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ESTIMATIONS OF THE LEHMER MEAN BY THE HERON MEAN AND THEIR GENERALIZATIONS INVOLVING REFINED HEINZ OPERATOR INEQUALITIES

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ABSTRACT. As generalizations of the arithmetic and the geometric means, for positive real numbers a and b, the power difference mean $J_q(a,b) = \frac{q}{q+1}\frac{a^{q+1}-b^{q+1}}{a^q-b^q}$, the Lehmer mean $L_q(a,b) = \frac{a^{q+1}+b^{q+1}}{a^q+b^q}$ and the Heron mean $K_q(a,b) = (1-q)\sqrt{ab} + q\frac{a+b}{2}$ are well known.

In this paper, concerning our recent results on estimations of the power difference mean, we obtain the greatest value $\alpha = \alpha(q)$ and the least value $\beta = \beta(q)$ such that the double inequality for the Lehmer mean

$$K_\alpha(a,b) < L_q(a,b) < K_\beta(a,b)$$

holds for any $q \in \mathbb{R}$. We also obtain an operator version of this estimation. Moreover, we discuss generalizations of the results on estimations of the power difference and the Lehmer means.This argument involves refined Heinz operator inequalities by Liang and Shi.

1. INTRODUCTION

Many researchers investigate means of positive numbers or operators. In what follows, we use the following notations for several means of two positive real

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numbers a and b. For $q \in \mathbb{R}$,

$$A(a,b) = \frac{a+b}{2} \text{ (arithmetic mean)}, \quad G(a,b) = \sqrt{ab} \text{ (geometric mean)},$$
$$H(a,b) = \frac{2ab}{a+b} \text{ (harmonic mean)}, \quad LM(a,b) = \frac{a-b}{\log a - \log b} \text{ (logarithmic mean)},$$
$$J_q(a,b) = \begin{cases} \dfrac{q}{q+1}\dfrac{a^{q+1}-b^{q+1}}{a^q-b^q} & \text{if } q \neq 0, -1, \\ \dfrac{a-b}{\log a - \log b} & \text{if } q = 0, \qquad \text{(power difference mean)}, \\ \dfrac{ab(\log a - \log b)}{a-b} & \text{if } q = -1, \end{cases}$$

$$L_q(a,b) = \frac{a^{q+1}+b^{q+1}}{a^q+b^q} \text{ (Lehmer mean)},$$

$$K_q(a,b) = (1-q)\sqrt{ab} + q\frac{a+b}{2} \text{ (Heron mean)}.$$

These means are symmetric, that is, $A(a,b) = A(b,a)$, $G(a,b) = G(b,a)$ and so on. We note that $J_q(a,a) \equiv \lim_{b \to a} J_q(a,b) = a$. It is well known that

$$\begin{aligned} &H(a,b) \leq G(a,b) \leq LM(a,b) \leq A(a,b), \\ &A(a,b) = J_1(a,b) = L_0(a,b) = K_1(a,b), \\ &LM(a,b) = J_0(a,b), \\ &G(a,b) = J_{\frac{-1}{2}}(a,b) = L_{\frac{-1}{2}}(a,b) = K_0(a,b), \\ &H(a,b) = J_{-2}(a,b) = L_{-1}(a,b), \end{aligned}$$

and also $J_q(a,b)$, $L_q(a,b)$ and $K_q(a,b)$ are monotone increasing on $q \in \mathbb{R}$.

As estimations of these means, the following relation is well known.

$$LM(a,b) \leq K_\alpha(a,b) \text{ for all } a,b > 0 \text{ if and only if } \alpha \geq \frac{1}{3}. \tag{1.1}$$

The inequality $LM(a,b) \leq K_{\frac{1}{3}}(a,b)$ is sometimes called the classical Pólya inequality [4, 14, 18]. In [1], Bhatia proved (1.1) by using Taylor expansion. Recently, inspired by (1.1) and its proof in [1], we obtained estimations of the power difference mean by the Heron mean as follows:

Theorem 1.A ([7]). *For all $a,b > 0$ with $a \neq b$, we have the following.*

(i) *Let $q \in (0, \frac{1}{2}) \cup (1, \infty)$. Then*

$$K_{\frac{2q}{q+1}}(a,b) < J_q(a,b) < K_{\frac{2q+1}{3}}(a,b).$$

(ii) *Let $q \in (\frac{1}{2}, 1)$. Then*

$$K_{\frac{2q+1}{3}}(a,b) < J_q(a,b) < K_{\frac{2q}{q+1}}(a,b).$$

(iii) *Let* $q \in (\frac{-1}{2}, 0]$. *Then*

$$G(a,b) = K_0(a,b) < J_q(a,b) < K_{\frac{2q+1}{3}}(a,b).$$

(iv) *Let* $q \in (-\infty, \frac{-1}{2})$. *Then*

$$K_{\frac{2q+1}{3}}(a,b) < J_q(a,b) < K_0(a,b) = G(a,b).$$

The given parameters of $K_\alpha(a,b)$ *in each case are best possible.*

A part of Theorem 1.A is shown by Xia, Hou, Wang and Chu [17]. We remark that equalities hold between $J_q(a,b)$ and $K_\alpha(a,b)$ for some $\alpha = \alpha(q)$ if $q = 1, \frac{1}{2}, \frac{-1}{2}$, and also Theorem 1.A implies (1.1) by putting $q = 0$.

Here, an operator means a bounded linear operator on a Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. A real-valued function f defined on $J \subset \mathbb{R}$ is said to be operator monotone if

$$A \leq B \ \text{ implies } \ f(A) \leq f(B)$$

for selfadjoint operators A and B whose spectra $\sigma(A), \sigma(B) \subset J$, where $A \leq B$ means $B - A \geq 0$. We remark that $xf(x^{-1})$, $f(x^{-1})^{-1}$ and $\frac{x}{f(x)}$ are operator monotone if $f > 0$ is operator monotone on $(0, \infty)$.

Kubo and Ando [10] investigated an axiomatic approach for operator means. In [10], they obtained that there exists a one-to-one correspondence between an operator mean $\mathfrak{M}$ and an operator monotone function $f \geq 0$ on $[0, \infty)$ with $f(1) = 1$. We remark that f is called the representing function of $\mathfrak{M}$, and also an operator mean $\mathfrak{M}$ can be defined by

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}} \tag{1.2}$$

if $A > 0$ and $B \geq 0$.

