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ISOMORPHISMS OF DISCRETE MULTIPLIER HOPF C^*-BIALGEBRAS: THE NONTRACIAL CASE

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ABSTRACT. We construct Hopf algebra isomorphisms of discrete (multiplier) Hopf C^*-bialgebras from K-theoretical data, without assuming that the Haar weight is tracial.

1. INTRODUCTION

Classification is a recurring theme in mathematics, and it seems that the most successful approach to classifying C^*-algebras is the Elliott program[7], whose aim is to use K-theory, often augumented by some additional information, as a classifying functor. In [12, 11], we considered the case of C^*-algebras with Hopf algebra structure, and we found that in many cases there is a (fusion) product structure on the K-theory group. We then addressed the problem of constructing Hopf algebra maps from algebra maps respecting the product structure on the K-theory group, with applications to constructing automorphisms and isomorphisms of Hopf algebras. These results are slightly complicated to establish, and have the significant limitation of requiring the Haar weight to be tracial. Moreover, they depend on the theory of linear preservers, which results in certain technical restrictions. In this article, we avoid using the theory of linear preservers, and extend the scope of the results to the case where the Haar weight is not necessarily tracial. A Hopf algebra is a bi-algebra with an antipode map S. See [1] for information on Hopf algebras. A multiplier Hopf algebra is a generalization where the co-product homomorphism takes values in a multiplier algebra. See [16] for more information on multiplier Hopf algebras. Consider a compact Hopf algebra A that is also, up to closure, a C^*-algebra; in the Baaj–Skandalis framework of

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C^*-bialgebras provided by [2], the dual object is both a discrete multiplier Hopf algebra and a C^*-algebra, B. The given algebra A is also known as a compact-type C^*-bialgebra, and the dual is called a discrete-type C^*-bialgebra. This dual object will be a possibly infinite direct sum of matrix blocks at the level of C^*-algebras, and the co-product homomorphism, $\Delta\colon B \to \mathcal{M}(B \otimes B)$, takes values in a direct product of matrix blocks (see [17] for details). Our main result is probably the set of related Corollaries 2.10, 2.11, and 2.12, which determine Hopf C^*-bialgebra (co-anti)isomorphism classes in terms of a certain type of K-theory (fusion) ring. We should mention that the K-theory group used in classification of C^*-algebras is an ordered group. Thus our K-theory ring also has an order structure. See [7], or [4, Section V.2.4], for more information on ordered K-theory of C^*-algebras, and see [3] for more information on fusion rings. Our notation is based on that of [13], denoting co-products by Δ, antipodes by S, pairings by $\beta(\cdot\,,\cdot)$, and co-units by ϵ. We denote the flip, on a tensor product, by σ.

2. K-theory of discrete multiplier Hopf C^*-bialgebras

We recall, see [11, pg.99] and [12, Prop. 2.1], that the K-theory of a discrete-type C^*-bialgebra is equipped with a product operation, $\square$. Thus, given two projective (sub)modules M_1 and M_2 over such an algebra, A, the product provides a projective A-module $M_1 \square M_2$, and is compatible with the K-theory equivalence relation. It is possible to define the K-theory group in terms of projections in $A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the algebra of compact operators, instead of modules (see, for example, [4, Section V.1]). Extending the domain of the product to finite linear combinations of projections, we have:

Definition 2.1. *We denote by $\square$ a binary operation on elements of $A \otimes \mathcal{K}$ with the following properties:*

1. *(Pullback) $a(b \square c) = \sum (a_{(1)}b) \square (a_{(2)}c)$, where $a \in A$ and $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$; and*
2. *(normalization) $(\varphi \otimes t)(b \square c) = (\varphi \otimes t)(b)(\varphi \otimes t)(c)$, where φ denotes the left Haar weight of A and t denotes the standard trace on $\mathcal{K}$.*

