{t} \in (\mathbf{0}, \mathbf{b}), \\ \frac{1}{t_1...t_n} \int_{b_1}^{t_1} ... \int_{b_n}^{t_n} u(\boldsymbol{x}) d\boldsymbol{x} < \infty, & \boldsymbol{t} \in (\mathbf{b}, \boldsymbol{\infty}). \end{cases} \tag{1.2}$$

2. Multidimensional Hardy-type inequalities for superquadratic functions

First, we state a definition and some results in [2], which are germane to the proofs of our propositions below.

Definition 2.1. (See [2, Definition 2.1].) A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C_x (y-x) \quad \text{for all } y \geq 0.$$

We say that φ is subquadratic if $-\varphi$ is superquadratic.

Lemma 2.2. *(See [2, Theorem 2.3].) Let (Ω, μ) be a probability measure space. The inequality*

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leq \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right) d\mu(s) \tag{2.1}$$

holds for all probability measures μ and all nonnegative $\mu-$integrable functions f if and only if φ is superquadratic. Moreover, (2.1) holds in the reversed direction if and only if φ is subquadratic.

Lemma 2.3. *(See [2, Lemma 3.1].) Suppose $\varphi : [0,\infty) \to \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\frac{\varphi'(x)}{x}$ is nondecreasing, then φ is superquadratic.*

Remark 2.4. According to Lemmas 2.2 and 2.3 it yields that if $\varphi(t) = t^p, p \geq 2$, in Lemma 2.2, then

$$\left(\int_\Omega f(s)d\mu(s)\right)^p \leq \int_\Omega (f(s))^p d\mu(s) - \int_\Omega \left|f(s) - \int_\Omega f(s)d\mu(s)\right|^p d\mu(s)$$

holds and the reversed inequality holds when $1 < p \leq 2$ (see also [1, Example 1, p. 1448]).

Proposition 2.5. *Let $\boldsymbol{b} \in (\boldsymbol{0},\boldsymbol{\infty})$, $u : (\boldsymbol{0},\boldsymbol{b}) \to \mathbb{R}$ be a weight which is locally integrable in $(\boldsymbol{0},\boldsymbol{b})$ and $v(\boldsymbol{x})$ be defined by (1.2). Suppose $I = (a,c)$, $0 \leq a < c \leq \infty$, $\varphi : I \to \mathbb{R}$, and $f : (\boldsymbol{0},\boldsymbol{b}) \to \mathbb{R}$ is an integrable function, such that $f(\boldsymbol{x}) \in I$, for all $\boldsymbol{x} \in (\boldsymbol{0},\boldsymbol{b})$.*

(i) If φ is superquadratic, then the following inequality holds:

$$\begin{aligned}
&\int_0^{b_1} ... \int_0^{b_n} u(\boldsymbol{x})\varphi\left(\frac{1}{x_1...x_n}\int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t})d\boldsymbol{t}\right)\frac{d\boldsymbol{x}}{x_1...x_n} \\
&+ \int_0^{b_1} ... \int_0^{b_n} \int_{t_1}^{b_1} ... \int_{t_n}^{b_n} \varphi\left(\left|f(\boldsymbol{t}) - \frac{1}{x_1...x_n}\int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t})d\boldsymbol{t}\right|\right)\frac{u(\boldsymbol{x})}{x_1^2...x_n^2}d\boldsymbol{x}d\boldsymbol{t} \\
&\leq \int_0^{b_1} ... \int_0^{b_n} v(\boldsymbol{x})\varphi(f(\boldsymbol{x}))\frac{d\boldsymbol{x}}{x_1...x_n}.
\end{aligned} \tag{2.2}$$

(ii) If φ is subquadratic, then (2.2) holds in the reversed direction.

Remark 2.6. If we consider Proposition 2.5 for $u(\boldsymbol{x}) \equiv 1$, then we have

$$v(\boldsymbol{x}) = x_1 ... x_n \int_{x_1}^{b_1} ... \int_{x_n}^{b_n} \frac{d\boldsymbol{t}}{t_1^2 ... t_n^2} = \prod_{i=1}^{n} \left(1 - \frac{x_i}{b_i}\right), \; \boldsymbol{x} \in (\boldsymbol{0}, \mathbf{b}),$$