For two strictly positive operators A and B, the arithmetic mean $\mathfrak{A}(A,B)$, the geometric mean $\mathfrak{G}(A,B)$ and the harmonic mean $\mathfrak{H}(A,B)$ are defined as follows:

$$\mathfrak{A}(A,B) = \frac{A+B}{2}, \ \mathfrak{G}(A,B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}, \ \mathfrak{H}(A,B) = \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}.$$

They are typical examples of operator means, and their representing functions are

$$A(1,x) = \frac{x+1}{2}, \quad G(1,x) = \sqrt{x}, \quad H(1,x) = \frac{2x}{x+1}.$$

Now it is permitted to consider binary operations given by (1.2) for general real-valued functions. The logarithmic mean $\mathfrak{LM}(A,B)$, the power difference mean $\mathfrak{J}_q(A,B)$, the Lehmer mean $\mathfrak{L}_q(A,B)$, and the Heron mean $\mathfrak{K}_q(A,B)$ are given by $LM(1,x)$, $J_q(1,x)$, $L_q(1,x)$ and $K_q(1,x)$, respectively. For $-2 \leq q \leq 1$, it is known in [2, 5, 6, 16] that $\mathfrak{J}_q(A,B)$ is increasing on q and $\mathfrak{J}_q(A,B)$ is an operator mean. For $-1 \leq q \leq 0$, it is known in [13] that $\mathfrak{L}_q(A,B)$ is increasing on q and $\mathfrak{L}_q(A,B)$ is an operator mean. Obviously $\mathfrak{K}_q(A,B)$ is an operator mean for $0 \leq q \leq 1$.

In [7], we obtain an operator version of Theorem 1.A as a generalization of Fujii, Furuichi and Nakamoto's result in [3].

In this paper, we obtain estimations of the Lehmer mean by the Heron mean concerning Theorem 1.A and its operator version. Moreover, we discuss generalizations of our results on estimations of the power difference and the Lehmer means. This argument involves refined Heinz inequalities by Liang and Shi [11].

2. Lemmas

In this section, as lemmas to prove our main results, we show two properties of functions $h_k(q)$ for $k = 1, 2, \dots$ and $q \in \mathbb{R}$ defined by

$$h_k(q) \equiv \frac{(q+1)^{2k} - q^{2k}}{\sum_{i=1}^{k} \binom{2k}{2i} q^{2(k-i)}} \tag{2.1}$$

and $h_k(0) \equiv 1$ for convenience' sake. Here, $\displaystyle\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is a binomial coefficient for nonnegative integers n and r such that $0 \le r \le n$. We remark that $h_1(q) = 2q + 1$ in particular.

Lemma 2.1. *The limit* $h_\infty(q) \equiv \lim\limits_{k\to\infty} h_k(q)$ *exists and* $h_\infty(q) = \begin{cases} 2 & (q > 0), \\ 1 & (q = 0), \\ 0 & (q < 0). \end{cases}$

Proof. Firstly, we state the following relation (2.2) which is important to prove results in this paper. By putting $j = k - i$,

$$\begin{aligned} 2\sum_{i=1}^{k} \binom{2k}{2i} q^{2(k-i)} &= 2\sum_{j=0}^{k-1} \binom{2k}{2(k-j)} q^{2j} \\ = 2\sum_{j=0}^{k} \binom{2k}{2j} q^{2j} - 2q^{2k} &= (q+1)^{2k} + (q-1)^{2k} - 2q^{2k}. \end{aligned} \tag{2.2}$$

If $q \neq 0$, the following holds by (2.2).

$$\begin{aligned} h_k(q) &= \frac{(q+1)^{2k} - q^{2k}}{\frac{1}{2}\left\{(q+1)^{2k} + (q-1)^{2k} - 2q^{2k}\right\}} \\ &= \frac{2\left\{1 - \left(\frac{q}{q+1}\right)^{2k}\right\}}{1 + \left(\frac{q-1}{q+1}\right)^{2k} - 2\left(\frac{q}{q+1}\right)^{2k}} \quad (\text{if } q \neq -1) \qquad (2.3) \\ &= \frac{2\left\{\left(\frac{q+1}{q}\right)^{2k} - 1\right\}}{\left(\frac{q+1}{q}\right)^{2k} + \left(\frac{q-1}{q}\right)^{2k} - 2}. \qquad (2.4) \end{aligned}$$

Now we divide the range of q into four cases.

(Case 1) If $q > 0$, then $-1 < \frac{q-1}{q+1} < 1$ and $0 < \frac{q}{q+1} < 1$. Therefore (2.3) implies $h_\infty(q) = 2$.

(Case 2) If $\frac{-1}{2} < q < 0$, then $\frac{q-1}{q+1} < -1$ and $-1 < \frac{q}{q+1} < 0$, so that we have $h_\infty(q) = 0$.

(Case 3) If $q < \frac{-1}{2}$, then $-1 < \frac{q+1}{q} < 1$ and $\frac{q-1}{q} > 1$. Therefore (2.4) implies $h_\infty(q) = 0$.

(Case 4) If $q = 0$, then $h_k(0) = 1 \to 1$ as $k \to \infty$.

Hence the proof is complete. $\square$

Lemma 2.2. *Let $h_k(q)$ for $q \in \mathbb{R}$ as in (2.1). Then the following assertions hold:*

(i) *If $k \geq 2$, then*

$$h_1(q) - h_k(q) = \frac{2q(2q+1)(2q-1)}{\sum_{i=1}^{k} \binom{2k}{2i} q^{2(k-i)}} \sum_{\substack{u,v,w \geq 0 \\ u+v+w=k-2}} (q+1)^{2u}(q-1)^{2v}q^{2w}.$$

(ii) *If $k \geq 1$ and $q > 0$, then*

$$h_k(q) - h_\infty(q) = \frac{2q-1}{\sum_{i=1}^{k} \binom{2k}{2i} q^{2(k-i)}} \sum_{\substack{v,w \geq 0 \\ v+w=k-1}} (q-1)^{2v}q^{2w}.$$

Proof. (i) We have only to show the case $q \neq 0$. Since we get

$$\begin{aligned} h_1(q) - h_k(q) &= 2q+1 - \frac{(q+1)^{2k} - q^{2k}}{\sum_{i=1}^{k} \binom{2k}{2i} q^{2(k-i)}} \\ &= \frac{(2q+1)\sum_{i=1}^{k} \binom{2k}{2i} q^{2(k-i)} - \left\{(q+1)^{2k} - q^{2k}\right\}}{\sum_{i=1}^{k} \binom{2k}{2i} q^{2(k-i)}}, \end{aligned}$$

we have only to show

$$\begin{aligned} \kappa_1(q) &\equiv (2q+1)\sum_{i=1}^{k} \binom{2k}{2i} q^{2(k-i)} - \{(q+1)^{2k} - q^{2k}\} \\ &= 2q(2q+1)(2q-1) \sum_{\substack{u,v,w \geq 0 \\ u+v+w=k-2}} (q+1)^{2u}(q-1)^{2v}q^{2w}. \end{aligned} \tag{2.5}$$

