

Weighted composition operators from Nevanlinna type spaces to weighted Bloch type spaces

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Abstract

In this paper, we characterize metrically compact weighted composition operators from Nevanlinna type spaces to weighted Bloch type spaces.

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1 Introduction

Let \mathbb{D} be the open unit disk in the complex plane \mathbb{C} , $H(\mathbb{D})$ the space of all analytic functions on \mathbb{D} , \mathbb{T} the boundary of \mathbb{D} and $d\sigma$ denote the normalized Lebesgue measure on \mathbb{T} . The Nevanlinna class N is defined as the set of all holomorphic functions f on \mathbb{D} such that

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}} \log(1 + |f(r\zeta)|) d\sigma(\zeta) < \infty.$$

It is well-known fact that every holomorphic function f in the class N has a finite non-tangential limit, denoted by f^* , almost everywhere on \mathbb{T} .

For each $1 < p < \infty$, the *Nevanlinna type space* N^p is a subspace of the Nevanlinna class N defined as

$$N^p = \left\{ f \in H(\mathbb{D}) : \|f\|_{N^p} = \sup_{0 \leq r < 1} \left(\int_{\mathbb{T}} (\log(1 + |f(r\zeta)|))^p d\sigma(\zeta) \right)^{1/p} < \infty \right\}.$$

N^p is an F -space with respect to the translation-invariant metric $d_{N^p}(f, g) = \|f - g\|_{N^p}$. Moreover, the subharmonicity of $(\log(1 + |f|))^p$ implies that $f \in N^p$ has the following growth estimation:

$$\log(1 + |f(z)|) \leq \frac{4^{1/p} \|f\|_{N^p}}{(1 - |z|^2)^{1/p}} \quad (1.1)$$

for $z \in \mathbb{D}$. Thus the convergence in N^p gives the uniform convergence on compact subsets of \mathbb{D} . For more informations on Nevanlinna type spaces, see [1, 2, 3, 5, 6, 14].

Let ν be a continuous (*weight*) on \mathbb{D} such that $\nu(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ and ν is normal, that is, there exist positive numbers η and τ , $0 < \eta < \tau$, and $\delta \in [0, 1)$ such that

$$\frac{\nu(r)}{(1-r)^\eta} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\nu(r)}{(1-r)^\eta} = 0;$$

$$\frac{\nu(r)}{(1-r)^\tau} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\nu(r)}{(1-r)^\tau} = \infty.$$

For such a weight ν , the *weighted Bloch-type space* \mathcal{B}_ν on \mathbb{D} is the space of all holomorphic functions f on \mathbb{D} such that

$$\sup_{z \in \mathbb{D}} \nu(z) |f'(z)| < \infty.$$

The *little weighted Bloch-type space* $\mathcal{B}_{\nu,0}$ consists of all $f \in \mathcal{B}_\nu$ such that

$$\lim_{|z| \rightarrow 1} \nu(z) |f'(z)| = 0.$$

Both spaces \mathcal{B}_ν and $\mathcal{B}_{\nu,0}$ are Banach spaces with the norm

$$\|f\|_{\mathcal{B}_\nu} = |f(0)| + \sup_{z \in \mathbb{D}} \nu(z) |f'(z)|,$$

and $\mathcal{B}_{\nu,0}$ is a closed subspace of \mathcal{B}_ν . The compactness of a closed subset $L \subset \mathcal{B}_{\nu,0}$ can be characterized as follows.

Lemma 1.1. A closed set L in $\mathcal{B}_{\nu,0}$ is compact if and only if it is bounded with respect to the norm $\|\cdot\|_{\mathcal{B}_\nu}$ and satisfies

$$\lim_{|z| \rightarrow 1} \sup_{f \in L} \nu(z) |f'(z)| = 0.$$

This result for the case $\nu(z) = (1 - |z|^2)$ was proved by Madigan and Matheson [7]. By a modification of their proof, we can prove the above lemma. Lemma 1.1 is a very useful tool in the study of the compactness of linear operators, when the range space is $\mathcal{B}_{\nu,0}$ (see [7, 8, 9, 10], *etc*).