The above definition is from [11, pg.99], but that reference only treats the finite-dimensional case. The extension to the possibly infinite-dimensional discrete case is justified by [12, Prop. 2.1]. The dual object of a discrete-type C^*-bialgebra is, in the setting provided by [2], a compact-type C*-bialgebra. There is a (densely defined) Fourier transform, see [9, Def. 3.4] that can be defined as $\mathcal{F}(b) := [\varphi(\cdot b) \otimes \mathrm{Id}]V$, where V is a multiplicative unitary coming from the left regular representation, and φ is the (extension of the) left Haar weight. In terms of (invariant) linear functionals, this Fourier transform can be defined, following Van Daele[16], see also [9], by

$$\beta(a, \mathcal{F}(b)) = \varphi(ab),$$

where the elements a and b belong to a Hopf C^*-algebra A, φ is the left Haar weight, and $\beta(\cdot\,,\cdot)$ is the pairing with the dual algebra. Following [9, Def. 3.10], an operator-valued convolution product, $\diamond$, can be defined by the property $\mathcal{F}(a \diamond b) = \mathcal{F}(a)\mathcal{F}(b)$, where a and b are elements of a Hopf C^*-algebra A, and $\mathcal{F}$ is the Fourier transform defined previously. We remark that the Fourier transform is invertible on its range, and that

$$\beta(a,w) = \varphi(a\mathcal{F}^{-1}(w))$$

for all w in the range of the Fourier transform.

The next Proposition relates $\square$ and $\diamond$.

Proposition 2.2. *Let A be a discrete Hopf C^*-algebra with faithful Haar weights. Let ℓ_1 and ℓ_2 be compactly supported elements of A^+. If t is the standard trace on $\mathcal{K}$, then $(\mathrm{Id}\otimes t)(\ell_1 \square \ell_2) = \ell_1 \diamond \ell_2$.*

Proof. Let us denote by φ the left invariant Haar functional on A.

Consider a linear functional ω of the form $\varphi(z\cdot)$ with $z \in A$ being supported in finitely many matrix blocks. If $\Delta(z) = \sum a_{(1)} \otimes a_{(2)}$, we use the normalization property $(\varphi\otimes t)(a \square b) = \varphi(a)\varphi(b)$ of the operation $\square$, to obtain:

$$\begin{aligned}
(\omega\otimes t)(\ell_1 \square \ell_2) &= \sum (\varphi\otimes t)((a_{(1)}\ell_1) \square (a_{(2)}\ell_2)) \\
&= \sum \varphi(a_{(1)}\ell_1)\varphi(a_{(2)}\ell_2) \\
&= \sum (\varphi\otimes\varphi)((a_{(1)}\ell_1) \otimes (a_{(2)}\ell_2)) \\
&= \sum \beta((a_{(1)} \otimes a_{(2)}, \mathcal{F}(\ell_1)\otimes\mathcal{F}(\ell_2)),
\end{aligned}$$

In the last step we used the equation $\beta(a,\mathcal{F}(b)) = \varphi(ab)$. But since $\Delta(z) = \sum a_{(1)} \otimes a_{(2)}$, we have

$$\begin{aligned}
\sum \beta((a_{(1)} \otimes a_{(2)}, \mathcal{F}(\ell_1)\otimes\mathcal{F}(\ell_2)) &= \beta(z, \mathcal{F}(\ell_1)\mathcal{F}(\ell_2)) \\
&= \varphi(z\mathcal{F}^{-1}(\mathcal{F}(\ell_1)\mathcal{F}(\ell_2))),
\end{aligned}$$

using the equation $\beta(a,w) = \varphi(a\mathcal{F}^{-1}(w))$. Thus it follows that $(\omega\otimes t)(\ell_1 \square \ell_2)$ is equal to $\varphi(z\mathcal{F}^{-1}(\mathcal{F}(\ell_1)\mathcal{F}(\ell_2)))$, and this last expression can be written $\omega(\ell_1 \diamond \ell_2)$. The linear functional ω is of the form $\varphi(z\cdot)$ with z supported in finitely many matrix blocks. Thus, it follows by taking weak limits that

$$(\mathrm{Id}\otimes t)(\ell_1 \square \ell_2) - \ell_1 \diamond \ell_2 \in A$$

is zero under linear functionals of the form $\omega := \varphi(z\cdot)$ with $z \in A$. The faithfulness of φ then implies that $(\mathrm{Id}\otimes t)(\ell_1 \square \ell_2) - \ell_1 \diamond \ell_2 = 0$ as claimed. ☐

Lemma 2.3. *Let A, B be Hopf C^*-algebras with faithful left Haar weights. Let $f : A \to B$ be a C^*-algebraic isomorphism that intertwines the left Haar weights. Then, we have the identity*

$$\mathcal{F}_B \circ f = f^{*-1} \circ \mathcal{F}_A,$$

where $\mathcal{F}$ denotes the Fourier transform, and f^{*-1}, which is continuous, is the inverse map for f^*, the map induced by f on the dual algebras.

