A characterization of admissible vectors related to representations on hypergroups

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Abstract

In this paper among the other things, we give some sufficient and necessary conditions for an element of $L^2(K)$ to be a Parseval admissible vector, where K is a locally compact hypergroup.

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1 Introduction and preliminaries

Locally compact hypergroups, as extensions of locally compact groups, were introduced in a series of papers by C. F. Dunkl [2], R. I. Jewett [5] and R. Spector [8] (see also [9]). Roughly speaking, a hypergroup is a locally compact Hausdorff space with a convolution and involution such that the space of its regular measures is an associative Banach algebra. Examples include locally compact groups, double coset spaces, polynomial hypergroups, and orbit spaces (for more examples see [1]). In the last decade, the theory of frame and wavelet has been extended in harmonic analysis on locally compact groups. In [11], we have initiated the study of admissible vectors on some function spaces related to hypergroups and we have generalized basic properties of coorbit spaces.

In this paper, by a version of Wiener's theorem which was proved in [10], we give a characterization of admissible vectors related to the left regular representation of hypergroups, and extend the main results of [6].

Let K be a locally compact Hausdorff space. We denote by $\mathcal{M}(K)$ the space of all regular complex Borel measures on K, by $\mathcal{C}_c(K)$ the set of all compact supported complex-valued continuous functions on K, and by δ_x the Dirac measure at $x \in K$. The support of a measure $\mu \in \mathcal{M}(K)$ is denoted by $\mathrm{supp}(\mu)$.

In this paper, we assume that K is a commutative *locally compact hypergroup* (or simply a *hypergroup*) together with a convolution $(\mu, \nu) \mapsto \mu * \nu$ from $\mathcal{M}(K) \times \mathcal{M}(K)$ into $\mathcal{M}(K)$, an involution $x \mapsto x^-$ from K onto K, and the identity element e. Also, we assume that m is a Haar measure for K. For definition and basic properties of hypergroups see [5] and [1].

For each complex-valued Borel function f on K, $\mu \in \mathcal{M}(K)$ and $x, y \in K$ we denote

$$f(x * y) := \int_K f d(\delta_x * \delta_y) \quad \text{and} \quad (\mu * f)(x) := \int_K f(y^- * x) d\mu(y).$$

A non-zero complex-valued bounded continuous function ξ on a commutative hypergroup K is called a *character* if for all $x, y \in K$, $\xi(x * y) = \xi(x)\xi(y)$ and $\xi(x^-) = \overline{\xi(x)}$. The set of all characters

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of K equipped with the uniform convergence topology on compact subsets of K, is denoted by $\hat{K}$ and is called *dual* of K. If $\hat{K}$ with the complex conjugation as involution and poinwise product, i.e.

$$\xi(x)\eta(x) = \int_{\hat{K}} \chi(x) d(\delta_\xi * \delta_\eta)(\chi), \qquad (x \in K \text{ and } \xi, \eta \in \hat{K}),$$

as convolution is a hypergroup, then K is called a *strong* hypergroup. In spite of the group case, in general, K is not necessarily a strong hypergroup. In the case that the second dual of K, $\hat{\hat{K}}$, is also a hypergroup, K is called *Pontryagin* hypergroup. If K is a Pontryagin hypergroup, then we have $\hat{\hat{K}} \cong K$ [1, Theorem 2.4.3].

For each $1 \leq p < \infty$, we denote the Lebesgue space $L^p(K, m)$ by $L^p(K)$, where m is the (left) Haar measure. By the Levitan Theorem [1], there exists a unique non-negative measure π for $\hat{K}$ (called *Plancherel* measure) such that for each $f \in L^1(K, m) \cap L^2(K, m)$, $\int_K |f|^2 dm = \int_{\hat{K}} |\hat{f}|^2 d\pi$, where

$$\hat{f}(\xi) := \int_K f(x)\overline{\xi(x)} dm(x), \quad (\xi \in \hat{K})$$

is the Fourier transform of f. The mapping $f \mapsto \hat{f}$ can be extended to an isometric isomorphism from $L^2(K, m)$ onto $L^2(\hat{K}, \pi)$.