By (2.2), the equation (2.5) holds since

$$\begin{aligned}
\kappa_1(q) &= (2q+1)\sum_{i=1}^{k}\binom{2k}{2i}q^{2(k-i)} - \{(q+1)^{2k}-q^{2k}\} \\
&= \frac{2q+1}{2}\left\{(q+1)^{2k}+(q-1)^{2k}-2q^{2k}\right\} - \{(q+1)^{2k}-q^{2k}\} \\
&= \frac{1}{2}\left\{(2q+1)(q+1)^{2k}+(2q+1)(q-1)^{2k}-2(2q+1)q^{2k}-2(q+1)^{2k}+2q^{2k}\right\} \\
&= \frac{1}{2}\left\{(2q-1)(q+1)^{2k}+(2q+1)(q-1)^{2k}-4q\cdot q^{2k}\right\} \\
&= \frac{1}{2}\left[(2q-1)\left\{(q+1)^{2k}-q^{2k}\right\}-(2q+1)\left\{q^{2k}-(q-1)^{2k}\right\}\right] \\
&\overset{(*)}{=} 2q(2q+1)(2q-1)\sum_{\substack{u,v,w\geq 0\\ u+v+w=k-2}}(q+1)^{2u}(q-1)^{2v}q^{2w},
\end{aligned}$$

and the last equality $(*)$ holds since

$$\begin{aligned}
&(2q-1)\left\{(q+1)^{2k}-q^{2k}\right\}-(2q+1)\left\{q^{2k}-(q-1)^{2k}\right\} \\
&= (2q-1)\{(q+1)^2-q^2\}\left\{(q+1)^{2(k-1)}+(q+1)^{2(k-2)}q^2+\cdots+(q+1)^2q^{2(k-2)}+q^{2(k-1)}\right\} \\
&\quad -(2q+1)\{q^2-(q-1)^2\}\left\{q^{2(k-1)}+q^{2(k-2)}(q-1)^2+\cdots+q^2(q-1)^{2(k-2)}+(q-1)^{2(k-1)}\right\} \\
&= (2q+1)(2q-1)\sum_{i=1}^{k-1}\left\{(q+1)^{2i}-(q-1)^{2i}\right\}q^{2(k-1-i)} \\
&= (2q+1)(2q-1)\sum_{i=1}^{k-1}\left\{(q+1)^2-(q-1)^2\right\} \\
&\quad \times\left\{(q+1)^{2(i-1)}+(q+1)^{2(i-2)}(q-1)^2+\cdots+(q-1)^{2(i-1)}\right\}q^{2(k-1-i)} \\
&= 4q(2q+1)(2q-1)\sum_{i=1}^{k-1}\left\{\sum_{j=0}^{i-1}(q+1)^{2j}(q-1)^{2(i-1-j)}\right\}q^{2(k-1-i)} \\
&= 4q(2q+1)(2q-1)\sum_{\substack{u,v,w\geq 0\\ u+v+w=k-2}}(q+1)^{2u}(q-1)^{2v}q^{2w}.
\end{aligned}$$

Therefore the desired result holds.

(ii) We get

$$\begin{aligned}
h_k(q)-h_\infty(q) &= \frac{(q+1)^{2k}-q^{2k}}{\sum_{i=1}^{k}\binom{2k}{2i}q^{2(k-i)}}-2 \\
&= \frac{(q+1)^{2k}-q^{2k}-2\sum_{i=1}^{k}\binom{2k}{2i}q^{2(k-i)}}{\sum_{i=1}^{k}\binom{2k}{2i}q^{2(k-i)}},
\end{aligned}$$

so we have only to show

$$\begin{aligned}\kappa_2(q) &\equiv (q+1)^{2k} - q^{2k} - 2\sum_{i=1}^{k}\binom{2k}{2i}q^{2(k-i)} \\ &= (2q-1)\sum_{\substack{v,w\geq 0\\ v+w=k-1}}(q-1)^{2v}q^{2w}.\end{aligned} \tag{2.6}$$

By (2.2), the equation (2.6) holds since

$$\begin{aligned}\kappa_2(q) &= (q+1)^{2k} - q^{2k} - 2\sum_{i=1}^{k}\binom{2k}{2i}q^{2(k-i)} \\ &= (q+1)^{2k} - q^{2k} - \left\{(q+1)^{2k} + (q-1)^{2k} - 2q^{2k}\right\} \\ &= q^{2k} - (q-1)^{2k} \\ &= \{q^2-(q-1)^2\}\left\{q^{2(k-1)} + q^{2(k-2)}(q-1)^2 + \cdots + q^2(q-1)^{2(k-2)} + (q-1)^{2(k-1)}\right\} \\ &= (2q-1)\sum_{\substack{v,w\geq 0\\ v+w=k-1}}(q-1)^{2v}q^{2w}.\end{aligned}$$

Hence the proof is complete. $\square$

3. Main results

Firstly, we obtain estimations of the Lehmer mean of two positive real numbers by the Heron mean.

Theorem 3.1. *For all $a, b > 0$ with $a \neq b$, we have the following.*

(i) *Let $q \in (\frac{1}{2}, \infty)$. Then*
$$K_2(a,b) < L_q(a,b) < K_{2q+1}(a,b).$$
(ii) *Let $q \in (0, \frac{1}{2})$. Then*
$$K_{2q+1}(a,b) < L_q(a,b) < K_2(a,b).$$
(iii) *Let $q \in (\frac{-1}{2}, 0)$. Then*
$$G(a,b) = K_0(a,b) < L_q(a,b) < K_{2q+1}(a,b).$$
(iv) *Let $q \in (-\infty, \frac{-1}{2})$. Then*
$$K_{2q+1}(a,b) < L_q(a,b) < K_0(a,b) = G(a,b).$$

The given parameters of $K_\alpha(a,b)$ in each case are best possible.

We remark that equalities hold between $L_q(a,b)$ and $K_\alpha(a,b)$ in the following cases.

$$\begin{aligned} L_q(a,b) &= K_{2q+1}(a,b) = K_2(a,b) \quad \text{for } q = \tfrac{1}{2}. \\ L_q(a,b) &= K_{2q+1}(a,b) = K_1(a,b) \quad \text{for } q = 0. \\ L_q(a,b) &= K_{2q+1}(a,b) = K_0(a,b) \quad \text{for } q = \tfrac{-1}{2}. \end{aligned}$$

To prove Theorem 3.1, we shall show the following propositions.