Proof. The Fourier transform on A can be written as $\mathcal{F}_A \colon a \mapsto \varphi_A(\cdot\, a)$. A similar statement holds for the Fourier transform on B. Then, using the property that

$\varphi_B(f(x)) = \varphi_A(x)$ for all $x \in A$, we get

$$\begin{aligned} f^*(\mathcal{F}_B(f(a))) &= f^* \circ (\varphi_B(\cdot\, f(a))) \\ &= \varphi_B(f(\cdot)f(a)) \\ &= \varphi_B(f((\cdot)a)) \\ &= \varphi_A(\cdot\, a). \end{aligned}$$

From the above it follows that $f^* \circ \mathcal{F}_B \circ f = \mathcal{F}_A$. That f^{*-1} exists follows from the fact that f is invertible and the inverse induces a map of dual algebras. That f^{*-1} is continuous is seen by applying the open mapping theorem to f^*. Composing $f^* \circ \mathcal{F}_B \circ f = \mathcal{F}_A$ with f^{*-1} on the left, we have the desired conclusion. □

Lemma 2.4. *Let A and B be discrete-type C^*-bialgebras that have a faithful left Haar weight. Let $f : A \to B$ be a C^*-isomorphism that intertwines left Haar weights, and whose induced map on K-theory respects the product $\square$. The map f^* induced by f on the dual algebra(s) satisfies, for all y_i in the dual algebra $\widehat{A}$,*

$$\widehat{\varphi}(f^{*-1}(y_1 y_2)) = \widehat{\varphi}(f^{*-1}(y_1) f^{*-1}(y_2))$$

where $\widehat{\varphi}$ is the Haar state of $\widehat{A}$.

Proof. Proposition 4.8 in [18] gives $\epsilon_B(a) = \widehat{\varphi} \circ \mathcal{F}_B(a)$ for all $a \in B$, where ϵ_B is the co-unit homomorphism of B, $\mathcal{F}_B$ is the Fourier transform defined by the left Haar weight of B, and $\widehat{\varphi}$ is the Haar state of the dual algebra, $\widehat{B}$. Since $\widehat{B}$ is compact-type, it is unimodular, and the left and right Haar states coincide.

Since ϵ_B is a *-homomorphism, $\epsilon_B \otimes t$, where t is the canonical trace on the compact operators, defines a state on the K-theory group of B. We denote by $f_* \colon K(A) \to K(B)$ the map induced on K-theory by $f : A \to B$. Since f_* respects the product $\square$, we have

$$(\epsilon_B \otimes t)(f_*(x_1 \,\square\, x_2)) = (\epsilon_B \otimes t)(f(x_1) \,\square\, f(x_2)),$$

where x_1, x_2 are projections in A.

By Proposition 2.2, it follows that

$$\epsilon_B(f(p) \diamond f(q)) = \epsilon_B(f(p \diamond q)),$$

where $\diamond$ denotes convolution of operators. We may replace the x_i in the above by finite linear combinations of projections, and since such linear combinations are dense in a discrete c_0-direct sum of matrix algebras, the above equation holds in general. The already established identities $\epsilon_B(a) = \widehat{\varphi} \circ \mathcal{F}_B(a)$ and $\mathcal{F}_B \circ f = f^{*-1} \circ \mathcal{F}_A$, from Lemma 2.3, together with the definition of convolution as $p \diamond q = \mathcal{F}^{-1}(\mathcal{F}(p)\mathcal{F}(q))$, let us rewrite the above equation as

$$\widehat{\varphi}(f^{*-1}(\widehat{p}\widehat{q})) = \widehat{\varphi}(f^{*-1}(\widehat{p}) f^{*-1}(\widehat{q})),$$

where $\widehat{p}$ and $\widehat{q}$ denote the Fourier transforms of p and q. But then the equation $\widehat{\varphi}(f^{*-1}(y_1 y_2)) = \widehat{\varphi}(f^{*-1}(y_1) f^{*-1}(y_2))$ holds for all y_i in a dense subset of the unital C^*-algebra $\widehat{A}$, and since $\widehat{\varphi}$ is bounded and f^{*-1} is continuous, the equation then holds for all $y \in \widehat{A}$. □

We recall that a linear and not necessarily multiplicative map f of algebras is said to be a Jordan map if it is true that $f(x)^2 = f(x^2)$ for all algebra elements x. We now show how to construct an algebra map of discrete Hopf C^*-algebras such that the induced pullback map on the compact duals is a Jordan map.