For each $1 \leq p < \infty$, we denote $L^p(\hat{K}) := L^p(\hat{K}, \pi)$, where π is the Plancherel measure on $\hat{K}$ associated with the (left) Haar measure m. If K is a strong hypergroup, then the Plancherel measure π is a Haar measure on $\hat{K}$ and $\text{supp}(\pi) = \hat{K}$ [1, 2.4.3].

For every $k \in L^1(\hat{K})$, the inverse Fourier transform $\check{k}$ of k is defined by

$$\check{k}(x) = \int_{\hat{K}} \xi(x) k(\xi) d\pi(\xi), \quad (x \in K).$$

Let H be a normal subhypergroup of K. Then, $K/H := \{x * H : x \in K\}$ equipped with quotient topology is a locally compact space. In general, K/H is not a hypergroup. For giving a reasonable convolution on K/H, the mappings

$$(\delta_x * \delta_y)(f) := \int_K f(z * H) d(\delta_x * \delta_y)(z) \qquad (x, y \in K,\ f \in \mathcal{C}_c(K/H)), \tag{1.1}$$

must be well-defined, i.e., independent of the representatives x and y of the cosets $x * H$ and $y * H$. In this case, the mapping 1.1 can be extended to a convolution on $\mathcal{M}(K/H)$, and K/H is called a *quotient hypergroup*; for details see [12]. By [12, Proposition 1.8], if a normal subhypergroup H of K is of compact type, then K/H is a quotient hypergroup.

A closed subhypergroup H of K is called a *Weil subhypergroup* if $T_H : f \mapsto T_H f$ defined by $(T_H f)(x * H) := \int_H f(x * t) dt, \quad (x \in K)$ is a well-defined linear mapping from $\mathcal{C}_c(K)$ into $\mathcal{C}_c(K/H)$. Let H be a Weil subhypergroup of K such that K/H is a quotient hypergroup. By [4, Proposition 1], if K/H admits a Haar measure, then the *Weil's formula*

$$\int_K f(z) dz = \int_{K/H} \int_H f(x * t) dt d(x * H),$$

holds. Also, by [12, Proposition 1.8 and Theorem 2.3], if H is a normal compact subhypergroup of K, then the Weil's formula holds.

2 Main results

Definition 2.1. A complex valued function f on K is called a *trigonometric polynomial* if for some $a_1, \ldots, a_n \in \mathbb{C}$ and $\xi_1, \ldots, \xi_n \in \hat{K}$ we have $f = \sum_{i=1}^{n} a_i \xi_i$. The set of all trigonometric functions on K is denoted by $\mathrm{Trig}(K)$.

By [1], $\mathrm{Trig}(K)$ is dense in $L^2(K)$.

Definition 2.2. If H is a subhypergroup of a hypergroup K, then

$$H^{\perp} := \{\xi \in \hat{K} : \xi(x) = 1 \text{ for all } x \in H\}$$

is called the *annihilator* of H in $\hat{K}$.

If K is a commutative hypergroup, then $H^{\perp}$ is closed in $\hat{K}$, and if K is a strong hypergroup, then $H^{\perp}$ is a subhypergroup of $\hat{K}$ (see [1, 2.2.45]).

Lemma 2.3. If K is a strong commutative hypergroup and H is a compact subhypergroup of K, then K/H is a hypergroup and $\widehat{K/H} \equiv H^{\perp}$. In addition, H is strong, $\hat{K}/H^{\perp} \cong \hat{H}$, and $\hat{H}$ has a left Haar measure.

Proof. See [1, 2.4.8, 2.4.10, 2.4.16 and 2.4.3]. Q.E.D.

In the sequel, we assume that K is a compact Pontryagin commutative hypergroup and H is a compact subhypergroup of K.