so (2.2) reads as follows: If φ is a superquadratic, then

$$\begin{aligned}
&\int_0^{b_1} ... \int_0^{b_n} \varphi\left(\frac{1}{x_1 ... x_n} \int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t}) d\boldsymbol{t}\right) \frac{d\boldsymbol{x}}{x_1 ... x_n} \\
&+ \int_0^{b_1} ... \int_0^{b_n} \int_{t_1}^{b_1} ... \int_{t_n}^{b_n} \varphi\left(\left| f(\boldsymbol{t}) - \frac{1}{x_1 ... x_n} \int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t}) d\boldsymbol{t}\right|\right) \frac{d\boldsymbol{x}}{x_1^2 ... x_n^2} d\boldsymbol{t} \\
&\leq \int_0^{b_1} ... \int_0^{b_n} \varphi(f(\boldsymbol{x})) \prod_{i=1}^{n} \left(1 - \frac{x_i}{b_i}\right) \frac{d\boldsymbol{x}}{x_1 ... x_n}, \qquad (2.3)
\end{aligned}$$

and (2.3) holds in the reversed direction when φ is subquadratic.

Proof. (i) Let φ be superquadratic. Then, by applying the refined Jensen's inequality (2.1) to the first term on the left hand side of (2.2) and then Fubini theorem repeatedly, we have that

$$\begin{aligned}
&\int_0^{b_1} ... \int_0^{b_n} u(\boldsymbol{x}) \varphi\left(\frac{1}{x_1 ... x_n} \int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t}) d\boldsymbol{t}\right) \frac{d\boldsymbol{x}}{x_1 ... x_n} \\
\leq\; & \int_0^{b_1} ... \int_0^{b_n} \frac{u(\boldsymbol{x})}{x_1^2 ... x_n^2} \left(\int_0^{x_1} ... \int_0^{x_n} \varphi(f(\boldsymbol{t})) d\boldsymbol{t}\right) d\boldsymbol{x} \\
&- \int_0^{b_1} ... \int_0^{b_n} \frac{u(\boldsymbol{x})}{x_1^2 ... x_n^2} \int_0^{x_1} ... \int_0^{x_n} \varphi\left(\left| f(\boldsymbol{t}) - \frac{1}{x_1 ... x_n} \int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t}) d\boldsymbol{t}\right|\right) d\boldsymbol{t} d\boldsymbol{x} \\
=\; & \int_0^{b_1} ... \int_0^{b_n} \varphi(f(\boldsymbol{t})) \int_{t_1}^{b_1} ... \int_{t_n}^{b_n} \frac{u(\boldsymbol{x})}{x_1^2 ... x_n^2} d\boldsymbol{x} d\boldsymbol{t} \\
&- \int_0^{b_1} ... \int_0^{b_n} \int_{t_1}^{b_1} ... \int_{t_n}^{b_n} \varphi\left(\left| f(\boldsymbol{t}) - \frac{1}{x_1 ... x_n} \int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t}) d\boldsymbol{t}\right|\right) \frac{u(\boldsymbol{x})}{x_1^2 ... x_n^2} d\boldsymbol{x} d\boldsymbol{t} \\
=\; & \int_0^{b_1} ... \int_0^{b_n} v(\boldsymbol{t}) \varphi(f(\boldsymbol{t})) \frac{d\boldsymbol{t}}{t_1 ... t_n} \\
&- \int_0^{b_1} ... \int_0^{b_n} \int_{t_1}^{b_1} ... \int_{t_n}^{b_n} \varphi\left(\left| f(\boldsymbol{t}) - \frac{1}{x_1 ... x_n} \int_0^{x_1} ... \int_0^{x_n} f(\boldsymbol{t}) d\boldsymbol{t}\right|\right) \frac{u(\boldsymbol{x})}{x_1^2 ... x_n^2} d\boldsymbol{x} d\boldsymbol{t},
\end{aligned}$$

from which (2.2) follows.

(ii) Similar to the proof of (i) and by making the same calculations with φ subquadratic we see that only the inequality sign will be reversed. The proof is complete. □

Proposition 2.7. *Let $\boldsymbol{b} \in [\boldsymbol{0}, \boldsymbol{\infty})$, $u : (\boldsymbol{b}, \boldsymbol{\infty}) \to \mathbb{R}$ be a weight which is locally integrable in $(\boldsymbol{0}, \boldsymbol{b})$ and $v(\boldsymbol{x})$ be defined by (1.2). Suppose $I = (a, c)$, $0 \leq a < c \leq \infty$, $\varphi : I \to \mathbb{R}$, and $f : (\boldsymbol{b}, \boldsymbol{\infty}) \to \mathbb{R}$ is an integrable function, such that $f(\boldsymbol{x}) \in I$, for all $\boldsymbol{x} \in (\boldsymbol{b}, \boldsymbol{\infty})$.*