The Haar weight of a discrete multiplier Hopf C^*-algebra does not in general itself define a state on the K-theory group of that algebra, except in the unimodular case. There is however the following useful K-theory state that can be defined using a Haar weight.

Definition 2.5. *The* dimension state *associated with a Haar weight* $\varphi\colon A \to \mathbb{C}$ *is the map* $d_\varphi\colon K(A) \to \mathbb{C}$ *defined on central minimal projections* p_x *by* $d_\varphi(p_x) := \varphi(p_x)$.

Since the central minimal projections generate the K-theory group, the above map extends by linearity to obtain a state on the K-theory group. Thus we have both a left and a right dimension state. The term dimension state seems appropriate because the state d_φ can be described in terms of the quantum dimensions of certain representations, see for example [8, Prop. 2.1].

Proposition 2.6. *Let* A *and* B *be separable discrete-type* C^**-bialgebras that have faithful Haar weights. Suppose that there exists an isomorphism of* K*-theory groups that respects the product* $\square$*, and intertwines the dimension states associated with the left Haar weights. Then there exists a* C^**-algebraic isomorphism of* A *and* B *that intertwines left Haar weights, such that its pullback is a Jordan map on the dual algebras.*

Proof. Let us suppose that we are given an isomorphism of ordered K-theory groups, $f\colon K(A) \to K(B)$ that respects the product, $\square$. Using the classification theory[4, Theorem V.2.4.19] for AF-algebras, we may lift this isomorphism to a C^*-algebraic isomorphism $f_*\colon A \to B$. This lifted map is not unique, because it may be replaced by any map in its approximate unitary equivalence class. In general, we expect that f_* will not intertwine the left Haar weights, φ_A, and φ_B. Thus, the linear functionals $\varphi_B \circ f_*$ and φ_A are in general distinct linear functionals on A. However, when we restrict both of these linear functionals to some matrix block of A, the condition of intertwining dimension states makes the restricted states equal on the identity element of the matrix block. We can then make the restricted states unitarily equivalent by some unitary of that matrix block (for example, using [4, Theorem III.2.6.7]). Since we may do this in every matrix block, we obtain a unitary in the multiplier algebra of A that makes the states $\varphi_B \circ f_*$ and φ_A unitarily equivalent. We can therefore adjust f_* by a unitary from the multiplier algebra in order to ensure that f_* will intertwine the left Haar weights.

By Lemma 2.4, we have

$$\widehat{\varphi}(f^{*-1}(b_1)f^{*-1}(b_2)) = \widehat{\varphi}(f^{*-1}(b_1b_2))$$

where f^{*-1} is the inverse of the pullback map on the duals induced by f_*, the b_i are elements of $\widehat{A}$, and $\widehat{\varphi}$ is the Haar state of $\widehat{B}$. The same proof shows that

$$\widehat{\varphi}(f^{*-1}(a)(f^{*-1}(b)^2 - f^{*-1}(b^2))) = 0$$

for all a and b in $\widehat{A}$. Since f^{*-1} is clearly a surjective map, we can deduce that

$$\omega(f^{*-1}(b)^2 - f^{*-1}(b^2)) = 0$$

for all linear functionals ω of the form $\widehat{\varphi}(z\cdot)$ where $z := f^{*-1}(a)$ can be chosen to be any element of $\widehat{B}$. But, linear functionals of this form separate the points of $\widehat{B}$, so therefore $f^{*-1}(b)^2 - f^{*-1}(b^2) = 0$. This equation shows that f^{*-1}, and therefore f^*, is a Jordan map. □

We now show that an algebra map of discrete-type C^*-bialgebras whose pull-back on the dual algebras is Jordan will necessarily intertwine antipodes. The next Proposition is a slight generalization of Proposition 3.1 from [12], which in turn is based on an idea from [6]. We give a detailed proof for the reader's convenience.

