Hermite-Hadamard type inequalities for generalized (s, m, φ)-preinvex functions via k-fractional integrals

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Abstract

In the present paper, the notion of generalized (s, m, φ)-preinvex function is applied to establish some new Hermite-Hadamard type inequalities via k-fractional Riemann-Liouville integrals. At the end, some applications to special means are given. These results not only extend the results appeared in the literature, but also provide new estimates on these types.

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1 Introduction

The following notation are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The following definition will be used in the sequel.

Definition 1.2. The hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

for $c > b > 0$ and $|z| < 1$, where $\beta(x, y)$ is the Euler beta function for all $x, y > 0$.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained ([3],[4]). For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions ([10],[11]).

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Fractional calculus ([11]), was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.3. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes ([11],[14]).

Definition 1.4. ([5]) If $k > 0$, then k-Gamma function Γ_k is defined as

$$\Gamma_k(\alpha) = \lim_{n \longrightarrow \infty} \frac{n! k^n (nk)^{\frac{\alpha}{k}} - 1}{(\alpha)_{n,k}}.$$

If $\mathrm{Re}(\alpha) > 0$ then k-Gamma function in integral form is defined as

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt,$$

with the property that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

Definition 1.5. ([12]) Let $f \in L_1[a,b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad b > x.$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals.

Now, let us recall some definitions of various convex functions.

Definition 1.6. ([8]) A function $f : \mathbb{R}_{\circ} \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for all $x, y \in \mathbb{R}_{\circ}$, $\lambda \in [0,1]$ and $s \in (0,1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_\circ$ as usual. The s-convex functions in the second sense have been investigated in ([8]).

Definition 1.7. ([1]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true ([1],[15]).

Definition 1.8. ([13]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f\left(x + t\eta(y, x)\right) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The aim of this paper is to establish some generalizations of Hermite-Hadamard type inequalities using new identity given in Section 2 for generalized (s, m, φ)-preinvex functions via k-fractional Riemann-Liouville integrals. In Section 3, some applications to special means are given. These results not only extend the results appeared in the literature ([7]), but also provide new estimates on these types.

2 Main results

Definition 2.1. ([6]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

Definition 2.3. ([9]) Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ and $\varphi : I \longrightarrow K$ a continuous function. For $f : K \longrightarrow \mathbb{R}$ and any fixed $s, m \in (0, 1]$, if

$$f\left(m\varphi(x) + \lambda\eta(\varphi(y), \varphi(x), m)\right) \leq m(1 - \lambda)^s f(\varphi(x)) + \lambda^s f(\varphi(y)) \tag{2.1}$$

is valid for all $x, y \in I, \lambda \in [0, 1]$, then we say that $f(x)$ is a generalized (s, m, φ)-preinvex function with respect to η.

Throughout this paper we denote

$$\begin{gathered} A_{\alpha,k}(\eta, \varphi, r, m, a, b) \\ = \left[f\left(m\varphi(b) + \frac{\eta(\varphi(a), \varphi(b), m)}{r + 1} \right) - f\left(m\varphi(a) + \frac{\eta(\varphi(b), \varphi(a), m)}{r + 1} \right) \right] \\ -(r + 1)^{\frac{\alpha}{k}} \Gamma_k(\alpha + k) \end{gathered}$$

$$\times\left[\frac{I^{\alpha,k}_{\left(m\varphi(b)+\frac{\eta(\varphi(a),\varphi(b),m)}{r+1}\right)^{-}}f(m\varphi(b))}{\eta^{\frac{\alpha}{k}}(\varphi(a),\varphi(b),m)}-\frac{I^{\alpha,k}_{\left(m\varphi(a)+\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\right)^{-}}f(m\varphi(a))}{\eta^{\frac{\alpha}{k}}(\varphi(b),\varphi(a),m)}\right].$$

In this section, in order to prove our main results regarding some Hermite-Hadamard type inequalities for generalized (s,m,φ)-preinvex function via k-fractional Riemann-Liouville integrals, we need the following new interesting lemma:

Lemma 2.4. Let $\varphi : I \longrightarrow K$ be a continuous function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $m \in (0,1]$, $r \in [0,1]$ and let $\eta(\varphi(y),\varphi(x),m) > 0$, $\forall\, x,y \in I$. Assume that $f : K \longrightarrow \mathbb{R}$ is a differentiable function on K° and $f' \in L_1[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$. Then for $\alpha,k>0$, the following identity for k-fractional integrals holds:

$$\begin{aligned}&A_{\alpha,k}(\eta,\varphi,r,m,a,b)\\ &=\frac{\eta(\varphi(a),\varphi(b),m)}{r+1}\int_0^1 t^{\frac{\alpha}{k}}f'\left(m\varphi(b)+\frac{t}{r+1}\eta(\varphi(a),\varphi(b),m)\right)dt\\ &-\frac{\eta(\varphi(b),\varphi(a),m)}{r+1}\int_0^1 t^{\frac{\alpha}{k}}f'\left(m\varphi(a)+\frac{t}{r+1}\eta(\varphi(b),\varphi(a),m)\right)dt. \qquad (2.2)\end{aligned}$$

Proof. A simple proof of the equality (2.2) can be done by performing an integration by parts in the integrals from the right side and changing the variables. The details are left to the interested reader. Q.E.D.

Remark 2.5. If we choose $m = r = 1$, $\varphi(x) = x$, $\forall x \in I$, $\eta(a,b,m) = a - mb$ and $\eta(b,a,m) = b - ma$, then we get ([7], Lemma 3.1).

Now we turn our attention to establish new inequalities of Hermite-Hadamard type for generalized (s,m,φ)-preinvex functions via k-fractional integrals. Using Lemma 2.4, the following results can be obtained for the corresponding version for power of the absolute value of the first derivative.

Theorem 2.6. Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s,m \in (0,1], r \in [0,1]$ and let $\eta(\varphi(y),\varphi(x),m) > 0$, $\forall\, x,y \in I$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s,m,φ)-preinvex function on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$ and $q>1$, $p^{-1}+q^{-1}=1$, then for any $\alpha,k>0$, the following inequality for k-fractional integrals holds:

$$\begin{aligned}|A_{\alpha,k}(\eta,\varphi,r,m,a,b)| &\le \frac{1}{(r+1)^{1+\frac{s}{q}}}\frac{1}{(s+1)^{\frac{1}{q}}}\left(\frac{k}{k+p\alpha}\right)^{\frac{1}{p}}\\ \times\Bigg\{&|\eta(\varphi(a),\varphi(b),m)|\Big[|f'(\varphi(a))|^q+m\left((r+1)^{s+1}-r^{s+1}\right)|f'(\varphi(b))|^q\Big]^{\frac{1}{q}}\\ +&|\eta(\varphi(b),\varphi(a),m)|\Big[m\left((r+1)^{s+1}-r^{s+1}\right)|f'(\varphi(a))|^q+|f'(\varphi(b))|^q\Big]^{\frac{1}{q}}\Bigg\}. \qquad (2.3)\end{aligned}$$

Proof. Suppose that $q > 1$. Using Lemma 2.4, the fact that $|f'|^q$ is a generalized (s, m, φ)-preinvex function, Hölder's inequality and property of the modulus, we have

$$|A_{\alpha,k}(\eta, \varphi, r, m, a, b)|$$

$$\leq \frac{|\eta(\varphi(a), \varphi(b), m)|}{r+1} \int_0^1 t^{\frac{\alpha}{k}} \left| f'\left(m\varphi(b) + \frac{t}{r+1}\eta(\varphi(a), \varphi(b), m)\right)\right| dt$$

$$+\frac{|\eta(\varphi(b), \varphi(a), m)|}{r+1} \int_0^1 t^{\frac{\alpha}{k}} \left| f'\left(m\varphi(a) + \frac{t}{r+1}\eta(\varphi(b), \varphi(a), m)\right)\right| dt$$