(i) If φ is superquadratic, then the following inequality holds:

$$\int_{b_1}^{\infty}...\int_{b_n}^{\infty} u(\boldsymbol{x})\varphi\left(x_1...x_n\int_{x_1}^{\infty}...\int_{x_n}^{\infty} f(\boldsymbol{t})\frac{d\boldsymbol{t}}{t_1^2...t_n^2}\right)\frac{d\boldsymbol{x}}{x_1...x_n}$$
$$+\int_{b_1}^{\infty}...\int_{b_n}^{\infty}\int_{b_1}^{t_1}...\int_{b_n}^{t_n}\varphi\left(\left|f(\boldsymbol{t})-x_1...x_n\int_{x_1}^{\infty}...\int_{x_n}^{\infty} f(\boldsymbol{t})\frac{d\boldsymbol{t}}{t_1^2...t_n^2}\right|\right)u(\boldsymbol{x})d\boldsymbol{x}\frac{d\boldsymbol{t}}{t_1^2...t_n^2}$$
$$\leq \int_{b_1}^{\infty}...\int_{b_n}^{\infty} v(\boldsymbol{x})\varphi(f(\boldsymbol{x}))\frac{d\boldsymbol{x}}{x_1...x_n}. \tag{2.4}$$

(ii) If φ is subquadratic, then the inequality sign in (2.4) is reversed.

Proof. The proof is similar to that of Proposition 2.5 so we omit the details. □

Remark 2.8. By setting $u(\boldsymbol{x}) \equiv 1$ in Proposition 2.7 we obtain that

$$v(\boldsymbol{x}) = \frac{1}{x_1...x_n}\int_{b_1}^{x_1}...\int_{b_n}^{x_n} d\boldsymbol{t} = \prod_{i=1}^{n}\left(1-\frac{b_i}{x_i}\right),\ \boldsymbol{x}\in(\boldsymbol{b},\boldsymbol{\infty}).$$

Thus, (2.4) becomes

$$\int_{b_1}^{\infty}...\int_{b_n}^{\infty}\varphi\left(x_1...x_n\int_{x_1}^{\infty}...\int_{x_n}^{\infty} f(\boldsymbol{t})\frac{d\boldsymbol{t}}{t_1^2...t_n^2}\right)\frac{d\boldsymbol{x}}{x_1...x_n}$$
$$+\int_{b_1}^{\infty}...\int_{b_n}^{\infty}\int_{b_1}^{t_1}...\int_{b_n}^{t_n}\varphi\left(\left|f(\boldsymbol{t})-x_1...x_n\int_{x_1}^{\infty}...\int_{x_n}^{\infty} f(\boldsymbol{t})\frac{d\boldsymbol{t}}{t_1^2...t_n^2}\right|\right)d\boldsymbol{x}\frac{d\boldsymbol{t}}{t_1^2...t_n^2}$$
$$\leq \int_{b_1}^{\infty}...\int_{b_n}^{\infty}\varphi(f(\boldsymbol{x}))\prod_{i=1}^{n}\left(1-\frac{b_i}{x_i}\right)\frac{d\boldsymbol{x}}{x_1...x_n}, \tag{2.5}$$

for any superquadratic function φ and the inequality sign in (2.5) is reversed when φ is subquadratic.

Remark 2.9. Note that in the one-dimensional case ($n = 1$), Propositions 2.5 and 2.7 reduce to the corresponding Propositions 2.1 and 2.2 in [11], respectively.

3. **Refined Multidimensional Hardy-type inequalities**

Our first result in this section reads:

Theorem 3.1. *Let $1 < p < \infty$, $\boldsymbol{k} = (k_1,...k_n) \in \mathbb{R}^n$ be such that $k_i > 1$ $(i = 1,...,n)$, $0 < \boldsymbol{b} \leq \infty$, and let f be locally integrable on $(\boldsymbol{0},\boldsymbol{b})$ such that $0 < \int_0^{b_1}...\int_0^{b_n}\prod_{i=1}^{n} x_i^{p-k_i} f^p(\boldsymbol{x})d\boldsymbol{x} < \boldsymbol{\infty}$.*