If $f \in L^2(K)$ and $x \in K$, we put $\tau_x f(y) := f(x^- * y)$, where $y \in K$.

Remark 2.4. For each $x \in K$, $f \in L^2(K)$ and $\xi \in \hat{K}$ we have $\widehat{\tau_x f}(\xi) = \hat{f}(\xi)\overline{\xi(x)}$.

Definition 2.5. For each $\varphi \in L^2(K)$ we denote $A_\varphi :=$ linear span $\{\tau_x \varphi : x \in K\}$, and $\|.\|_2$–closure of A_φ is denoted by V_φ. In this definition, if the elements x are considered from a subhypergroup H of K, then V_φ would be correspondent to H.

Definition 2.6. Let $\varphi \in L^2(K)$. We denote by $L^2(\hat{H}, w_\varphi)$ the space of all functions $r : \hat{H} \to \mathbb{C}$ with $\int_{\hat{H}} |r(\xi)|^2 w_\varphi(\xi) d\xi < \infty$, where

$$w_\varphi(\xi) := \int_{H^{\perp}} |\hat{\varphi}(\xi * \eta)|^2 d\eta.$$

In this case, the mapping

$$\|r\|_\varphi := \left(\int_{\hat{H}} |r(\xi)|^2 w_\varphi(\xi) d\xi \right)^{\frac{1}{2}} \qquad (r \in L^2(\hat{H}, w_\varphi))$$

is a norm on $L^2(\hat{H}, w_\varphi)$.

Remark 2.7. Under above notations, we have $w_\varphi \in L^1(\hat{H})$. In fact,

$$\begin{aligned}
\int_{\hat{H}} |w_\varphi(\xi)| d\xi &= \int_{\hat{H}} \int_{H^\perp} |\hat{\varphi}(\xi * \eta)|^2 d\eta\, d\xi \\
&\leq \int_{\hat{H}} \int_{H^\perp} |\hat{\varphi}|^2(\xi * \eta) d\eta\, d\xi \quad \text{(by Holder inequality)} \\
&= \int_{\hat{K}/H^\perp} \int_{H^\perp} |\hat{\varphi}|^2(\xi * \eta) d\eta\, d(\xi * H^\perp) \quad \text{(by Lemma 2.3)} \\
&= \int_{\hat{K}} |\hat{\varphi}(\xi)|^2 d\xi = \|\hat{\varphi}\|_2^2 = \|\varphi\|_2^2 < \infty.
\end{aligned}$$

Lemma 2.8. Let $\varphi \in L^2(K)$. Then, $f \in A_\varphi$ if and only if for some $r \in \mathrm{Trig}(\hat{K})$, $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$ $(\xi \in \hat{K})$.

Proof. Let $f \in A_\varphi$. Then, there are $y_1, \ldots, y_n \in H$ and $a_1, \ldots, a_n \in \mathbb{C}$ such that

$$f(x) = \sum_{i=1}^n a_i \tau_{y_i} \varphi(x) = \sum_{i=1}^n a_i \varphi(y_i^- * x).$$

So by [5, 5.1D],

$$\begin{aligned}
\hat{f}(\xi) &= \sum_{i=1}^n a_i \int_K \varphi(y_i^- * x)\overline{\xi(x)} dm(x) \\
&= \sum_{i=1}^n a_i \int_K \varphi(x)\overline{\xi(y_i * x)} dm(x) \\
&= \sum_{i=1}^n a_i \overline{\xi(y_i)} \int_K \varphi(x)\overline{\xi(x)} dm(x) \\
&= \sum_{i=1}^n a_i \overline{\xi(y_i)} \hat{\varphi}(\xi) = \hat{\varphi}(\xi) r(\xi),
\end{aligned}$$

where $r(\xi) := \sum_{i=1}^n a_i \overline{\xi(y_i)}$, and so $r \in \mathrm{Trig}(\hat{K})$.