Let $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of \mathbb{D} , the *weighted composition operator* $W_{\psi,\varphi}$ is a linear operator on $H(\mathbb{D})$ defined by $W_{\psi,\varphi} f = \psi \cdot f \circ \varphi$ for $f \in H(\mathbb{D})$. It is of interest to provide function-theoretic characterizations involving ψ and φ of boundedness and compactness of $W_{\psi,\varphi}$ acting between different function spaces. Given two linear topological vector spaces X and Y , a linear operator $T : X \rightarrow Y$ is *metrically bounded* if there exists a constant $C > 0$ such that $d_Y(Tf, 0) \leq C d_X(f, 0)$ for $f \in X$. In general, the boundedness of T and the metrical boundedness of T do not coincide. However, if X and Y are Banach spaces, then the metrical boundedness of T coincides with the boundedness of T . For more informations on the metrical boundedness, we can refer to papers [2, 3]. Also recall that a linear operator $T : X \rightarrow Y$ is *bounded with respect to metric balls* if it takes every metric ball in X into a metric ball in Y . Since a metric ball is also a bounded set, so boundedness with respect to metric balls also coincides with metrical boundedness, if X and Y are Banach spaces. For the compactness of linear operators, for example, composition operators or weighted composition operators on these spaces, we use the metrical compactness. Namely, $T : X \rightarrow Y$ is *metrically compact* if it takes every metric ball in X into a relatively compact subset in Y . These operators on Nevanlinna type spaces have been studied by several authors (see [1, 2, 3, 8, 9, 10, 11, 12, 13]). Motivated by their work, in this paper, we will give characterizations for the boundedness with metric balls and the metrical compactness of $W_{\psi,\varphi}$ acting from Nevanlinna type spaces into weighted Bloch-type spaces.

2 Boundedness and compactness

We first give sufficient conditions for $W_{\psi,\varphi}$ to be bounded with respect to metric balls.

Proposition 2.1. If $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of \mathbb{D} such that

$$\sup_{z \in \mathbb{D}} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} < \infty \tag{2.1}$$

and

$$\sup_{z \in \mathbb{D}} \nu(z)|\psi'(z)| \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} < \infty \tag{2.2}$$

for every $c > 0$, then $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is bounded with respect to metric balls.

Proof. For $z \in \mathbb{D}$ and $\zeta \in \mathbb{T}$, we have

$$1 - |z + (1 - |z|)\zeta/2|^2 \geq 1 - (1 + |z|)^2/4 \geq (1 - |z|^2)/4.$$

Thus by Cauchy's integral formula and (1.1), we obtain that

$$(1 - |z|^2)|f'(z)| \leq \frac{2}{\pi} \int_{\mathbb{T}} |f(z + (1 - |z|)\zeta/2)| |d\zeta| \leq 4 \exp \left\{ \frac{4^{1+1/p} \|f\|_{N^p}}{(1 - |z|^2)^{1/p}} \right\} \tag{2.3}$$

for each $z \in \mathbb{D}$. Hence we have that

$$\begin{aligned} \|W_{\psi,\varphi} f\|_{\mathcal{B}_\nu} &= |\psi(0)f(\varphi(0))| + \sup_{z \in \mathbb{D}} \nu(z)|\psi'(z)f(\varphi(z)) + \psi(z)\varphi'(z)f'(\varphi(z))| \\ &\leq |\psi(0)| \exp \left\{ \frac{4^{1/p} \|f\|_{N^p}}{(1 - |\varphi(0)|^2)^{1/p}} \right\} + \sup_{z \in \mathbb{D}} \nu(z)|\psi'(z)| \exp \left\{ \frac{4^{1/p} \|f\|_{N^p}}{(1 - |\varphi(z)|^2)^{1/p}} \right\} \\ &\quad + 4 \sup_{z \in \mathbb{D}} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{4^{1+1/p} \|f\|_{N^p}}{(1 - |\varphi(z)|^2)^{1/p}} \right\}. \end{aligned}$$

Combining the above inequality and condition (2.1) and condition (2.2), we see that $W_{\psi,\varphi}$ takes every metric ball in N^p into a metric ball in \mathcal{B}_ν , namely $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is bounded with respect to metric balls. Q.E.D.