(i) If $p \geq 2$, then

$$\int_0^{b_1}\cdots\int_0^{b_n}\prod_{i=1}^n x_i^{-k_i}\left(\int_0^{x_1}\cdots\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right)^p d\boldsymbol{x}$$
$$+\left(\prod_{i=1}^n\frac{k_i-1}{p}\right)\int_0^{b_1}\cdots\int_0^{b_n}\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\left|\prod_{i=1}^n\frac{p}{k_i-1}\left(\frac{t_i}{x_i}\right)^{1-\frac{k_i-1}{p}}f(\boldsymbol{t})\right.$$
$$\left.-\frac{1}{x_1...x_n}\int_0^{x_1}\cdots\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right|^p\prod_{i=1}^n x_i^{p-k_i-\frac{k_i-1}{p}}d\boldsymbol{x}\prod_{i=1}^n t_i^{\frac{k_i-1}{p}-1}d\boldsymbol{t}$$
$$\leq\left(\prod_{i=1}^n\frac{p}{k_i-1}\right)^p\int_0^{b_1}\cdots\int_0^{b_n}\prod_{i=1}^n\left(1-\left[\frac{x_i}{b_i}\right]^{\frac{k_i-1}{p}}\right)x_i^{p-k_i}f^p(\boldsymbol{x})d\boldsymbol{x}. \tag{3.1}$$

(ii) If $1<p\leq 2$, then inequality (3.1) holds in the reversed direction.

Remark 3.2. For the case $n=1$ Theorem 3.1 coincides with the corresponding Theorem 3.1 in [11].

Proof. (i) Applying Proposition 2.5 with the superquadratic function $\varphi(x)=x^p$, $p\geq 2$, and $u(\boldsymbol{x})\equiv 1$ (cf. Remark 2.6), we find that

$$\int_0^{b_1}\cdots\int_0^{b_n}\left(\frac{1}{x_1...x_n}\int_0^{x_1}\cdots\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right)^p\frac{d\boldsymbol{x}}{x_1...x_n}$$
$$+\int_0^{b_1}\cdots\int_0^{b_n}\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\left|f(\boldsymbol{t})-\frac{1}{x_1...x_n}\int_0^{x_1}\cdots\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right|^p\frac{d\boldsymbol{x}}{x_1^2...x_n^2}d\boldsymbol{t}$$
$$\leq\int_0^{b_1}\cdots\int_0^{b_n}\prod_{i=1}^n\left(1-\frac{x_i}{b_i}\right)f^p(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1...x_n}. \tag{3.2}$$

Denote the first and second terms on the left hand side of (3.2) by I_1 and I_2 and the term on the right hand side by I_3, respectively. Replace the parameter b_i by $a_i=b_i^{\frac{k_i-1}{p}}$, $i=1,2,...,n$, and choose for f the function $\boldsymbol{x}\mapsto f(x_1^{\frac{p}{k_1-1}},...,x_n^{\frac{p}{k_n-1}})\prod_{i=1}^n x_i^{\frac{p}{k_i-1}-1}$. Thereafter, use the substitutions $y_i=x_i^{\frac{p}{k_i-1}}$ and $s_i=t_i^{\frac{p}{k_i-1}}$, $i=1,...,n$. Then

$$I_1=\int_0^{a_1}\cdots\int_0^{a_n}\left(\frac{1}{x_1...x_n}\int_0^{x_1}\cdots\int_0^{x_n}f(t_1^{\frac{p}{k_i-1}},...,t_n^{\frac{p}{k_n-1}})\prod_{i=1}^n t_i^{\frac{p}{k_i-1}-1}d\boldsymbol{t}\right)^p\frac{d\boldsymbol{x}}{x_1...x_n}$$
$$=\left(\prod_{i=1}^n\frac{k_i-1}{p}\right)^p\int_0^{a_1}\cdots\int_0^{a_n}\left(\frac{1}{x_1...x_n}\int_0^{x_1^{\frac{p}{k_1-1}}}\cdots\int_0^{x_n^{\frac{p}{k_n-1}}}f(\boldsymbol{s})d\boldsymbol{s}\right)^p\frac{d\boldsymbol{x}}{x_1...x_n}$$
$$=\left(\prod_{i=1}^n\frac{k_i-1}{p}\right)^{p+1}\int_0^{b_1}\cdots\int_0^{b_n}\prod_{i=1}^n y_i^{-k_i}\left(\int_0^{y_1}\cdots\int_0^{y_n}f(\boldsymbol{s})d\boldsymbol{s}\right)^p d\boldsymbol{y}, \tag{3.3}$$