Conversely, let for some $r \in \mathrm{Trig}(\hat{K})$, $\underline{\hat{f}(\xi)} = r(\xi)\hat{\varphi}(\xi)$ $\quad (\xi \in \hat{K})$. Then, there are $y_1, \ldots, y_n \in K$ and $a_1, \ldots, a_n \in \mathbb{C}$ such that $r = \sum_{i=1}^n a_i \xi(y_i)$. For each $i = 1, \ldots, n$ we put $M_i := \int_{\hat{K}} |\chi(y_i)|^2 d\pi(\chi)$. So by [13], $0 < M_i < \infty$ and for each $y \in K$,

$$\int_{\hat{K}} \xi(y)\overline{\xi(y_i)} d\pi(\xi) := \begin{cases} M_i & \text{if } y = y_i, \\ 0 & \text{if } y \neq y_i. \end{cases}$$

Then, for each $y \in K$,

$$\check{r}(y) = \int_{\hat{K}} \xi(y) r(\xi) d\pi(\xi) = \sum_{i=1}^n a_i \int_{\hat{K}} \xi(y)\overline{\xi(y_i)} d\pi(\xi) = \sum_{i=1}^n a_i M_i \chi_{\{y_i\}},$$

where χ_A is the characteristic function on a set A.

So,

$$\begin{aligned} f(x) = (\check{r} * \varphi)(x) &= \int_K \check{r}(y)\varphi(y^- * x)dm(y) \\ &= \sum_{i=1}^n a_i M_i \int_K \chi_{\{y_i\}}(y)\varphi(y^- * x)dm(y) \\ &= \sum_{i=1}^n a_i M_i \varphi(y_i^- * x)m(\{y_i\}) \\ &= \sum_{i=1}^n b_i \tau_{y_i}\varphi(x), \end{aligned}$$

where $b_i := a_i M_i m(\{y_i\})$. Hence, $f \in A_\varphi$. Q.E.D.

Remark 2.9. In the following result, we put the condition of boundedness on the dual hypergroup $\hat{K}$. In fact, there are several classes of hypergroups satisfying this condition. For example, for every character ξ of the compact countable hypergroup $\mathbb{Z}_+^*$, introduced by C. F. Dunkl and C. E. Ramirez [3], we have $|\xi| \leq 1$ (see [3, 3.6 and 3.7]).

Lemma 2.10. Let $\hat{K}$ be bounded (i.e. there is a constant number $M > 0$ such that for all $\xi \in \hat{K}$, $|\xi| \leq M$), and $\varphi \in L^2(K)$. Then, $\mathrm{Trig}(\hat{K}) \subseteq L^2(\hat{H}, w_\varphi)$.

Proof. The proof is similar to the Remark 2.7. Q.E.D.

Here, we recall the following theorem from [10].

Theorem 2.11. Let K be a locally compact commutative strong hypergroup with a Haar measure m and associated Plancherel measure π, $\varepsilon > 0$, and $A := \{\sum_{j=1}^n \lambda_j \langle x_j, \cdot\rangle : n \in \mathbb{N}, \lambda_j \in \mathbb{C}, x_j \in K\ (j = 1, \ldots, n)\}$, where for each $x \in K$ and $\xi \in \hat{K}$, $\langle x, \xi\rangle := \xi(x)$. If $k_1 \in L^2(\hat{K}, \pi)$ has a null zeros set with respect to Plancherel measure π, then for each $k_2 \in L^2(\hat{K}, \pi)$ there exists an element $\psi \in A$ such that $\|k_2 - \psi k_1\|_2 < \varepsilon$.

Corollary 2.12. Let $\varphi \in L^2(K)$, and $\hat{\varphi} \neq 0$ a.e. on $\hat{K}$. $f \in V_\varphi$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$ for some $r \in L^2(\hat{H}, w_\varphi)$.

Proof. The proof follows from Theorem 2.11 and Lemma 2.8. Q.E.D.