Proposition 2.2. If $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of \mathbb{D} satisfying the conditions (2.1) and (2.2) for every $c > 0$, then $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is bounded.

Proof. Suppose that conditions (2.1) and (2.2) hold, then proceeding as in the proof of Proposition 2.1, we see that $W_{\psi,\varphi}(N^p) \subset \mathcal{B}_\nu$. Since N^p is an F -space, hence by the closed graph theorem, we have that $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is also bounded. Q.E.D.

The main result of this paper, characterizes the metrical compactness of $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$. In fact, we prove that boundedness of $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ with respect to metric balls is equivalent to metrical compactness of $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$.

To prove the main result, we need the following lemma.

Lemma 2.3. Suppose that $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of \mathbb{D} such that $W_{\psi,\varphi}(N^p) \subset \mathcal{B}_\nu$. Then $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is metrically compact if and only if for any sequence $\{f_j\}$ in N^p with $\sup_j \|f_j\|_{N^p} \leq K$ and which converges to zero uniformly on compact subsets of \mathbb{D} , $\{W_{\psi,\varphi}f_j\}$ converges to zero in \mathcal{B}_ν .

Proof. This is an extension of a well-known result on the compactness of weighted composition operators on holomorphic function spaces. By (1.1), we see that any metrical bounded sequence in N^p form a normal family. Hence an argument by using the Montel theorem proves this lemma proceeding on the same lines as the proof of Proposition 3.11 in [4]. Q.E.D.

Theorem 2.4. Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of \mathbb{D} . Then the following conditions are equivalent;

- (i) $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is bounded with respect to metric balls,
- (ii) $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is metrically compact,
- (iii) $\psi \in \mathcal{B}_\nu$,

$$\sup_{z \in \mathbb{D}} \nu(z) |\psi(z)\varphi'(z)| < \infty, \quad (2.4)$$

$$\lim_{|\varphi(z)| \rightarrow 1} \nu(z) |\psi'(z)| \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} = 0 \quad (2.5)$$

and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{\nu(z) |\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} = 0 \quad (2.6)$$

for any $c > 0$.

Proof. Since \mathcal{B}_ν is a Banach space, so each bounded set is also in a metric ball in \mathcal{B}_ν . Hence the metrical compactness of $W_{\psi,\varphi}$ from N^p to \mathcal{B}_ν gives the boundedness with respect to metric balls. So implication (ii) \Rightarrow (i) holds.

Now we will prove (iii) \Rightarrow (ii). Assume that $\psi \in \mathcal{B}_\nu$ and that the conditions (2.4), (2.5) and (2.6) hold for every $c > 0$. Let $\varepsilon > 0$ be arbitrary. Then we can choose $r_0 \in (0, 1)$ such that

$$\sup_{|\varphi(z)| > r_0} \nu(z) |\psi'(z)| \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} < \varepsilon$$

and

$$\sup_{|\varphi(z)| > r_0} \frac{\nu(z) |\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} < \varepsilon$$

for any $c > 0$. Choose a sequence $\{f_j\}$ in N^p such that $\sup_j \|f_j\|_{N^p} \leq K$ and $\{f_j\}$ converges to zero uniformly on compact subsets of \mathbb{D} . Since $\psi \in \mathcal{B}_\nu$ and (2.5) imply (2.2), and (2.4) and (2.6) imply (2.1), so by Proposition 2.2, we see that $W_{\psi,\varphi}(N^p) \subset \mathcal{B}_\nu$. The assumptions $\psi \in \mathcal{B}_\nu$ and (2.4) also imply that