$$
\begin{aligned}
I_2 = &\int_0^{a_1} \cdots \int_0^{a_n} \int_{t_1}^{a_1} \cdots \int_{t_n}^{a_n} \Bigg| f(t_1^{\frac{p}{k_1-1}}, \dots, t_n^{\frac{p}{k_n-1}}) \prod_{i=1}^n t_i^{\frac{p}{k_i-1}-1} \\
&- \frac{1}{x_1 \dots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1^{\frac{p}{k_1-1}}, \dots, t_n^{\frac{p}{k_n-1}}) \prod_{i=1}^n t_i^{\frac{p}{k_i-1}-1} d\boldsymbol{t} \Bigg|^p \frac{d\boldsymbol{x}}{x_1^2 \dots x_n^2} d\boldsymbol{t} \\
= &\left(\prod_{i=1}^n \frac{k_i-1}{p} \right)^{p+1} \int_0^{b_1} \cdots \int_0^{b_n} \int_{s_1^{\frac{k_1-1}{p}}}^{a_1} \cdots \int_{s_n^{\frac{k_n-1}{p}}}^{a_n} \Bigg| \prod_{i=1}^n \frac{p}{k_i-1} s_i^{1-\frac{k_i-1}{p}} f(\boldsymbol{s}) \\
&- \frac{1}{x_1 \dots x_n} \int_0^{x^{\frac{p}{k_1-1}}} \cdots \int_0^{x^{\frac{p}{k_n-1}}} f(\boldsymbol{s}) d\boldsymbol{s} \Bigg|^p \frac{d\boldsymbol{x}}{x_1^2 \dots x_n^2} \prod_{i=1}^n s_i^{\frac{k_i-1}{p}-1} d\boldsymbol{s} \\
= &\left(\prod_{i=1}^n \frac{k_i-1}{p} \right)^{p+2} \int_0^{b_1} \cdots \int_0^{b_n} \int_{s_1}^{b_1} \cdots \int_{s_n}^{b_n} \Bigg| \prod_{i=1}^n \frac{p}{k_i-1} s_i^{1-\frac{k_i-1}{p}} f(\boldsymbol{s}) \\
&- \frac{1}{y_1^{\frac{k_1-1}{p}} \dots y_n^{\frac{k_n-1}{p}}} \int_0^{y_1} \cdots \int_0^{y_n} f(\boldsymbol{s}) d\boldsymbol{s} \Bigg|^p \prod_{i=1}^n y_i^{\frac{1-k_i}{p}-1} d\boldsymbol{y} \prod_{i=1}^n s_i^{\frac{k_i-1}{p}-1} d\boldsymbol{s} \\
= &\left(\prod_{i=1}^n \frac{k_i-1}{p} \right)^{p+2} \int_0^{b_1} \cdots \int_0^{b_n} \int_{s_1}^{b_1} \cdots \int_{s_n}^{b_n} \Bigg| \prod_{i=1}^n \frac{p}{k_i-1} \left(\frac{s_i}{y_i} \right)^{1-\frac{k_i-1}{p}} f(\boldsymbol{s}) \\
&- \frac{1}{y_1 \dots y_n} \int_0^{y_1} \cdots \int_0^{y_n} f(\boldsymbol{s}) d\boldsymbol{s} \Bigg|^p \prod_{i=1}^n y_i^{p-k_i-\frac{k_i-1}{p}} d\boldsymbol{y} \prod_{i=1}^n s_i^{\frac{k_i-1}{p}-1} d\boldsymbol{s} \qquad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_0^{a_1} \cdots \int_0^{a_n} f^p(x_1^{\frac{p}{k_1-1}}, \dots, x_n^{\frac{p}{k_n-1}}) \prod_{i=1}^n x_i^{p\left(\frac{p}{k_i-1}-1\right)} \left(1 - \frac{x_i}{a_i}\right) \frac{d\boldsymbol{x}}{x_1 \dots x_n} \\
&= \left(\prod_{i=1}^n \frac{k_i-1}{p} \right) \int_0^{b_1} \cdots \int_0^{b_n} \prod_{i=1}^n \left(1 - \left[\frac{y_i}{b_i} \right]^{\frac{k_i-1}{p}} \right) y_i^{p-k_i} f^p(\boldsymbol{y}) d\boldsymbol{y}. \qquad (3.5)
\end{aligned}
$$

The proof of (3.1) follows by combining (3.2)-(3.5).