Proposition 3.2. *The following statements hold:*

(i) *Let* $q \in (\frac{-1}{2}, 0) \cup (\frac{1}{2}, \infty)$. *Then*

$$L_q(1,x) < K_\alpha(1,x) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \geq 2q+1.$$

(ii) *Let* $q \in (-\infty, \frac{-1}{2}) \cup (0, \frac{1}{2})$. *Then*

$$L_q(1,x) > K_\alpha(1,x) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \leq 2q+1.$$

Proposition 3.3. *The following statements hold:*

(i-1) *Let* $q \in (\frac{1}{2}, \infty)$. *Then*

$$L_q(1,x) > K_\alpha(1,x) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \leq 2.$$

(i-2) *Let* $q \in (\frac{-1}{2}, 0)$. *Then*

$$L_q(1,x) > K_\alpha(1,x) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \leq 0.$$

(ii-1) *Let* $q \in (0, \frac{1}{2})$. *Then*

$$L_q(1,x) < K_\alpha(1,x) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \geq 2.$$

(ii-2) *Let* $q \in (-\infty, \frac{-1}{2})$. *Then*

$$L_q(1,x) < K_\alpha(1,x) \text{ for all } x > 0 \text{ with } x \neq 1 \text{ if and only if } \alpha \geq 0.$$

Proof of Proposition 3.2. (i) Let $q \in (\frac{-1}{2}, 0) \cup (\frac{1}{2}, \infty)$. Firstly we show that $\alpha \geq 2q+1$ ensures

$$L_q(1,x) = \frac{x^{q+1}+1}{x^q+1} < (1-\alpha)\sqrt{x} + \alpha\frac{x+1}{2} = K_\alpha(1,x) \quad \text{for all } x > 0 \text{ with } x \neq 1. \tag{3.1}$$

By putting $x = e^{2t}$, (3.1) holds if and only if

$$\frac{e^{(q+1)t} + e^{-(q+1)t}}{e^{qt} + e^{-qt}} < (1-\alpha) + \alpha\frac{e^t + e^{-t}}{2} \quad \text{for all } t \in \mathbb{R} \setminus \{0\}. \tag{3.2}$$

Since both sides of (3.2) are even functions, we have only to consider the case $t > 0$. Then, (3.2) for $t > 0$ is equivalent to

$$\begin{aligned} f(t) &\equiv (e^{qt} + e^{-qt})\left\{(1-\alpha) + \alpha\frac{e^t+e^{-t}}{2}\right\} - (e^{(q+1)t} + e^{-(q+1)t}) \\ &= 2\cosh(qt)\left\{(1-\alpha) + \alpha\cosh t\right\} - 2\cosh((q+1)t) > 0 \quad \text{for all } t > 0. \end{aligned} \tag{3.3}$$

Therefore we prove (3.3). By Taylor expansion, we have

$$\begin{aligned}
f(t) &= 2\left(1+\frac{q^2t^2}{2!}+\frac{q^4t^4}{4!}+\cdots\right)\left\{(1-\alpha)+\alpha\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\cdots\right)\right\}\\
&\quad -2\left\{1+\frac{(q+1)^2t^2}{2!}+\frac{(q+1)^4t^4}{4!}+\cdots\right\}\\
&= 2\left(1+\frac{q^2}{2!}t^2+\frac{q^4}{4!}t^4+\cdots\right)\left(1+\frac{\alpha}{2!}t^2+\frac{\alpha}{4!}t^4+\cdots\right)\\
&\quad -2\left\{1+\frac{(q+1)^2}{2!}t^2+\frac{(q+1)^4}{4!}t^4+\cdots\right\}\\
&= 2\sum_{k=1}^{\infty}\left\{\frac{q^{2k}}{(2k)!}+\sum_{i=1}^{k}\frac{q^{2(k-i)}\alpha}{(2i)!(2k-2i)!}-\frac{(q+1)^{2k}}{(2k)!}\right\}t^{2k}\\
&= 2\sum_{k=1}^{\infty}\phi_{k,q}(\alpha)\,t^{2k},
\end{aligned}$$

where

$$\phi_{k,q}(\alpha)\equiv\frac{q^{2k}}{(2k)!}+\sum_{i=1}^{k}\frac{q^{2(k-i)}\alpha}{(2i)!(2k-2i)!}-\frac{(q+1)^{2k}}{(2k)!}\quad\text{for } k=1,2,\ldots. \tag{3.4}$$

Then $\phi_{k,q}(\alpha)>0$ if and only if

$$\alpha>\frac{(q+1)^{2k}-q^{2k}}{\sum_{i=1}^{k}\binom{2k}{2i}q^{2(k-i)}}=h_k(q).$$

By (i) in Lemma 2.2, $q\in(\frac{-1}{2},0)\cup(\frac{1}{2},\infty)$ ensures that $h_1(q)>h_k(q)$ for all $k\geq 2$. Therefore, if $\alpha\geq 2q+1=h_1(q)$, then $\phi_{1,q}(\alpha)\geq 0$ and $\phi_{k,q}(\alpha)>0$ for all $k\geq 2$, that is, (3.3) holds.

On the other hand, if $\alpha<2q+1=h_1(q)$, then $\phi_{1,q}(\alpha)<0$ holds, that is, $f(t)<0$ for sufficiently small $t>0$. Therefore (3.3) assures $\alpha\geq 2q+1$.

We can prove (ii) similarly, so the proof is complete. □

Proof of Proposition 3.3. (i) Let $q\in(\frac{-1}{2},0)\cup(\frac{1}{2},\infty)$. Then by the same way to the proof of Proposition 3.2, we have only to consider the case that

$$f(t)=2\sum_{k=1}^{\infty}\phi_{k,q}(\alpha)\,t^{2k}<0\quad\text{holds for all } t>0, \tag{3.5}$$

that is, $\alpha<h_k(q)$ for $k=1,2,\ldots$, where $\phi_{k,q}(\alpha)$ is defined in (3.4), and also $h_k(q)$ is in (2.1).

(i-1) Let $q\in(\frac{1}{2},\infty)$. By (ii) in Lemma 2.2, $q\in(\frac{1}{2},\infty)$ ensures that $h_k(q)>h_\infty(q)$ for all $k\geq 1$, so that (3.5) holds if $\alpha\leq 2=h_\infty(q)$ by Lemma 2.1.

On the other hand, for any $\epsilon>0$, there exists a natural number n_0 such that $n\geq n_0$ implies $h_\infty(q)<h_n(q)<h_\infty(q)+\epsilon$. If $\alpha_\epsilon\equiv h_\infty(q)+\epsilon>2=h_\infty(q)$, then $\phi_{n,q}(\alpha_\epsilon)>0$ holds for $n\geq n_0$, that is, $f(t)>0$ for sufficiently large t. Therefore (3.5) assures $\alpha\leq 2$.

(i-2) Let $q \in (\frac{-1}{2}, 0)$. Then $h_k(q) > h_\infty(q) = 0$ for all $k \geq 2$ by Lemma 2.1. Therefore (3.5) holds if $\alpha \leq 0$. We can show "only if" part by the same way to (i-1).

We can prove (ii-1) and (ii-2) similarly, so the proof is complete. □

Proof of Theorem 3.1. By putting $x = \frac{b}{a}$ in Propositions 3.2 and 3.3, we immediately obtain the desired result. □

We remark that we obtain the following inequalities on hyperbolic functions by (3.3) in the proof of Proposition 3.2. This proposition corresponds to [7, Proposition 3.4]. Of course, we can produce related inequalities from other results in Propositions 3.2 and 3.3.

Proposition 3.4. *Let $q > \frac{1}{2}$. Then the following inequalities hold.*

(i) *If $\alpha \geq 2q+1$, then*

$$(\alpha - 2)\cosh((q+1)t) + 2(1-\alpha)\cosh(qt) + \alpha\cosh((q-1)t) > 0$$

holds for all $t > 0$.