$$\sup_{|\varphi(z)| \leq r_0} \nu(z) |\psi'(z)| \|f_j(\varphi(z))\| \leq \sup_{z \in \mathbb{D}} \nu(z) |\psi'(z)| \cdot \max_{|w| \leq r_0} |f_j(w)| \rightarrow 0$$

and

$$\sup_{|\varphi(z)| \leq r_0} \nu(z) |\psi(z) \varphi'(z)| |f'_j(\varphi(z))| \leq \sup_{z \in \mathbb{D}} \nu(z) |\psi(z) \varphi'(z)| \cdot \max_{|w| \leq r_0} |f'_j(w)| \rightarrow 0$$

as $j \rightarrow \infty$. Therefore, we have

$$\sup_{|\varphi(z)| \leq r_0} \nu(z) |(W_{\psi, \varphi} f_j)'(z)| \rightarrow 0$$

as $j \rightarrow \infty$. On the other hand, it follows from (1.1) and (2.3) that

$$\begin{aligned} \sup_{|\varphi(z)| > r_0} \nu(z) |(W_{\psi, \varphi} f)'(z)| &\leq \sup_{|\varphi(z)| > r_0} \nu(z) |\psi'(z)| \exp \left\{ \frac{4^{1/p} K}{(1 - |\varphi(z)|^2)^{1/p}} \right\} \\ &+ 4 \sup_{|\varphi(z)| > r_0} \frac{\nu(z) |\psi(z) \varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{4^{1+1/p} K}{(1 - |\varphi(z)|^2)^{1/p}} \right\} < \varepsilon. \end{aligned}$$

Thus we see that

$$\limsup_{j \rightarrow \infty} \|W_{\psi, \varphi} f_j\|_{\mathcal{B}_\nu} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that $\|W_{\psi, \varphi} f_j\|_{\mathcal{B}_\nu} \rightarrow 0$ as $j \rightarrow \infty$. Hence Lemma 2.3 shows that $W_{\psi, \varphi} : N^p \rightarrow \mathcal{B}_\nu$ is metrically compact.

Finally we prove the implication (i) \Rightarrow (iii). By taking the constant function $f(z) = 1$ in N^p , we have that $\psi \in \mathcal{B}_\nu$. Again by taking $f(z) = z$ in N^p and using facts that $|\varphi(z)| < 1$ and $\psi \in \mathcal{B}_\nu$, we have that

$$\sup_{z \in \mathbb{D}} \nu(z) |\psi(z) \varphi'(z)| < \infty.$$

Fix $c > 0$ and put $w = \varphi(z)$. We define the following function

$$f_w(v) = \left\{ 3 \frac{1 - |w|^2}{(1 - \bar{w}v)^2} - 2 \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^3} \right\} \exp \left\{ c \left\{ 3 \frac{1 - |w|^2}{(1 - \bar{w}v)^2} - 2 \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^3} \right\}^{1/p} \right\}.$$

Then we can easily see that the family $\{f_w\}$ forms a metric ball in N^p . Also we have that

$$\begin{aligned} f'_w(v) &= \left\{ \frac{6c\bar{w}}{p} \left\{ 3 \frac{1 - |w|^2}{(1 - \bar{w}v)^2} - 2 \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^3} \right\}^{1/p} + 6\bar{w} \right\} \left\{ \frac{1 - |w|^2}{(1 - \bar{w}v)^3} - \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^4} \right\} \\ &\times \exp \left\{ c \left\{ 3 \frac{1 - |w|^2}{(1 - \bar{w}v)^2} - 2 \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^3} \right\}^{1/p} \right\}. \end{aligned}$$