Example 2.13. Let $\mathbb{Z}_+^*$ be the one-point compactification of $\mathbb{Z}_+ = \{0, 1, 2, ...\}$, and p be a prime number. A convolution, with ∞ as the identity, is defined on $M(\mathbb{Z}_+^*)$ by:

$$\delta_m * \delta_n := \begin{cases} \delta_{min(n,m)} & n \neq m \\ \frac{p-2}{p-1}\delta_n + \sum_{k=1}^\infty \frac{1}{p^k}\delta_{k+n} & n = m, \end{cases}$$

where $m, n \in \mathbb{Z}_+$. Then, $\mathbb{Z}_+^*$ is a Hermitian hypergroup. This compact countable hyprgroup was introduced by Dunkl and Ramirez [3]. A Haar measure on $\mathbb{Z}_+^*$ is given by

$$m(\{k\}) := \begin{cases} (\frac{1}{p})^k(1-\frac{1}{P}) & k = 0, 1, 2, ... \\ 0 & k = \infty. \end{cases}$$

Also, we have $\widehat{\mathbb{Z}}_+^* = \{\chi_n : n = 0, 1, 2, ...\}$, where

$$\chi_n(m) := \begin{cases} 1 & m \geq n \ \ or \ \ m = \infty \\ \frac{-1}{p-1} & m = n-1 \\ 0 & m \leq n-2. \end{cases}$$

Actually, $\widehat{\mathbb{Z}}_+^* \cong \mathbb{Z}_+$ is again a Hermitian hypergroup, with $\chi_0 = 1$ as the identity, and the convolution defined by

$$\delta_{\chi_n} * \delta_{\chi_m} := \begin{cases} \delta_{\chi_{max(n,m)}} & n \neq m, \ \ n, m \in \mathbb{Z}_+ \\ \frac{1}{p^{n-1}(p-1)}\delta_0 + \sum_{k=1}^{n-1} p^{k-n}\delta_k + \frac{p-2}{p-1}\delta_n & n = m, \ \ n \in \mathbb{Z}_+. \end{cases}$$

The Plancherel measure on $\widehat{\mathbb{Z}}_+^*$ is given by

$$\pi(\{\chi_k\}) := \begin{cases} \frac{1-\frac{1}{p}}{(\frac{1}{p})^k} & k = 1, 2, ... \\ 1 & k = 0. \end{cases}$$

For more details see [3]. We have $(\widehat{\mathbb{Z}}_+^*)^\perp = \{\chi_0\}$.

Let $K = H = \mathbb{Z}_+^*$ and $\varphi \in L^2(\mathbb{Z}_+^*)$. Then,

$$w_\varphi(\xi) = \int_{\{\chi_0\}} |\hat{\varphi}|^2(\xi * \eta) d\eta = |\hat{\varphi}|^2(\xi)\pi(\{\chi_0\}) = |\hat{\varphi}|^2(\xi), \qquad (\xi \in \widehat{\mathbb{Z}}_+^*).$$

Since $\widehat{\mathbb{Z}}_+^*$ is discrete, $\hat{\varphi} \neq 0$ a.e. on $\widehat{\mathbb{Z}}_+^*$ if and only if $\hat{\varphi}(\xi) \neq 0$ for all $\xi \in \widehat{\mathbb{Z}}_+^*$. So, if for all $n \in \mathbb{Z}_+$,

$$\begin{aligned} \hat{\varphi}(\chi_n) &= \sum_{k=0}^{[\infty]} \varphi(k)\overline{\chi_n(k)} m(\{k\}) \\ &= \frac{-\varphi(n-1)}{p-1}(\frac{1}{p})^{n-1}(1-\frac{1}{p}) + \sum_{k=1}^{\infty} \varphi(k)(\frac{1}{p})^k(1-\frac{1}{p}) \neq 0, \end{aligned}$$

then, by Corollary 2.12, $f \in V_\varphi$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$ for some $r \in L^2(\mathbb{Z}_+, |\hat{\varphi}|^2)$.