Proposition 2.7. *Let A and B be discrete-type Hopf C^*-bialgebras. Let $f\colon A \longrightarrow B$ be an C^*-algebraic *-isomorphism that intertwines left Haar states and co-units. Let the induced map $f^*\colon \widehat{B} \longrightarrow \widehat{A}$ on the duals be Jordan. It then follows that f^* is a Jordan *-isomorphism.*

Proof. The properties of Jordan maps of C^*-algebras[14] provide a central projection P of the enveloping von Neumann bialgebra of $\widehat{A}$ such that $Pf^*\colon \widehat{B} \to \widehat{A}$ is multiplicative, and $(1-P)f^*\colon \widehat{B} \to \widehat{A}$ is antimultiplicative. (A similar statement is true for f^{*-1}.) Thus f^* can be written as the sum of a C^*-algebraic homomorphism and a C*-algebraic anti-homomorphism, each having range and support in a complemented ideal of an enveloping von Neumann bi-algebra.

Recall that $\widehat{A}$ and $\widehat{B}$ contain dense compact algebraic quantum groups, $\widehat{A}_0$ and $\widehat{B}_0$. (See [15, Theorem 5.4.1], for example). The duals of $\widehat{A}_0$ and $\widehat{B}_0$ are multiplier Hopf algebras. As algebras, they are algebraic direct sums inside A and B, in other words, they are the subalgebras of compactly supported elements. Since f is a C^*-isomorphism, it maps compactly supported elements to compactly supported elements, and therefore f^* maps $\widehat{B}_0$ to $\widehat{A}_0$.

The decomposition of the restricted map $f^*\colon \widehat{B}_0 \to \widehat{A}_0$ into a homomorphism and an anti-homomorphism — still supported on complemented ideals — gives, by duality, complementary linear subspaces of A_0, denoted $\mathcal{S}_1(A_0)$ and $\mathcal{S}_2(A_0)$, such that f restricted to $\mathcal{S}_1(A_0)$ is comultiplicative and f restricted to $\mathcal{S}_2(A_0)$ is anti-comultiplicative. Recalling that the dual of a complemented ideal is a sub-co-algebra[1, Theorem 2.3.1.ii], the $\mathcal{S}_i(A_0)$ are sub-co-algebras.

Now consider the convolution algebra $\mathcal{C}(\mathcal{S}_1(A_0), B_0)$, where convolution is defined, following [1, pg. 61], by $f{*}g := m_{B_0}[(f \otimes g)\Delta_{A_0}]$.

Taking w in $\mathcal{S}_1(A_0)$, and writing $\Delta(w) = \sum w_{(1)} \otimes w_{(2)}$, the co-multiplicativity of f on w gives

$$\begin{aligned}
[(S_{B_0} \circ f){*}f] &= \sum S_{B_0}(f(w_{(1)}))f(w_{(2)}) \\
&= \epsilon_{B_0}(f(w))1_{\mathcal{M}(B_0)} \\
&= \epsilon_{A_0}(w)1_{\mathcal{M}(B_0)}.
\end{aligned}$$

We next consider $[f*(f\circ S_A)](w)$. In this case, for all $w\in A_0$, writing $\Delta(w)=\sum w_{(1)}\otimes w_{(2)}$, we have

$$\begin{aligned}[f*(f\circ S_{A_0})](w) &= \sum f(w_{(1)})f(S_{A_0}(w_{(2)}))\\ &= f\left(\sum w_{(1)}S_{A_0}(w_{(2)})\right)\\ &= f(\epsilon_{B_0}(w)1_{\mathcal{M}(A_0)})\\ &= \epsilon_{A_0}(w)1_{\mathcal{M}(B_0)}.\end{aligned}$$

We thus conclude that in the convolution algebra $\mathcal{C}(\mathcal{S}_1(A_0),B_0)$, we have

$$f*(f\circ S_{A_0})=1_{\mathcal{C}}$$

and

$$(S_{B_0}\circ f)*f=1_{\mathcal{C}}.$$

It follows by the associativity of the operation $*$ that $S_{B_0}\circ f$ is in fact equal to $f\circ S_{A_0}$, on $\mathcal{S}_1(A_0)$.