Since $\{W_{\psi, \varphi} f_w\}$ is a metric ball in \mathcal{B}_ν , there is a positive constant C which is independent of $w = \varphi(z)$ such that $\|W_{\psi, \varphi} f_w\|_{\mathcal{B}_\nu} \leq C$. Thus by using the fact that $f'_w(w) = 0$, we obtain that

$$\begin{aligned} C &\geq \nu(z) |(W_{\psi, \varphi} f_w)'(z)| \\ &= \frac{\nu(z) |\psi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\}, \end{aligned}$$

and so

$$\nu(z)|\psi'(z)| \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} \leq C(1 - |\varphi(z)|^2).$$

Taking limit as $|\varphi(z)| \rightarrow 1$ on both sides of the above inequality, we get (2.5). Again fix $c > 0$, put $w = \varphi(z)$ and consider the following function

$$h_w(v) = \left\{ \frac{1 - |w|^2}{(1 - \bar{w}v)^2} - \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^3} \right\} \exp \left\{ c \left\{ 3 \frac{1 - |w|^2}{(1 - \bar{w}v)^2} - 2 \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^3} \right\}^{1/p} \right\}.$$

Then once again we can easily see that $\{h_w\}$ forms a metric ball in N^p and $h_w(w) = 0$. Also we have that

$$\begin{aligned} h'_w(v) &= \left[\bar{w} \left\{ 2 \frac{1 - |w|^2}{(1 - \bar{w}v)^3} - 3 \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^4} \right\} + \frac{6c}{p} \bar{w} \left\{ \frac{1 - |w|^2}{(1 - \bar{w}v)^2} - \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^3} \right\} \right] \\ &\times \left\{ 3 \frac{(1 - |w|^2)}{(1 - \bar{w}v)^2} - 2 \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^3} \right\}^{-1+1/p} \left\{ \frac{1 - |w|^2}{(1 - \bar{w}v)^3} - \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^4} \right\} \\ &\times \exp \left\{ c \left\{ 3 \frac{1 - |w|^2}{(1 - \bar{w}v)^2} - 2 \frac{(1 - |w|^2)^2}{(1 - \bar{w}v)^3} \right\}^{1/p} \right\}. \end{aligned}$$

Therefore, we have that

$$h'_w(w) = \frac{-\bar{w}}{(1 - |w|^2)^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\}.$$

Since $\{W_{\psi,\varphi}h_w\}$ is a metric ball in \mathcal{B}_ν , there is a positive constant C which is independent of $w = \varphi(z)$ such that $\|W_{\psi,\varphi}h_w\|_{\mathcal{B}_\nu} \leq C$. Thus we obtain that

$$\begin{aligned} C &\geq \nu(z)|(W_{\psi,\varphi}f_w)'(z)| \\ &= \frac{\nu(z)|\psi(z)\varphi'(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\}, \end{aligned}$$

and so

$$\frac{\nu(z)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} \leq \frac{Cp(1 - |\varphi(z)|^2)}{|\varphi(z)|}.$$

Taking limit as $|\varphi(z)| \rightarrow 1$ on both sides of the above inequality, we get (2.6). This completes the proof. Q.E.D.

Next we will investigate the compactness of operators $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_{\nu,0}$.

Proposition 2.5. If $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of \mathbb{D} such that

$$\lim_{|z| \rightarrow 1} \nu(z)|\psi'(z)| \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} = 0 \tag{2.7}$$

$$\lim_{|z| \rightarrow 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} = 0 \tag{2.8}$$

for all $c > 0$, then $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_{\nu,0}$ is bounded with respect to metric balls. Also we obtain that $W_{\psi,\varphi}(N^p) \subset \mathcal{B}_{\nu,0}$.