$$\leq \frac{|\eta(\varphi(a), \varphi(b), m)|}{r+1} \left(\int_0^1 t^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left| f'\left(m\varphi(b) + \frac{t}{r+1}\eta(\varphi(a), \varphi(b), m)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(b), \varphi(a), m)|}{r+1} \left(\int_0^1 t^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left| f'\left(m\varphi(a) + \frac{t}{r+1}\eta(\varphi(b), \varphi(a), m)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(a), \varphi(b), m)|}{r+1} \left(\int_0^1 t^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left(m\left(1 - \frac{t}{r+1}\right)^s |f'(\varphi(b))|^q + \left(\frac{t}{r+1}\right)^s |f'(\varphi(a))|^q\right) dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(b), \varphi(a), m)|}{r+1} \left(\int_0^1 t^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \left(m\left(1 - \frac{t}{r+1}\right)^s |f'(\varphi(a))|^q + \left(\frac{t}{r+1}\right)^s |f'(\varphi(b))|^q\right) dt\right)^{\frac{1}{q}}$$

$$= \frac{1}{(r+1)^{1+\frac{s}{q}}} \frac{1}{(s+1)^{\frac{1}{q}}} \left(\frac{k}{k+p\alpha}\right)^{\frac{1}{p}}$$

$$\times \left\{ |\eta(\varphi(a), \varphi(b), m)| \Big[|f'(\varphi(a))|^q + m\left((r+1)^{s+1} - r^{s+1}\right) |f'(\varphi(b))|^q \Big]^{\frac{1}{q}} \right.$$

$$\left. + |\eta(\varphi(b), \varphi(a), m)| \Big[m\left((r+1)^{s+1} - r^{s+1}\right) |f'(\varphi(a))|^q + |f'(\varphi(b))|^q \Big]^{\frac{1}{q}} \right\}.$$

The proof of Theorem 2.6 is completed. Q.E.D.

Corollary 2.7. Under the same conditions as in Theorem 2.6, if we choose $m = s = r = 1$, $\varphi(x) = x$, $\forall x \in I$, $\eta(a, b, m) = a - mb$ and $\eta(b, a, m) = b - ma$, then we get ([7], Theorem 3.2).

Theorem 2.8. Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1], r \in [0, 1]$ and let $\eta(\varphi(y), \varphi(x), m) > 0$, $\forall\, x, y \in I$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s, m, φ)-preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$ and $q \geq 1$, then for any $\alpha, k > 0$, the following inequality for k-fractional integrals holds:

$$|A_{\alpha,k}(\eta, \varphi, r, m, a, b)| \leq \frac{1}{(r+1)^{1+\frac{s}{q}}} \left(\frac{k}{k+\alpha}\right)^{1-\frac{1}{q}} \times \Bigg\{ |\eta(\varphi(a), \varphi(b), m)|$$

$$\times \left[\frac{k|f'(\varphi(a))|^q}{k(s+1)+\alpha} + \frac{mk(r+1)^s}{k+\alpha} \cdot {}_2F_1\left(-s, \frac{\alpha}{k}+1; \frac{\alpha}{k}+2; \frac{1}{r+1}\right) |f'(\varphi(b))|^q\right]^{\frac{1}{q}}$$

$$+|\eta(\varphi(b), \varphi(a), m)|$$

$$\times \left[\frac{mk(r+1)^s}{k+\alpha} \cdot {}_2F_1\left(-s, \frac{\alpha}{k}+1; \frac{\alpha}{k}+2; \frac{1}{r+1}\right) |f'(\varphi(a))|^q + \frac{k|f'(\varphi(b))|^q}{k(s+1)+\alpha}\right]^{\frac{1}{q}} \Bigg\}. \tag{2.4}$$