(ii) $(2q-1)\cosh((q+1)t) + (2q+1)\cosh((q-1)t) > 4q\cosh(qt)$ *for all $t > 0$.*

Proof. (i) is shown by applying the product-to-sum formula to (3.3). We have (ii) by putting $\alpha = 2q+1$ in (i). □

Next, we state estimations of the Lehmer mean of two strictly positive operators. In [7], we obtained estimations of the power difference mean for positive operators as a generalization of Fujii, Furuichi and Nakamoto's result in [3].

Theorem 3.A ([7]). *Let A and B be positive invertible operators.*

(i) *Let $q \in (0, \frac{1}{2}) \cup (1, \infty)$. Then*

$$\mathfrak{K}_{\frac{2q}{q+1}}(A,B) \leq \mathfrak{J}_q(A,B) \leq \mathfrak{K}_{\frac{2q+1}{3}}(A,B).$$

(ii) *Let $q \in (\frac{1}{2}, 1)$. Then*

$$\mathfrak{K}_{\frac{2q+1}{3}}(A,B) \leq \mathfrak{J}_q(A,B) \leq \mathfrak{K}_{\frac{2q}{q+1}}(A,B).$$

(iii) *Let $q \in (\frac{-1}{2}, 0]$. Then*

$$\mathfrak{G}(A,B) = \mathfrak{K}_0(A,B) \leq \mathfrak{J}_q(A,B) \leq \mathfrak{K}_{\frac{2q+1}{3}}(A,B).$$

(iv) *Let $q \in (-\infty, \frac{-1}{2})$. Then*

$$\mathfrak{K}_{\frac{2q+1}{3}}(A,B) \leq \mathfrak{J}_q(A,B) \leq \mathfrak{K}_0(A,B) = \mathfrak{G}(A,B).$$

The given parameters of $\mathfrak{K}_\alpha(A,B)$ in each case are best possible.

Similarly, we have the following result on the Lehmer mean $\mathfrak{L}_q(A,B)$.

Theorem 3.5. *Let A and B be positive invertible operators.*

(i) *Let $q \in (\frac{1}{2}, \infty)$. Then*

$$\mathfrak{K}_2(A,B) \leq \mathfrak{L}_q(A,B) \leq \mathfrak{K}_{2q+1}(A,B).$$

(ii) *Let* $q \in (0, \frac{1}{2})$. *Then*

$$\mathfrak{K}_{2q+1}(A, B) \leq \mathfrak{L}_q(A, B) \leq \mathfrak{K}_2(A, B).$$

(iii) *Let* $q \in (\frac{-1}{2}, 0)$. *Then*

$$\mathfrak{G}(A, B) = \mathfrak{K}_0(A, B) \leq \mathfrak{L}_q(A, B) \leq \mathfrak{K}_{2q+1}(A, B).$$

(iv) *Let* $q \in (-\infty, \frac{-1}{2})$. *Then*

$$\mathfrak{K}_{2q+1}(A, B) \leq \mathfrak{L}_q(A, B) \leq \mathfrak{K}_0(A, B) = \mathfrak{G}(A, B).$$

The given parameters of $\mathfrak{K}_\alpha(A, B)$ *in each case are best possible.*

Proof. We have Theorem 3.5 by putting $x = A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$ and applying the standard operational calculus in Propositions 3.2 and 3.3. □

4. Generalizations involving refined Heinz operator inequalities

In this section, we discuss generalizations of the results in section 3. We consider extensions of the power difference and the Lehmer means for two positive real numbers a and b as follows: For $p, q \in \mathbb{R}$,

$$J_{p,q}(a,b) = \begin{cases} (ab)^{\frac{1-p+q}{2}} \dfrac{q}{p} \dfrac{a^p - b^p}{a^q - b^q} & \text{if } p \neq 0 \text{ and } q \neq 0, \\ (ab)^{\frac{1+q}{2}} \dfrac{q(\log a - \log b)}{a^q - b^q} & \text{if } p = 0 \text{ and } q \neq 0, \\ (ab)^{\frac{1-p}{2}} \dfrac{a^p - b^p}{p(\log a - \log b)} & \text{if } p \neq 0 \text{ and } q = 0, \\ \sqrt{ab} & \text{if } p = q = 0, \end{cases}$$

$$L_{p,q}(a,b) = (ab)^{\frac{1-p+q}{2}} \frac{a^p + b^p}{a^q + b^q}.$$

We remark that $J_{p,q}(a, b)$ and $L_{p,q}(a, b)$ are symmetric, and also $J_{q+1,q}(a, b) = J_q(a, b)$ and $L_{q+1,q}(a, b) = L_q(a, b)$ hold. For positive real numbers a, b and for $q \in \mathbb{R}$, the Heinz mean is defined by

$$HZ_q(a, b) = \frac{a^{1-q} b^q + a^q b^{1-q}}{2}.$$

We remark that $HZ_0(a, b) = HZ_1(a, b) = A(a, b)$, $HZ_{\frac{1}{2}}(a, b) = G(a, b)$ and $HZ_{1-q}(a, b) = HZ_q(a, b)$ hold, and also $HZ_q(a, b)$ is decreasing for $q \leq \frac{1}{2}$ and increasing for $q \geq \frac{1}{2}$. These properties ensure $G(a, b) \leq HZ_q(a, b) \leq A(a, b)$ for $q \in [0, 1]$.

We consider opertor versions of these mean, and they are denoted by $\mathfrak{J}_{p,q}(A, B)$, $\mathfrak{L}_{p,q}(A, B)$ and $\mathfrak{H}\mathfrak{Z}_q(A, B)$ for $A, B > 0$. Here, we call inequalities

$$\mathfrak{G}(A, B) \leq \mathfrak{H}\mathfrak{Z}_q(A, B) \leq \mathfrak{A}(A, B) \tag{4.1}$$

for $q \in [0, 1]$ the Heinz operator inequalities [9, 11]. We state in the next section that we have the results on operator monotonicity of the representing functions of $\mathfrak{J}_{p,q}(A, B)$ and $\mathfrak{L}_{p,q}(A, B)$ by using Nagisa and Wada's result in [12].

Now we are ready to get generalizations of Theorems 3.A and 3.5. We obtain estimations of $\mathfrak{J}_{p,q}(A,B)$ and $\mathfrak{L}_{p,q}(A,B)$ by extensions of the Heron mean

$$\mathfrak{K}_{\alpha,q}(A,B) = (1-\alpha)\mathfrak{G}(A,B) + \alpha\,\mathfrak{H}\mathfrak{Z}_q(A,B).$$

We remark that $\mathfrak{K}_{\alpha,q}(A,B)$ are increasing for $\alpha \in \mathbb{R}$. We can also obtain generalizations for positive real numbers by using

$$K_{\alpha,q}(a,b) = (1-\alpha)\sqrt{ab} + \alpha\frac{a^{1-q}b^q + a^q b^{1-q}}{2},$$

but we omit them.