Proof. Since $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is bounded with respect to metric balls by Proposition 2.1, we only need to prove that $W_{\psi,\varphi}(L) \subset \mathcal{B}_{\nu,0}$ for any metric balls L in N^p . However, the inequalities (1.1) and (2.3) show that

$$\begin{aligned} \nu(z)|(W_{\psi,\varphi}f)'(z)| &\leq \nu(z)|\varphi'(z)| \exp \left\{ \frac{4^{1/p}\|f\|_{N^p}}{(1-|\varphi(z)|^2)^{1/p}} \right\} \\ &+ 4 \frac{\nu(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp \left\{ \frac{4^{1+1/p}\|f\|_{N^+}}{(1-|\varphi(z)|^2)^{1/p}} \right\} \end{aligned}$$

which holds for each $f \in L$. Thus the conditions (2.7) and (2.8) imply that $\nu(z)|(W_{\psi,\varphi}f)'(z)| \rightarrow 0$ as $|z| \rightarrow 1$. This completes the proof. Q.E.D.

Theorem 2.6. Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of \mathbb{D} . Then the following conditions are equivalent;

- (i) $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_{\nu,0}$ is bounded with respect to metric balls,
- (ii) $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is bounded with respect to metric balls, $\psi \in \mathcal{B}_{\nu,0}$ and $\lim_{|z| \rightarrow 1} \nu(z)|\psi(z)\varphi'(z)| = 0$,
- (iii) $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_{\nu,0}$ is metrically compact,
- (vii) ψ and φ satisfy the conditions (2.7) and (2.8).

Proof. The implication (iii) \Rightarrow (i) is obvious. Also we can easily see that (i) \Rightarrow (ii) are holds. In fact, we may consider the function $f(z) = 1$ in N^p . Then $\|f\|_{N^p} = \log 2$, and so f is in some metric balls in N^p . This shows that $\psi \in \mathcal{B}_{\nu,0}$. Again, consider the function $f(z) = z$ in N^p . Once again $\|f\|_{N^p} \leq \log 2$, and so f is in some metric balls in N^p . Thus using $\psi \in \mathcal{B}_{\nu,0}$, we can shows that $\lim_{|z| \rightarrow 1} \nu(z)|\psi(z)\varphi'(z)| = 0$.

To prove (ii) \Rightarrow (iv), we take a sequence $\{z_j\}$ in \mathbb{D} with $|z_j| \rightarrow 1$ as $j \rightarrow \infty$. Then

$$\begin{aligned} &\limsup_{|z| \rightarrow 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp \left\{ \frac{c}{(1-|\varphi(z)|^2)^{1/p}} \right\} \\ &= \lim_{j \rightarrow \infty} \frac{\nu(z_j)|\psi(z_j)\varphi'(z_j)|}{1-|\varphi(z_j)|^2} \exp \left\{ \frac{c}{(1-|\varphi(z_j)|^2)^{1/p}} \right\}. \end{aligned} \tag{2.9}$$

If $\sup_{j \geq 1} |\varphi(z_j)| < 1$, then the assumption $\psi \in \mathcal{B}_{\nu,0}$ implies that the right limit in the equation (2.9) is equal to 0, and so we obtain the condition (2.7). If $\sup_{j \geq 1} |\varphi(z_j)| = 1$, then we can choose a subsequence $\{z_{j_k}\} \subset \{z_j\}$ such that $|\varphi(z_{j_k})| \rightarrow 1$ as $k \rightarrow \infty$. Since $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_\nu$ is bounded with respect to metric balls, by Theorem 2.4, ψ and φ satisfy

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp \left\{ \frac{c}{(1-|\varphi(z)|^2)^{1/p}} \right\} = 0 \tag{2.10}$$

for any $c > 0$. By (2.9) and (2.10) we have that

$$\begin{aligned} & \limsup_{|z| \rightarrow 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} \\ &= \lim_{k \rightarrow \infty} \frac{\nu(z_{j_k})|\psi(z_{j_k})\varphi'(z_{j_k})|}{1 - |\varphi(z_{j_k})|^2} \exp \left\{ \frac{c}{(1 - |\varphi(z_{j_k})|^2)^{1/p}} \right\} \\ &\leq \limsup_{|\varphi(z)| \rightarrow 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{c}{(1 - |\varphi(z)|^2)^{1/p}} \right\} = 0. \end{aligned}$$

This implies that (2.7) holds. Similarly, we can show that (2.8) also holds.