Definition 2.14. The mapping $\tau : \mathcal{M}(K) \to B(L^2(K))$ defined by $\tau(\mu)(f) := \mu * f$, where $f \in L^2(K)$ and $\mu \in \mathcal{M}(K)$, is a representation of the hypergroup K called *left regular representation.*

Proposition 2.15. Let $\varphi \in L^2(K)$. The set $\{\tau_x\varphi : x \in K\}$ is an orthogonal system in $L^2(K)$ if $|\hat{\varphi}| = 1$ a.e. on $\hat{K}$.

Proof. For each $x, y \in K$ we have

$$\begin{aligned}\langle \tau_x\varphi, \tau_y\varphi\rangle &= \langle \widehat{\tau_x\varphi}, \widehat{\tau_y\varphi}\rangle \qquad \text{(by [1, 2.2.20])}\\ &= \int_{\hat{K}} \widehat{\tau_x\varphi}(\xi)\overline{\widehat{\tau_y\varphi}(\xi)}d\xi\\ &= \int_{\hat{K}} \hat{\varphi}(\xi)\overline{\xi(x)\hat{\varphi}(\xi)}\xi(y)d\xi\\ &= \int_{\hat{K}} |\hat{\varphi}(\xi)|^2\xi(x^- * y)d\xi.\end{aligned}$$

Now, let $|\hat{\varphi}| = 1$ a.e. on $\hat{K}$. Then, by above relation and orthogonality of the elements of $\hat{\hat{K}} \equiv K$ [13], $\{\tau_x\varphi : x \in K\}$ is an orthogonal system. Q.E.D.

Definition 2.16. Let K be a hypergroup with a (left) Haar measure m, H be a subhypergroup of K, $\pi : \mathcal{M}(K) \to B(\mathcal{H}_\pi)$ be a representation of K on a Hilbert space $\mathcal{H}_\pi$, and $V \subseteq \mathcal{H}_\pi$. A vector $h_0 \in \mathcal{H}_\pi$ is called a (π, V)*–admissible vector* with respect to H if there are constant numbers $A, B > 0$ such that for every $h \in V$,

$$A\|h\|^2 \le \int_H |\langle \pi_x(h_0), h\rangle|^2 dm_H(x) \le B\|h\|^2,$$

where m_H is a left Haar measure on H and $\pi_x := \pi(\delta_x)$.

If $A = B = 1$, h_0 is called *Parseval* admissible.

Definition 2.17. Let K be a hypergroup. The *center* of K is defined by

$$Z(K) := \{x \in K : \delta_x * \delta_{x^-} = \delta_e = \delta_{x^-} * \delta_x\}.$$

In fact, $Z(K)$ is the maximal subgroup of K, and was defined in [2, 1.6] by Dunkl and named maximal subgroup in an equivalent definition in [5, 10.4] by Jewett (see [5, 10.4B] and [7]).

Remark 2.18. In particular, if $H := \{x_k\}_{k=1}^n$ is a subhypergroup of K and $H \subseteq Z(K)$, then $h_0 \in \mathcal{H}_\pi$ is a π–admissible vector with respect to H if and only if there exist constant numbers $A, B > 0$ such that for every $h \in \mathcal{H}_\pi$,

$$A\|h\|^2 \le \sum_{k=1}^n |\langle \pi_{x_k}(h_0), h\rangle|^2 \le B\|h\|^2,$$

since in this case m_H is the counting measure.

Theorem 2.19. A function $\varphi \in L^2(K)$ is a Parseval (τ, V_φ)–admissible vector if and only if $\hat{\varphi} = \chi_{\Omega_\varphi}$ a.e. on $\hat{K}$, where $\Omega_\varphi := \mathrm{supp}(\hat{\varphi})$.