Theorem 4.1. *Let $A, B > 0$ and $p, q \in \mathbb{R}$ with $p \neq q$.*

(i) *If $\frac{q}{p-q} \in (0, \frac{1}{2}) \cup (1, \infty)$, then*

$$\mathfrak{K}_{\frac{2q}{p},\frac{1-p+q}{2}}(A,B) \le \mathfrak{J}_{p,q}(A,B) \le \mathfrak{K}_{\frac{p+q}{3(p-q)},\frac{1-p+q}{2}}(A,B).$$

(ii) *If $\frac{q}{p-q} \in (\frac{1}{2}, 1)$, then*

$$\mathfrak{K}_{\frac{p+q}{3(p-q)},\frac{1-p+q}{2}}(A,B) \le \mathfrak{J}_{p,q}(A,B) \le \mathfrak{K}_{\frac{2q}{p},\frac{1-p+q}{2}}(A,B).$$

(iii) *If $\frac{q}{p-q} \in (\frac{-1}{2}, 0]$, then*

$$\mathfrak{G}(A,B) = \mathfrak{K}_{0,\frac{1-p+q}{2}}(A,B) \le \mathfrak{J}_{p,q}(A,B) \le \mathfrak{K}_{\frac{p+q}{3(p-q)},\frac{1-p+q}{2}}(A,B).$$

(iv) *If $\frac{q}{p-q} \in (-\infty, \frac{-1}{2})$, then*

$$\mathfrak{K}_{\frac{p+q}{3(p-q)},\frac{1-p+q}{2}}(A,B) \le \mathfrak{J}_{p,q}(A,B) \le \mathfrak{K}_{0,\frac{1-p+q}{2}}(A,B) = \mathfrak{G}(A,B).$$

The given parameters $\alpha = \alpha(p,q)$ of $\mathfrak{K}_{\alpha,q}(A,B)$ in each case are best possible.

Theorem 4.2. *Let $A, B > 0$ and $p, q \in \mathbb{R}$ with $p \neq q$.*

(i) *If $\frac{q}{p-q} \in (\frac{1}{2}, \infty)$, then*

$$\mathfrak{K}_{2,\frac{1-p+q}{2}}(A,B) \le \mathfrak{L}_{p,q}(A,B) \le \mathfrak{K}_{\frac{p+q}{p-q},\frac{1-p+q}{2}}(A,B).$$

(ii) *If $\frac{q}{p-q} \in (0, \frac{1}{2})$, then*

$$\mathfrak{K}_{\frac{p+q}{p-q},\frac{1-p+q}{2}}(A,B) \le \mathfrak{L}_{p,q}(A,B) \le \mathfrak{K}_{2,\frac{1-p+q}{2}}(A,B).$$

(iii) *If $\frac{q}{p-q} \in (\frac{-1}{2}, 0)$, then*

$$\mathfrak{G}(A,B) = \mathfrak{K}_{0,\frac{1-p+q}{2}}(A,B) \le \mathfrak{L}_{p,q}(A,B) \le \mathfrak{K}_{\frac{p+q}{p-q},\frac{1-p+q}{2}}(A,B).$$

(iv) *If $\frac{q}{p-q} \in (-\infty, \frac{-1}{2})$, then*

$$\mathfrak{K}_{\frac{p+q}{p-q},\frac{1-p+q}{2}}(A,B) \le \mathfrak{L}_{p,q}(A,B) \le \mathfrak{K}_{0,\frac{1-p+q}{2}}(A,B) = \mathfrak{G}(A,B).$$

The given parameters $\alpha = \alpha(p,q)$ of $\mathfrak{K}_{\alpha,q}(A,B)$ in each case are best possible.

Proof of Theorem 4.1. (i) For $r \in (0, \frac{1}{2}) \cup (1, \infty)$ and $t > 0$,

$$K_{\frac{2r}{r+1}}(1,t) < J_r(1,t) < K_{\frac{2r+1}{3}}(1,t) \tag{4.2}$$

holds by (i) in Theorem 1.A. Put $r = \frac{q}{p-q}$ and $t = x^{p-q}$ for $p, q \in \mathbb{R}$ with $p \neq q$. Then (4.2) is equivalent to

$$K_{\frac{2q}{p}}(1, x^{p-q}) < J_{\frac{q}{p-q}}(1, x^{p-q}) < K_{\frac{p+q}{3(p-q)}}(1, x^{p-q}). \tag{4.3}$$

Since

$$x^{\frac{1-p+q}{2}} J_{\frac{q}{p-q}}(1, x^{p-q}) = x^{\frac{1-p+q}{2}} \frac{\frac{q}{p-q}}{\frac{q}{p-q}+1} \frac{x^{q+(p-q)}-1}{x^q-1} = x^{\frac{1-p+q}{2}} \frac{q}{p} \frac{x^p-1}{x^q-1} = J_{p,q}(1,x)$$

and

$$\begin{aligned} x^{\frac{1-p+q}{2}} K_\alpha(1, x^{p-q}) &= x^{\frac{1-p+q}{2}} \left\{(1-\alpha)x^{\frac{p-q}{2}} + \alpha \frac{x^{p-q}+1}{2}\right\} \\ &= (1-\alpha)\sqrt{x} + \alpha \frac{x^{\frac{1+p-q}{2}} + x^{\frac{1-p+q}{2}}}{2} = K_{\alpha, \frac{1-p+q}{2}}(1,x), \end{aligned}$$

(4.3) is equivalent to

$$K_{\frac{2q}{p}, \frac{1-p+q}{2}}(1,x) < J_{p,q}(1,x) < K_{\frac{p+q}{3(p-q)}, \frac{1-p+q}{2}}(1,x). \tag{4.4}$$

By putting $x = A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$ and applying the standard operational calculus in (4.4), we obtain

$$\mathfrak{K}_{\frac{2q}{p}, \frac{1-p+q}{2}}(A,B) \leq \mathfrak{J}_{p,q}(A,B) \leq \mathfrak{K}_{\frac{p+q}{3(p-q)}, \frac{1-p+q}{2}}(A,B).$$

We obviously get the best possibility on $\alpha = \alpha(p,q)$ of $\mathfrak{K}_{\alpha,q}(A,B)$ by the above argument.

We can show (ii), (iii) and (iv) similarly. Hence the proof is complete. □

Proof of Theorem 4.2. Noting that

$$x^{\frac{1-p+q}{2}} L_{\frac{q}{p-q}}(1, x^{p-q}) = x^{\frac{1-p+q}{2}} \frac{x^p+1}{x^q+1} = L_{p,q}(1,x),$$

we obtain the desired result by the same way as the proof of Theorem 4.1. □

Kittaneh and Krnić [9] obtained refined Heinz inequalities via the Hermite-Hadamard inequality by considering the parameterized class of functions $F_\nu : (0,\infty) \to \mathbb{R}$, $\nu \in [0,1]$ defined by

$$F_\nu(x) = \begin{cases} \dfrac{x^\nu - x^{1-\nu}}{\log x} & \text{if } x \neq 1, \\ 2\nu - 1 & \text{if } x = 1. \end{cases} \tag{4.5}$$

Liang and Shi [11] showed the following result on the Heinz mean and $F_\nu(x)$ as an improvement of the results in [9].