Finally we will prove the implication (iv) \Rightarrow (iii). Take any metric ball L_{N^p} in N^p . Then there is a constant $K > 0$ such that $\|f\|_{N^p} \leq K$ for any $f \in L_{N^p}$. For any $f \in L_{N^p}$ and $z \in \mathbb{D}$ we have that

$$\begin{aligned} \nu(z)|(W_{\psi,\varphi}f)'(z)| &\leq \nu(z)|\psi'(z)| \exp \left\{ \frac{4^{1/p}K}{(1 - |\varphi(z)|^2)^{1/p}} \right\} \\ &+ 4 \frac{\nu(z)|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} \exp \left\{ \frac{4^{1+1/p}K}{1 - |\varphi(z)|^2} \right\}. \end{aligned}$$

Combining this with (2.7) and (2.8), we obtain

$$\lim_{|z| \rightarrow 1} \sup_{f \in L_{N^p}} \nu(z)|(W_{\psi,\varphi}f)'(z)| = 0,$$

and so Lemma 1.1 shows that $W_{\psi,\varphi}(L_{N^p})$ is compact in $\mathcal{B}_{\nu,0}$ for any metric ball L_{N^p} . This means that $W_{\psi,\varphi} : N^p \rightarrow \mathcal{B}_{\nu,0}$ is metrically compact. The proof is accomplished. Q.E.D.

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References

- [1] J.S. Choa and H.O. Kim, *Composition operators between the Nevanlinna-type spaces*, J. Math. Anal. Appl. **257** (2001), 378–402.
- [2] J.S. Choa, H.O. Kim and J.H. Shapiro, *Compact composition operators on the Smirnov class*, Proc. Amer. Math. Soc. **128** (2000), 2297–2308.
- [3] B.R. Choe, H. Koo and W. Smith, *Carleson measures for the area Nevanlinna spaces and applications*, J. Anal. Math. **104** (2008), 207–233.
- [4] C.C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, CRC press, Boca Raton, New York, 1995.
- [5] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
- [6] H.O. Kim, *On an F -algebra of holomorphic functions*, Can. J. Math. **40** (1988), 718–741.

- [7] K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc. **347** (1995), 2679–2687.
- [8] A.K. Sharma, *Products of multiplication, composition and differentiation operators between weighted Bergman-Nevanlinna and Bloch-type spaces*, Turkish J. Math. **35** (2011), 275–291.
- [9] A.K. Sharma, *Weighted composition operators from Cauchy integral transforms to logarithmic weighted-type spaces*. Ann. Funct. Anal. **4** (2013), 163–174.
- [10] A.K. Sharma and S. Ueki, *Composition operators from Nevanlinna type spaces to Bloch type spaces*, Banach J. Math. Anal., **6** (2012), 112–123.
- [11] S. Stević, *Weighted differentiation composition operators from H^∞ and Bloch spaces to n th weighted-type spaces on the unit disk*, Appl. Math. Comput. **216** (2010), 3634–3641.
- [12] S. Stević, *Weighted composition operators from Bergman-Privalov-type spaces to weighted-type spaces on the unit ball*, Appl. Math. Comput. **217** (2010), 1939–1943.
- [13] S. Stević and A.K. Sharma, *Composition operators from Bergman Privalov spaces to Zygmund spaces*, Ann. Polon. Math. **105** (2012), 77–86.
- [14] N. Yanagihara, *Bounded subsets of some spaces of holomorphic functions*, Sci. Papers College Gen. Ed. Univ. Tokyo **23** (1973), 19–28.