(ii) The proof for the case $1 < p \leq 2$ is similar and the only difference is that in this case all the inequalities signs are reversed. □

In the next result we state the dual of Theorem 3.1.

Theorem 3.3. *Let* $1 < p < \infty$, $\boldsymbol{k} = (k_{1,} \dots k_n) \in \mathbb{R}^n$ *be such that* $k_i < 1$, $i = 1, 2, \dots, n$, $0 \leq \boldsymbol{b} < \infty$, *and let* f *be locally integrable on* $(\boldsymbol{b}, \boldsymbol{\infty})$ *and such that* $0 < \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \prod_{i=1}^n x_i^{p-k_i} f^p(\boldsymbol{x}) d\boldsymbol{x} < \infty$.

(iii) If $p \geq 2$, then

$$\int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \prod_{i=1}^{n} x_i^{-k_i} \left(\int_{x_1}^{\infty} \ldots \int_{x_n}^{\infty} f(\boldsymbol{t}) d\boldsymbol{t} \right)^p d\boldsymbol{x}$$
$$+ \left(\prod_{i=1}^{n} \frac{1-k_i}{p} \right) \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \int_{b_1}^{t_1} \ldots \int_{b_n}^{t_n} \left| \prod_{i=1}^{n} \frac{p}{1-k_i} \left(\frac{t_i}{x_i} \right)^{\frac{1-k_i}{p}+1} f(\boldsymbol{t}) \right.$$
$$\left. - \frac{1}{x_1 \ldots x_n} \int_{x_1}^{\infty} \ldots \int_{x_n}^{\infty} f(\boldsymbol{t}) d\boldsymbol{t} \right|^p \prod_{i=1}^{n} x_i^{\frac{1-k_i}{p}+p-k_i} d\boldsymbol{x} \prod_{i=1}^{n} t_i^{\frac{k_i-1}{p}-1} d\boldsymbol{t}$$
$$\leq \left(\prod_{i=1}^{n} \frac{p}{1-k_i} \right)^p \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \prod_{i=1}^{n} \left(1 - \left[\frac{b_i}{x_i} \right]^{\frac{1-k_i}{p}} \right) x_i^{p-k_i} f^p(\boldsymbol{x}) d\boldsymbol{x}. \tag{3.6}$$

(iv) If $1 < p \leq 2$, then inequality (3.6) holds in the reversed direction.

Remark 3.4. Note that for the case $n = 1$ Theorem 3.3 reduces to Theorem 3.2 in [11].

Proof. *(iii)* By applying Proposition 2.7 with the superquadratic function $\varphi(x) = x^p$, $p \geq 2$ and $u(\boldsymbol{x}) \equiv 1$ (cf. Remark 2.8), we find that

$$\int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \left(x_1 \ldots x_n \int_{x_1}^{\infty} \ldots \int_{x_n}^{\infty} f(\boldsymbol{t}) \frac{d\boldsymbol{t}}{t_1^2 \ldots t_n^2} \right)^p \frac{d\boldsymbol{x}}{x_1 \ldots x_n}$$
$$+ \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \int_{b_1}^{t_1} \ldots \int_{b_n}^{t_n} \left| f(\boldsymbol{t}) - x_1 \ldots x_n \int_{x_1}^{\infty} \ldots \int_{x_n}^{\infty} f(\boldsymbol{t}) \frac{d\boldsymbol{t}}{t_1^2 \ldots t_n^2} \right|^p d\boldsymbol{x} \frac{d\boldsymbol{t}}{t_1^2 \ldots t_n^2}$$
$$\leq \int_{b_1}^{\infty} \ldots \int_{b_n}^{\infty} \prod_{i=1}^{n} \left(1 - \frac{b_i}{x_i} \right) f^p(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \ldots x_n}. \tag{3.7}$$

Again, denote the first and second terms on the left hand side of (3.7) by I_1 and I_2 and the term on the right hand side by I_3, respectively. Then, in (3.7) replace the parameter b_i by $a_i = b_i^{\frac{1-k_i}{p}}$, $i = 1, 2, \ldots, n$, and the function f by $\boldsymbol{x} \mapsto f(x_1^{\frac{p}{1-k_1}}, \ldots, x_n^{\frac{p}{1-k_n}}) \prod_{i=1}^{n} x_i^{\frac{p}{1-k_i}+1}$. The rest of the proof is similar to the proof of Theorem 3.1.