Theorem 4.A ([11]). *Let $A, B > 0$. If $s, t \in [0,1]$ satisfy*

$$s, t \neq \frac{1}{2}, \quad \left|s - \frac{1}{2}\right| \geq \left|t - \frac{1}{2}\right|$$

and $k \geq \frac{1}{3}$, then

$$\mathfrak{HZ}_t(A,B) \leq \left(1 - \frac{(1-2t)^2}{(1-2s)^2}\right)\mathfrak{G}(A,B) + \frac{(1-2t)^2}{(1-2s)^2}\mathfrak{HZ}_s(A,B) \tag{4.6}$$

and

$$\frac{1}{2t-1}A^{\frac{1}{2}}F_t(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \leq \left(1 - k\frac{(1-2t)^2}{(1-2s)^2}\right)\mathfrak{G}(A,B) + k\frac{(1-2t)^2}{(1-2s)^2}\mathfrak{HZ}_s(A,B). \tag{4.7}$$

It is the essential part of (4.7) that

$$\frac{1}{2t-1}A^{\frac{1}{2}}F_t(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \leq (1-k)\mathfrak{G}(A,B) + k\,\mathfrak{HZ}_t(A,B) = \mathfrak{K}_{k,t}(A,B) \tag{4.8}$$

for $t \in [0,1] \setminus \{\frac{1}{2}\}$ and $k \geq \frac{1}{3}$ since the inequality (4.7) is led by (4.6) and (4.8). We remark that (4.8) is a refinement of the first inequality in the Heinz operator inequality (4.1) in the sense that (4.8) assures

$$\mathfrak{G}(A,B) \leq \frac{1}{2t-1}A^{\frac{1}{2}}F_t(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \leq \mathfrak{K}_{\frac{1}{3},t}(A,B) \leq \mathfrak{HZ}_t(A,B). \tag{4.9}$$

Theorems 4.1 and 4.2 assures the following refinements of the Heinz operator inequalities including (4.9).

Corollary 4.3. *Let $A, B > 0$ and $p, q \in \mathbb{R}$ with $p \neq q$.*

(i) *If $pq > 0$ and $|q| \leq \frac{|p|}{3}$, then*

$$\mathfrak{G}(A,B) \leq \mathfrak{K}_{\frac{2q}{p},\frac{1-p+q}{2}}(A,B) \leq \mathfrak{J}_{p,q}(A,B) \leq \mathfrak{K}_{\frac{p+q}{3(p-q)},\frac{1-p+q}{2}}(A,B) \leq \mathfrak{HZ}_{\frac{1-p+q}{2}}(A,B).$$

(ii) *If $pq > 0$ and $\frac{|p|}{3} < |q| < \frac{|p|}{2}$, then*

$$\mathfrak{G}(A,B) \leq \mathfrak{K}_{\frac{p+q}{3(p-q)},\frac{1-p+q}{2}}(A,B) \leq \mathfrak{J}_{p,q}(A,B) \leq \mathfrak{K}_{\frac{2q}{p},\frac{1-p+q}{2}}(A,B) \leq \mathfrak{HZ}_{\frac{1-p+q}{2}}(A,B).$$

(iii) *If $pq \leq 0$ and $|q| < |p|$, then*

$$\mathfrak{G}(A,B) = \mathfrak{K}_{0,\frac{1-p+q}{2}}(A,B) \leq \mathfrak{J}_{p,q}(A,B) \leq \mathfrak{K}_{\frac{p+q}{3(p-q)},\frac{1-p+q}{2}}(A,B) \leq \mathfrak{HZ}_{\frac{1-p+q}{2}}(A,B).$$

Corollary 4.4. *Let $A, B > 0$ and $p, q \in \mathbb{R}$ with $p \neq q$. If $pq < 0$ and $|q| < |p|$, then*

$$\mathfrak{G}(A,B) = \mathfrak{K}_{0,\frac{1-p+q}{2}}(A,B) \leq \mathfrak{L}_{p,q}(A,B) \leq \mathfrak{K}_{\frac{p+q}{p-q},\frac{1-p+q}{2}}(A,B) \leq \mathfrak{HZ}_{\frac{1-p+q}{2}}(A,B).$$

Proof of Corollaries 4.3 and 4.4. Here, we give a proof of (i) in Corollary 4.3. If $p = 3q$, that is $\frac{q}{p-q} = \frac{1}{2}$, then

$$\mathfrak{G}(A,B) \leq \mathfrak{J}_{3q,q}(A,B) = \mathfrak{K}_{\frac{2}{3},\frac{1-2q}{2}}(A,B) \leq \mathfrak{HZ}_{\frac{1-2q}{2}}(A,B)$$

holds. If $pq > 0$ and $0 < |q| < \frac{|p|}{3}$ hold, then it is equivalent to $\frac{q}{p-q} \in (0, \frac{1}{2})$. Therefore (i) in Theorem 4.1 implies the desired inequalities since $\frac{2q}{p} > 0$ and $\frac{p+q}{3(p-q)} < \frac{2}{3}$.

(ii), (iii) in Corollary 4.3 and Corollary 4.4 are shown similarly by (ii), (iii) in Theorem 4.1 and (iii) in Theorem 4.2, respectively. $\square$

We remark that $\mathfrak{HZ}_{\frac{1-p+q}{2}}(A,B) \leq \mathfrak{A}(A,B)$ holds if $|p-q| \leq 1$. We recognize that (iii) in Corollary 4.3 implies (4.9) as follows: By putting $p = 1-2t$ and $q = 0$, we have

$$\mathfrak{G}(A,B) = \mathfrak{K}_{0,t}(A,B) \leq \mathfrak{J}_{1-2t,0}(A,B) \leq \mathfrak{K}_{\frac{1}{3},t}(A,B) \leq \mathfrak{HZ}_t(A,B),$$

so that we obtain (4.9) since

$$J_{1-2t,0}(1,x) = x^t \frac{1}{1-2t} \frac{x^{1-2t}-1}{\log x} = \frac{1}{2t-1} F_t(x).$$

5. Operator monotonicity of $J_{p,q}(1,x)$ and $L_{p,q}(1,x)$

In this section, we discuss operator monotonicity of the representing functions of $\mathfrak{J}_{p,q}(A,B)$ and $\mathfrak{L}_{p,q}(A,B)$, that is,

$$J_{p,q}(1,x) = \begin{cases} x^{\frac{1-p+q}{2}} \dfrac{q}{p} \dfrac{x^p-1}{x^q-1} & \text{if } p \neq 0 \text{ and } q \neq 0, \\ x^{\frac{1+q}{2}} \dfrac{q \log x}{x^q-1} & \text{if } p = 0 \text{ and } q \neq 0, \\ x^{\frac{1-p}{2}} \dfrac{x^p-1}{p \log x} & \text{if } p \neq 0 \text{ and } q = 0, \\ \sqrt{x} & \text{if } p = q = 0, \end{cases}$$

$$L_{p,q}(1,x) = x^{\frac{1-p+q}{2}} \frac{x^p+1}{x^q+1} = x^{\frac{1-p+q}{2}} \frac{x^{2p}-1}{x^{2q}-1} \frac{x^q-1}{x^p-1}.$$

We want to point out that $L_{p,q}(1,x)$ can be expressed as the last form.