Proof. Suppose that $q \geq 1$. Using Lemma 2.4, the fact that $|f'|^q$ is a generalized (s, m, φ)-preinvex function, the well-known power mean inequality and property of the modulus, we have

$$|A_{\alpha,k}(\eta, \varphi, r, m, a, b)|$$

$$\leq \frac{|\eta(\varphi(a), \varphi(b), m)|}{r+1} \int_0^1 t^{\frac{\alpha}{k}} \left| f'\left(m\varphi(b) + \frac{t}{r+1}\eta(\varphi(a), \varphi(b), m)\right)\right| dt$$

$$+\frac{|\eta(\varphi(b), \varphi(a), m)|}{r+1} \int_0^1 t^{\frac{\alpha}{k}} \left| f'\left(m\varphi(a) + \frac{t}{r+1}\eta(\varphi(b), \varphi(a), m)\right)\right| dt$$

$$\leq \frac{|\eta(\varphi(a), \varphi(b), m)|}{r+1} \left(\int_0^1 t^{\frac{\alpha}{k}} dt\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 t^{\frac{\alpha}{k}} \left| f'\left(m\varphi(b) + \frac{t}{r+1}\eta(\varphi(a), \varphi(b), m)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(b), \varphi(a), m)|}{r+1} \left(\int_0^1 t^{\frac{\alpha}{k}} dt\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 t^{\frac{\alpha}{k}} \left| f'\left(m\varphi(a) + \frac{t}{r+1}\eta(\varphi(b), \varphi(a), m)\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(a), \varphi(b), m)|}{r+1} \left(\int_0^1 t^{\frac{\alpha}{k}} dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 t^{\frac{\alpha}{k}}\left(m\left(1-\frac{t}{r+1}\right)^s|f'(\varphi(b))|^q+\left(\frac{t}{r+1}\right)^s|f'(\varphi(a))|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(b),\varphi(a),m)|}{r+1}\left(\int_0^1 t^{\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 t^{\frac{\alpha}{k}}\left(m\left(1-\frac{t}{r+1}\right)^s|f'(\varphi(a))|^q+\left(\frac{t}{r+1}\right)^s|f'(\varphi(b))|^q\right)dt\right)^{\frac{1}{q}}$$

$$=\frac{1}{(r+1)^{1+\frac{s}{q}}}\left(\frac{k}{k+\alpha}\right)^{1-\frac{1}{q}}\times\Bigg\{|\eta(\varphi(a),\varphi(b),m)|$$

$$\times\left[\frac{k|f'(\varphi(a))|^q}{k(s+1)+\alpha}+\frac{mk(r+1)^s}{k+\alpha}\cdot{}_2F_1\left(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{r+1}\right)|f'(\varphi(b))|^q\right]^{\frac{1}{q}}$$

$$+|\eta(\varphi(b),\varphi(a),m)|$$

$$\times\left[\frac{mk(r+1)^s}{k+\alpha}\cdot{}_2F_1\left(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{r+1}\right)|f'(\varphi(a))|^q+\frac{k|f'(\varphi(b))|^q}{k(s+1)+\alpha}\right]^{\frac{1}{q}}\Bigg\}.$$

The proof of Theorem 2.8 is completed. Q.E.D.

Corollary 2.9. Under the same conditions as in Theorem 2.8, if we choose $m=s=r=1$, $\varphi(x)=x$, $\forall x\in I$, $\eta(a,b,m)=a-mb$ and $\eta(b,a,m)=b-ma$, then we get ([7], Theorem 3.1).

Remark 2.10. For $M>0$ and $q\geq 1$, if $|f'|^q\leq M$, then by our Theorems 2.6 and 2.8, we can get some special kinds of Hermite-Hadamard type inequalities via k-fractional integrals. For $k=1$, we obtain special kinds of Hermite-Hadamard type inequalities via fractional integrals. Also, for different choices of values r and continuous functions φ, for example: $r=0,\frac{1}{2},\frac{1}{3}$, and $\varphi(x)=e^x$, $\ln(x+1)$, x^β, $\forall x>0$ and $\forall\beta>1$, by our Theorems 2.6 and 2.8, we can get some interesting integral inequalities of these types.

3 Applications to special means

Definition 3.1. ([2]) A function $M:\mathbb{R}_+^2\longrightarrow\mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
2. Symmetry: $M(x,y)=M(y,x)$,
3. Reflexivity: $M(x,x)=x$,
4. Monotonicity: If $x\leq x'$ and $y\leq y'$, then $M(x,y)\leq M(x',y')$,
5. Internality: $\min\{x,y\}\leq M(x,y)\leq\max\{x,y\}$.