The algebraic subgroups inherit a $*$-involution from the enveloping C^*-algebras. In Lemma 3.3 of [9], see also [19], it is shown that the C^*-involution of the dual can be written in terms of the linear functional picture of the dual as

$$\omega(\cdot)\mapsto\omega^{\#}(\cdot)$$

where the linear functional $\omega^{\#}$ is defined by $\omega^{\#}(x)=\overline{\omega([S(x)]^*)}$. Let $y\in\widehat{B}_0$ be in the multiplicative domain of f^*. Thus, $f^*(y)$ is in $P\widehat{A}_0$. Written as a linear functional on $\mathcal{S}_1(A)$, the element $f^*(y)$ has the form $\omega(f(\cdot))\colon\mathcal{S}_1(A)\to\mathbb{C}$. We now show that $[f^*(\omega)]^{\#}(x)=\omega^{\#}(f(x))$ for all $x\in\mathcal{S}_1(x)$. We have

$$\begin{aligned}\omega^{\#}(f(x)) &= \overline{\omega(S_{B_0}(f(x))^*)}\\ &= \overline{\omega(f(S_{A_0}(x))^*)}\\ &= \overline{\omega(f(S_{A_0}(x)^*))}\\ &= [\omega(f(\,\cdot\,))]^{\#}(x).\end{aligned}$$

This shows that the multiplicative part of $f^*\colon\widehat{B}_0\to\widehat{A}_0$ is a *-homomorphism. Replacing A by the co-opposite algebra A^{cop}, it follows similarly that the multiplicative part of $f^*\colon\widehat{B}_0\to(\widehat{A}_0)^{op}$ is a *-homomorphism, and so the anti-multiplicative part of $f^*\colon\widehat{B}_0\to\widehat{A}_0$ is a *-antihomomorphism.

It follows that $f^*\colon\widehat{B}_0\to\widehat{A}_0$ respects the $*$-involution inherited from the enveloping C^*-algebra. At the level of C^*-algebras, we have that the (necessarily bounded) Jordan homomorphism $f^*\colon\widehat{B}\to\widehat{A}$ is *-preserving on the dense set $\widehat{B}_0$, and therefore is a Jordan *-homomorphism. □

We now give a generalization of [10, Lemma 2.7]. We note that in the following the pullback is required to be a Jordan *-homomorphism, not just a Jordan homomorphism.

Lemma 2.8. *Let A and B be discrete-type C^*-bialgebras. Let $\alpha\colon A \to B$ be a *-algebra isomorphism, and let $\alpha^*\colon \widehat{B} \to \widehat{A}$ be its action on the dual. We suppose the action α^* on the dual is a Jordan *-isomorphism. Then either α^* is multiplicative, or α^* is anti-multiplicative.*

Proof. A Jordan *-isomorphism maps the C^*-norm unit ball onto itself. If α^* maps the unit ball onto itself, this implies that the map that it induces on linear functionals is an isometry with respect to the usual dual norm (on linear functionals.) Thus, the map α is an isometry with respect to this norm, which makes α a bijective isometric algebra homomorphism in the sense of [5, Theorem 4.5]. It follows that α^* is either multiplicative or anti-multiplicative, as claimed. $\square$

In the next Theorem, the term co-unit state refers to the state on K-theory induced by the co-unit homomorphism.

Theorem 2.9. *Let A and B be reduced discrete separable C^*-bialgebras. Let $f\colon A \to B$ be a C^*-isomorphism that intertwines left dimension states and co-unit states. We suppose that the induced map on K-theory intertwines the products $\square_A$ and $\square_B$. Then A and B are isomorphic or co-anti-isomorphic as Hopf algebras. If the K-theory rings are not commutative, then A and B are isomorphic as Hopf algebras.*

Proof. By the proof of Proposition 2.6, the pullback $f^*\colon \widehat{B} \longrightarrow \widehat{A}$ of f is a Jordan isomorphism at the level of C^*-algebras. Proposition 2.7 ensures that the pullback map f^* is a Jordan *-isomorphism. By Lemma 2.8 the pullback map f^* is either multiplicative or anti-multiplicative. We thus have, by duality, that f is either an isomorphism or a co-anti-isomorphism of bi-algebras. It follows from uniqueness of the Hopf algebra antipode(s) that f is a Hopf algebra (co-anti)isomorphism. We note that a co-anti-isomorphism reverses the product $\square$, and by hypothesis, f intertwines the products $\square_A$ and $\square_B$. Therefore, either f cannot be a co-anti-isomorphism or the K-theory rings must be commutative. $\square$

From the viewpoint of the Elliott classification program[7], it is of interest to obtain C^*-(bi)algebra isomorphisms from purely K-theoretical data. Thus we mention the following related Corollaries, which come from lifting a K-theory map to a C^*-algebraic isomorphism, as in the proof of Proposition 2.6, and then applying Theorem 2.9.