Nagisa and Wada [12] investigated operator monotonicity of the following functions $f_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ on $(0,\infty)$ including $J_{p,q}(1,x)$ and $L_{p,q}(1,x)$ (see also [8, 15]).

$$f_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x) = \begin{cases} x^{\gamma} \displaystyle\prod_{i=1}^{n} \frac{x^{\alpha_i}-1}{x^{\beta_i}-1} & \text{if } x \neq 1, \\ \displaystyle\prod_{i=1}^{n} \frac{\alpha_i}{\beta_i} & \text{if } x = 1, \end{cases} \tag{5.1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$. In [12], they stated that we have only to consider the setting $|\gamma| \leq 2$, $\alpha_i, \beta_i \in (0,2]$ and $\alpha_i \neq \beta_j$ $(i,j = 1,2,\ldots,n)$, and they obtained the following Theorem 5.A. We remark that they also gave an equivalent condition for operator monotonicity, but it seems to be difficult to calculate in general.

Theorem 5.A ([12]). *Let $|\gamma| \leq 2$, $\alpha_i, \beta_i \in (0,2]$, $\alpha_i \neq \beta_j$ $(i,j = 1,2,\ldots,n)$ and $\alpha_i \leq \alpha_j$, $\beta_i \leq \beta_j$ if $1 \leq i < j \leq n$. If it satisfies*

$$0 \leq \gamma - \sum_{i=1}^{n} F(\beta_i, \alpha_i) \quad \textit{and} \quad \gamma + \sum_{i=1}^{n} F(\alpha_i, \beta_i) \leq 1, \tag{5.2}$$

then we have that $f_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)$ *in* (5.1) *is operator monotone on* $(0,\infty)$*, where the function* $F:[0,2]\times[0,2]\to\mathbb{R}$ *is defined as follows:*

$$F(a,b)=\begin{cases} a-b & \text{if } a\ge b,\ 0\le b\le 1,\\ a-1 & \text{if } 1<a,b\le 2,\\ 0 & \text{if } a<b,\ 0\le a\le 1.\end{cases}$$

By applying Theorem 5.A, we know the domains of (p,q) for which $J_{p,q}(1,x)$ and $L_{p,q}(1,x)$ are operator monotone.

Proposition 5.1. *Let* $p,q\in[-2,2]$.

(i) *If* (p,q) *belongs to the domain* $\{(p,q):\big||p|-|q|\big|\le 1,\ |p|+|q|\le 3\}$*, then* $J_{p,q}(1,x)$ *is operator monotone on* $(0,\infty)$.

(ii) *If* (p,q) *belongs to the domain*

$$\left\{(p,q):\frac{|p|}{2}\le|q|\le 2|p|,\ \big||p|-|q|\big|\le\frac{1}{3},\ 3|p|+|q|\le 3,\ |p|+3|q|\le 3\right\}$$

$$\cup\,\{(p,q):|p|+|q|\le 1\}\,,$$

$L_{p,q}(1,x)$ *is operator monotone on* $(0,\infty)$.

Corollary 5.2. *Let* $t\in[0,1]$*. Then* $J_{1-2t,0}(1,x)=\dfrac{1}{2t-1}F_t(x)$ *is operator monotone on* $(0,\infty)$*, where* $F_t(x)$ *is defined as in* (4.5).

Proof of Proposition 5.1. We only give a proof of (ii) since (i) is shown by the same way. Put $L_{p,q}(x)=L_{p,q}(1,x)$. We remark that

$$L_{p,0}(x)=\frac{x^{\frac{1+p}{2}}+x^{\frac{1-p}{2}}}{2}$$

is operator monotone if $p\in[0,1]$. By the definition of $L_{p,q}(x)$ and the relations

$$L_{p,q}(x)=L_{-p,q}(x)=L_{p,-q}(x)=L_{-p,-q}(x)\quad\text{and}\quad L_{q,p}(x)=L_{p,q}(x^{-1})^{-1},$$

we have only to consider the case $0<q\le p\le 1$.

(a) Case $p\le 2q$. Put $\boldsymbol{\alpha}=(q,2p)$ and $\boldsymbol{\beta}=(p,2q)$ in (5.1). Then we have

$$\sum_{i=1}^{n}F(\beta_i,\alpha_i)=F(p,q)+F(2q,2p)=\begin{cases}p-q & (q\le\frac{1}{2}),\\ p+q-1 & (q>\frac{1}{2}),\end{cases}$$

$$\sum_{i=1}^{n}F(\alpha_i,\beta_i)=F(q,p)+F(2p,2q)=\begin{cases}2p-2q & (q\le\frac{1}{2}),\\ 2p-1 & (q>\frac{1}{2}).\end{cases}$$

Put $\gamma=\frac{1-p+q}{2}$. If $q\le\frac{1}{2}$, then

$$0\le\gamma-\sum_{i=1}^{n}F(\beta_i,\alpha_i)\quad\text{if and only if}\quad p-q\le\frac{1}{3},$$

$$\gamma+\sum_{i=1}^{n}F(\alpha_i,\beta_i)\le 1\quad\text{if and only if}\quad p-q\le\frac{1}{3}$$

hold, so that the conditions in (5.2) hold if and only if $p - q \leq \frac{1}{3}$. Similarly, if $q > \frac{1}{2}$, then the conditions in (5.2) hold if and only if $3p + q \leq 3$.

(b) Case $2q < p$. Put $\boldsymbol{\alpha} = (q, 2p)$ and $\boldsymbol{\beta} = (2q, p)$ in (5.1). Then we have

$$\sum_{i=1}^{n} F(\beta_i, \alpha_i) = F(2q, q) + F(p, 2p) = q,$$
$$\sum_{i=1}^{n} F(\alpha_i, \beta_i) = F(q, 2q) + F(2p, p) = p,$$

so the conditions in (5.2) hold if and only if $p + q \leq 1$.

Therefore $L_{p,q}(x) = f_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)$ is operator monotone if (p, q) belongs to the domain

$$\left\{(p, q) : 0 < q \leq p \leq 2q,\ p - q \leq \frac{1}{3},\ 3p + q \leq 3\right\} \cup \{(p, q) : 0 < 2q < p,\ p + q \leq 1\},$$

by Theorem 5.A, so that we have the desired result. $\square$

Proof of Corollary 5.2. Put $p = 1 - 2t$ and $q = 0$ in (i) of Proposition 5.1. $\square$

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