(iv) The proof for the case $1 < p \leq 2$ is similar and the only difference is that in this case all the inequalities signs are reversed. □

4. **Concluding remarks and examples**

Example 4.1. In Theorem 3.1 if we let $1 < p < \infty$, $k_1 = \ldots = k_n = k$, where $k > 1 \in \mathbb{R}$, $0 < \boldsymbol{b} \leq \infty$, and the function f be locally integrable on $(\boldsymbol{0}, \boldsymbol{b})$ such that $0<\int_0^{b_1}...\int_0^{b_n}\prod_{i=1}^{n}x_i^{p-k}f^p(\boldsymbol{x})d\boldsymbol{x}<\infty$, then for $p\geq 2$, inequality (3.1) reads

$$\int_0^{b_1}...\int_0^{b_n}\prod_{i=1}^{n}x_i^{-k}\left(\int_0^{x_1}...\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right)^p d\boldsymbol{x}$$
$$+\left(\frac{k-1}{p}\right)^n\int_0^{b_1}...\int_0^{b_n}\int_{t_1}^{b_1}...\int_{t_n}^{b_n}\left|\left(\frac{p}{k-1}\right)^n\prod_{i=1}^{n}\left(\frac{t_i}{x_i}\right)^{1-\frac{k-1}{p}}f(\boldsymbol{t})\right.$$
$$\left.-\frac{1}{x_1...x_n}\int_0^{x_1}...\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right|^p\prod_{i=1}^{n}x_i^{p-k-\frac{k-1}{p}}d\boldsymbol{x}\prod_{i=1}^{n}t_i^{\frac{k-1}{p}-1}d\boldsymbol{t}$$
$$\leq\left(\frac{p}{k-1}\right)^{np}\int_0^{b_1}...\int_0^{b_n}\prod_{i=1}^{n}\left(1-\left[\frac{x_i}{b_i}\right]^{\frac{k-1}{p}}\right)x_i^{p-k}f^p(\boldsymbol{x})d\boldsymbol{x}. \tag{4.1}$$

The sign of the inequality in (4.1) is reversed if $1<p\leq 2$.

Remark 4.2. By applying Example 4.1 with $p=2$ we obtain the following interesting identity: If $k>1$, then

$$\int_0^{b_1}...\int_0^{b_n}\prod_{i=1}^{n}x_i^{-k}\left(\int_0^{x_1}...\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right)^2 d\boldsymbol{x}$$
$$+\left(\frac{k-1}{2}\right)^n\int_0^{b_1}...\int_0^{b_n}\int_{t_1}^{b_1}...\int_{t_n}^{b_n}\left|\left(\frac{2}{k-1}\right)^n\prod_{i=1}^{n}\left(\frac{t_i}{x_i}\right)^{1-\frac{k-1}{2}}f(\boldsymbol{t})\right.$$
$$\left.-\frac{1}{x_1...x_n}\int_0^{x_1}...\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right|^2\prod_{i=1}^{n}x_i^{2-k-\frac{k-1}{2}}d\boldsymbol{x}\prod_{i=1}^{n}t_i^{\frac{k-1}{2}-1}d\boldsymbol{t}$$
$$=\left(\frac{2}{k-1}\right)^{2n}\int_0^{b_1}...\int_0^{b_n}\prod_{i=1}^{n}\left(1-\left[\frac{x_i}{b_i}\right]^{\frac{k-1}{2}}\right)x_i^{2-k}f^2(\boldsymbol{x})d\boldsymbol{x}.$$

Note that for the case $n=1$ this identity coincides with that pointed out in Remark 3.1 in [11].

Remark 4.3. For the special case $k=p$ in Example 4.1, the inequality (4.1) takes the form: If $p\geq 2$, then

$$\int_0^{b_1}...\int_0^{b_n}\left(\frac{1}{x_1...x_n}\int_0^{x_1}...\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right)^p d\boldsymbol{x}$$
$$+\left(\frac{p-1}{p}\right)^n\int_0^{b_1}...\int_0^{b_n}\int_{t_1}^{b_1}...\int_{t_n}^{b_n}\left|\left(\frac{p}{p-1}\right)^n\prod_{i=1}^{n}\left(\frac{t_i}{x_i}\right)^{\frac{1}{p}}f(\boldsymbol{t})\right.$$
$$\left.-\frac{1}{x_1...x_n}\int_0^{x_1}...\int_0^{x_n}f(\boldsymbol{t})d\boldsymbol{t}\right|^p\prod_{i=1}^{n}x_i^{-\frac{p-1}{p}}d\boldsymbol{x}\prod_{i=1}^{n}t_i^{-\frac{1}{p}}d\boldsymbol{t}$$
$$\leq\left(\frac{p}{p-1}\right)^{np}\int_0^{b_1}...\int_0^{b_n}\prod_{i=1}^{n}\left(1-\left[\frac{x_i}{b_i}\right]^{\frac{p-1}{p}}\right)f^p(\boldsymbol{x})d\boldsymbol{x}. \tag{4.2}$$

The sign of the inequality in (4.2) is reversed if $1 < p \leq 2$.