Corollary 2.10. *Let A and B be reduced discrete separable Hopf C^*-bialgebras. Let $f\colon K(A) \to K(B)$ be a K-theory ring isomorphism that intertwines left dimension states and co-unit states. Then A and B are isomorphic or co-anti-isomorphic as Hopf algebras.*

Corollary 2.11. *Let A and B be reduced discrete separable Hopf C^*-bialgebras with K-theory rings that are not commutative. Then the following are equivalent:*

(1) *there exists a K-theory ring isomorphism $f\colon K(A) \to K(B)$ that intertwines left dimension states and co-unit states, and*
(2) *there exists a Hopf algebra isomorphism $f\colon A \to B$.*

We can also consider ring anti-isomorphisms that interchange the left and right dimension states. Applying Corollary 2.10 to either A and B or A and B^{cop}, we have:

Corollary 2.12. *Let A and B be reduced discrete separable Hopf C^*-bialgebras. Then the following are equivalent:*

(1) *there exists a K-theory ring (anti)isomorphism $f\colon K(A) \to K(B)$ that intertwines dimension states and co-unit states, and*
(2) *there exists a Hopf algebra (co-anti)isomorphism $f\colon A \to B$.*

The above corollaries give an abstract classification, in terms of an Elliott invariant, of discrete quantum groups in the C^*-algebraic picture. An isomorphism of discrete quantum groups can equally be regarded as an isomorphism of the compact duals, thus by passing to the dual we implicitly obtain an abstract classification of compact quantum groups.

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References

1. E. Abe, *Hopf Algebras*, Cambridge University Press, London, 1980.
2. S. Baaj and G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres*, Ann. Sci. École Norm. Sup. **4** (1993), 425–488.
3. T. Banica, *Fusion Rules for Representations of Compact Quantum Groups,* Expo. Math., **17** (1999), 313–338.
4. B. Blackadar, *Operator algebras: Theory of C^*-algebras and von Neumann algebras,* Springer-Verlag, Berlin, 2006.
5. M. Daws and H. Le Pham, *Isometries between quantum convolution algebras*, Q. J. Math. **64** (2013), 373–396.
6. S. Dăscălescu, C. Năstăsescu, and Ş. Raianu, *Hopf Algebras,* Marcel Dekker Inc., New York, 2001.
7. G.A. Elliott, *The classification problem for amenable C^*-algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zrich, 1994), 922–932, Birkhuser, Basel, 1995.
8. P. Fima, *Kazhdan's property T for discrete quantum groups,* Internat. J. Math. **21** (2010), no. 1, 47–65.
9. B.-J. Kahng, *Fourier transform on locally compact quantum groups*, J. Operator Theory **64** no. 1 (2010), 69–87.
10. D. Kučerovský, *On convolution products and automorphisms in Hopf C^*-algebras*, Positivity **18** (2014), no. 3, 595–601.
11. D. Kučerovský, *An abstract classification of finite-dimensional Hopf C^*-algebras*, Comptes Rendus Math. (Canada), **36** (4) 2014, 97–105.
12. D. Kučerovský, *Isomorphisms and automorphisms of discrete multiplier Hopf C^*-algebras*, Positivity **19** (2015), no. 1, 161–175.
13. J. Kustermans and S. Vaes, *Locally compact quantum groups,* Ann. Scient. École Norm. Sup., (4) **33** (2000), no. 6, 837–934.
14. E. Størmer, *On the Jordan structure of C^*-algebras,* Trans. Amer. Math. Soc. **120** (1965) 438–447.
15. T. Timmermann, *An Invitation to Quantum Groups and Duality,* EMS Textbooks in Mathematics, EMS Publishing House, Zürich, Switzerland, 2008.

16. A. Van Daele, *Multiplier Hopf algebras*, Trans. Amer. Math. Soc. **342** (1994), 917–932.
17. A. Van Daele, *Discrete quantum groups*, J. Algebra **180** (1996), 431–444.
18. A. Van Daele, *An Algebraic Framework for Group Duality*, Advances in Math. **140** (1998), 323–366.
19. A. Van Daele, *Locally compact quantum groups: the von Neumann algebra versus the C^*-algebra approach*, J Bull. Kerala Math. Assoc. (2007), 153–177.

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