Proof. Let $\varphi \in L^2(K)$ and $f \in V_\varphi$. By Corollary 2.12, there is $r \in L^2(\hat{K}, w_\varphi)$ such that $\hat{f} = r\hat{\varphi}$. Then, for each $x \in K$ we have

$$\begin{aligned}
\langle \tau_x\varphi, f\rangle = \langle \widehat{\tau_x\varphi}, \hat{f}\rangle &= \int_{\hat{K}} \widehat{\tau_x\varphi}(\xi)\overline{\hat{f}(\xi)}\, d\xi \\
&= \int_{\hat{K}} |\hat{\varphi}(\xi)|^2\overline{\xi(x)r(\xi)}\, d\xi \\
&= \int_{\hat{K}} F(\xi)\overline{\xi(x)}\, d\xi \\
&= \hat{F}(x),
\end{aligned}$$

where $F(\xi) := |\hat{\varphi}(\xi)|^2\,\overline{r(\xi)}$. So,

$$\int_K |\langle \tau_x\varphi, f\rangle|^2\, dm(x) = \int_K |\hat{F}(x)|^2\, dm(x) = \int_{\hat{K}} |F(\xi)|^2\, d\xi = \int_{\hat{K}} |r(\xi)|^2|\hat{\varphi}(\xi)|^4\, d\xi.$$

Also, we have

$$\|f\|_2^2 = \|\hat{f}\|_2^2 = \int_{\hat{K}} |\hat{f}(\xi)|^2\, d\xi = \int_{\hat{K}} |r(\xi)|^2|\hat{\varphi}(\xi)|^2\, d\xi.$$

Then, by Definition 2.16, φ is a Parseval (τ, V_φ)–admissible vector if and only if

$$\int_{\hat{K}} |r(\xi)|^2|\hat{\varphi}(\xi)|^2\left(|\hat{\varphi}(\xi)|^2 - 1\right)\, d\xi = 0,$$

and this holds if and only if $\hat{\varphi} = \chi_{\Omega_\varphi}$ a.e. on $\hat{K}$. For this, one can put $r = \chi_L$ where $L := \{\xi \in \Omega_\varphi : |\hat{\varphi}(\xi)|^2 > 1\}$ or $L := \{\xi \in \Omega_\varphi : |\hat{\varphi}(\xi)|^2 < 1\}$. Q.E.D.

References

[1] W. R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, De Gruyter, Berlin, 1995.

[2] C. F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. 179 (1973), 331–348.

[3] C. F. Dunkl and D. E. Ramirez, A family of countably compact P_*-hypergroups, *Trans. Amer. Math. Soc.*, 202, 339–356 (1975).

[4] P. Hermann, *Induced representations and hypergroup homomorphisms*, Monatsh. Math. 116 (1993), 245–262.

[5] R. I. Jewett, *Spaces with an abstract convolution of measures*, Adv. Math. 18 (1975), 1–101.

[6] R. A. Kamyabi Gol and R. Raisi Tousi, *The structure of shift inveriant spaces on a locally compact abelian group*, J. Math. Anal. Appl. 340 (2008), 219–225.

[7] K. A. Ross, *Centers of hypergroups*, Trans. Amer. Math. Soc. 243 (1978), 251–269.

[8] R. Spector, *Apercu de la theorie des hypergroups*, In: Analyse Harmonique sur les Groups de Lie, 643–673, Lec. Notes Math. Ser., 497, Springer, 1975.

[9] R. Spector, *Measures invariantes sur les hypergroups*, Trans. Amer. Math. Soc. 239 (1978), 147–165.

[10] S. M. Tabatabaie, *The problem of density on $L^2(G)$*, Acta Math. Hungar. 150 (2016), 339–345.

[11] S. M. Tabatabaie and B. H. Sadathoseyni, *Coorbit Spaces Related to Locally Compact Hypergroups*, Acta Math. Hungar. 153 (2017), 177-196.

[12] M. Voit, *Properties of subhypergroups*, Semigroup Forum, 56 (1997), 373–391.

[13] R. C. Vrem, *Lacunarity on compact hypergroups*, Math. Z., 164 (1978), 93–104.