Note that in the one-dimensional case ($n = 1$) and $b_1 = b_2 = ...b_n = \infty$, these inequalities coincide with those given in Example 4.3 in [11].

Example 4.4. In Theorem 3.3 if we let $1 < p < \infty$, $k_1 = ... = k_n = k$, where $k < 1 \in \mathbb{R}$, $0 \leq \boldsymbol{b} < \infty$, and the function f be locally integrable in $(\boldsymbol{b}, \boldsymbol{\infty})$ and such that $0 < \int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \prod_{i=1}^{n} x_i^{p-k} f^p(\boldsymbol{x}) d\boldsymbol{x} < \infty$. Then for $p \geq 2$, we obtain that

$$\int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \prod_{i=1}^{n} x_i^{-k} \left(\int_{x_1}^{\infty} ... \int_{x_n}^{\infty} f(\boldsymbol{t}) d\boldsymbol{t} \right)^p d\boldsymbol{x}$$
$$+ \left(\frac{1-k}{p} \right)^n \int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \int_{b_1}^{t_1} ... \int_{b_n}^{t_n} \left| \left(\frac{p}{1-k} \right)^n \prod_{i=1}^{n} \left(\frac{t_i}{x_i} \right)^{\frac{1-k}{p}+1} f(\boldsymbol{t}) \right.$$
$$\left. - \frac{1}{x_1 ... x_n} \int_{x_1}^{\infty} ... \int_{x_n}^{\infty} f(\boldsymbol{t}) d\boldsymbol{t} \right|^p \prod_{i=1}^{n} x_i^{\frac{1-k}{p}+p-k} d\boldsymbol{x} \prod_{i=1}^{n} t_i^{\frac{k-1}{p}-1} d\boldsymbol{t}$$
$$\leq \left(\frac{p}{1-k} \right)^{np} \int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \prod_{i=1}^{n} \left(1 - \left[\frac{b_i}{x_i} \right]^{\frac{1-k}{p}} \right) x_i^{p-k} f^p(\boldsymbol{x}) d\boldsymbol{x}. \tag{4.3}$$

Inequality (4.3) holds in the reversed direction if $1 < p \leq 2$.

Remark 4.5. In Example 4.4 by setting $p = 2$, inequality (4.3) can be replaced by the following interesting identity: If $k < 1$, then

$$\int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \prod_{i=1}^{n} x_i^{-k} \left(\int_{x_1}^{\infty} ... \int_{x_n}^{\infty} f(\boldsymbol{t}) d\boldsymbol{t} \right)^2 d\boldsymbol{x}$$
$$+ \left(\frac{1-k}{2} \right)^n \int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \int_{b_1}^{t_1} ... \int_{b_n}^{t_n} \left(\left(\frac{2}{1-k} \right)^n \prod_{i=1}^{n} \left(\frac{t_i}{x_i} \right)^{\frac{1-k}{2}+1} f(\boldsymbol{t}) \right.$$
$$\left. - \frac{1}{x_1 ... x_n} \int_{x_1}^{\infty} ... \int_{x_n}^{\infty} f(\boldsymbol{t}) d\boldsymbol{t} \right)^2 \prod_{i=1}^{n} x_i^{\frac{1-k}{2}+2-k} d\boldsymbol{x} \prod_{i=1}^{n} t_i^{\frac{k-1}{2}-1} d\boldsymbol{t}$$
$$= \left(\frac{2}{1-k} \right)^{2n} \int_{b_1}^{\infty} ... \int_{b_n}^{\infty} \prod_{i=1}^{n} \left(1 - \left[\frac{b_i}{x_i} \right]^{\frac{1-k}{2}} \right) x_i^{2-k} f^2(\boldsymbol{x}) d\boldsymbol{x}. \tag{4.4}$$

In particular, for the one dimensional case ($n = 1$) and $b_1 = b_2 = ... = b_n = 0$ inequality (4.4) reduces to the identity in Remark 3.2 in [11]

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