We consider some means for arbitrary positive real numbers α, β ($\alpha \neq \beta$).

1. The arithmetic mean:
$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$

2. The geometric mean:
$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$

3. The harmonic mean:
$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

4. The power mean:
$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:
$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:
$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln(\beta) - \ln(\alpha)}.$$

7. The generalized log-mean:
$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

8. The weighted p-power mean:
$$M_p\left(\begin{array}{cccc} \alpha_1, & \alpha_2, & \cdots & , \alpha_n \\ u_1, & u_2, & \cdots & , u_n \end{array}\right) = \left(\sum_{i=1}^{n} \alpha_i u_i^p\right)^{\frac{1}{p}}$$

   where $0 \leq \alpha_i \leq 1$, $u_i > 0\,(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $\overline{M} := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \longrightarrow A$ be a continuous function, therefore one can obtain various inequalities using the results of Section 2 for these means. Replace $\eta(\varphi(y), \varphi(x), m)$ with $\eta(\varphi(y), \varphi(x))$ and setting $\eta(\varphi(a), \varphi(b)) = \eta(\varphi(b), \varphi(a)) = \overline{M}$ for value $m = 1$ in (2.3) and (2.4), one can obtain the following interesting inequalities involving means:

$$\left|A_{\alpha,k}(M(\cdot, \cdot), \varphi, r, 1, a, b)\right|$$

$$= \Bigg| \left[f\left(\varphi(b) + \frac{\overline{M}}{r+1}\right) - f\left(\varphi(a) + \frac{\overline{M}}{r+1}\right)\right]$$

$$- \frac{(r+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)}{\overline{M}^{\frac{\alpha}{k}}}$$

$$\times \left[I^{\alpha,k}_{\left(\varphi(b)+\frac{\overline{M}}{r+1}\right)^-} f(\varphi(b)) - I^{\alpha,k}_{\left(\varphi(a)+\frac{\overline{M}}{r+1}\right)^-} f(\varphi(a))\right] \Bigg|$$

$$\leq \frac{\overline{M}}{(r+1)^{1+\frac{s}{q}}} \frac{1}{(s+1)^{\frac{1}{q}}} \left(\frac{k}{k+p\alpha}\right)^{\frac{1}{p}}$$

$$\times \Bigg\{ \left[|f'(\varphi(a))|^q + \left((r+1)^{s+1} - r^{s+1}\right) |f'(\varphi(b))|^q \right]^{\frac{1}{q}}$$

$$+ \left[\left((r+1)^{s+1} - r^{s+1}\right) |f'(\varphi(a))|^q + |f'(\varphi(b))|^q \right]^{\frac{1}{q}} \Bigg\}, \tag{3.1}$$

$$\left| A_{\alpha,k}(M(\cdot,\cdot), \varphi, r, 1, a, b)\right| \leq \frac{\overline{M}}{(r+1)^{1+\frac{s}{q}}} \left(\frac{k}{k+\alpha}\right)^{1-\frac{1}{q}}$$

$$\times \Bigg\{ \left[\frac{k|f'(\varphi(a))|^q}{k(s+1)+\alpha} + \frac{k(r+1)^s}{k+\alpha} \cdot {}_2F_1\left(-s, \frac{\alpha}{k}+1; \frac{\alpha}{k}+2; \frac{1}{r+1}\right) |f'(\varphi(b))|^q \right]^{\frac{1}{q}}$$

$$+ \left[\frac{k(r+1)^s}{k+\alpha} \cdot {}_2F_1\left(-s, \frac{\alpha}{k}+1; \frac{\alpha}{k}+2; \frac{1}{r+1}\right) |f'(\varphi(a))|^q + \frac{k|f'(\varphi(b))|^q}{k(s+1)+\alpha} \right]^{\frac{1}{q}} \Bigg\}. \tag{3.2}$$

Letting $\overline{M} = A, G, H, P_r, I, L, L_p, M_p$ in (3.1) and (3.2), we get inequalities involving means for a particular choice of a differentiable generalized $(s, 1, \varphi)$-preinvex functions f. The details are left to the interested reader.

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