

A note on the k -tuple total domination number of a graph

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Abstract

For every positive integer k and every graph $G = (V, E)$ with minimum degree at least k , a vertex set S is a k -tuple total dominating set (resp. k -tuple dominating set) of G , if for every vertex $v \in V$, $|N_G(v) \cap S| \geq k$, (resp. $|N_G[v] \cap S| \geq k$). The k -tuple total domination number $\gamma_{\times k, t}(G)$ (resp. k -tuple domination number $\gamma_{\times k}(G)$) is the minimum cardinality of a k -tuple total dominating set (resp. k -tuple dominating set) of G .

In this paper, we first prove that if m is a positive integer, then for which graphs G , $\gamma_{\times k, t}(G) = m$ or $\gamma_{\times k}(G) = m$, and give a necessary and sufficient condition for $\gamma_{\times k, t}(G) = \gamma_{\times(k+1)}(G)$. Then we show that if G is a graph of order n with $\delta(G) \geq k + 1 \geq 2$, then $\gamma_{\times k, t}(G)$ has the lower bound $2\gamma_{\times(k+1)}(G) - n$ and characterize graphs that equality holds for them. Finally we present two upper bounds for the k -tuple total domination number of a graph in terms of its order, minimum degree and k .

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1 Introduction

1.1 Definitions and notations

All graphs considered here are finite, undirected, and simple. For standard graph theory terminology not given here we refer to [3]. Let $G = (V, E)$ be a graph with vertex set V of order $n(G) = n$ and edge set E of size $m(G) = m$. The *minimum* and *maximum degree* of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$ and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is $deg(v) = deg_G(v) = |N_G(v)|$. We write K_n and $G[S]$ respectively for the *complete graph* of order n and the *induced subgraph* of G by a vertex set S .

Let $S \subseteq V$ be a set with cardinality at least k . For each k -element subset $S' \subseteq S$, we define the (S, k) -*private neighborhood* $pn_k(S', S)$ of S' to be the set of all vertices v in V such that $N(v) \cap S = S'$ (possibly, $v \in S$). More exactly, the *external* (S, k) -*private neighborhood* $epn_k(S', S)$ of S' is the set of all vertices $v \in V - S$ such that $N(v) \cap S = S'$, and also the *inner* (S, k) -*private neighborhood* $ipn_k(S', S)$ of S' is the set of all vertices $v \in S$ such that $N(v) \cap S = S'$. Further, we define the *open k -boundary* $OB_k(S)$ of S to be the set of all vertices v in G such that $v \in pn_k(S', S)$ for some k -element subset $S' \subseteq S$. Obviously, $OB_k(S) = \cup_{S'} pn_k(S', S)$, where S' is a k -element subset of S .

Let $k \geq 1$ be an integer. The k -*join* $G \circ_k H$ of a graph G to a graph H of order at least k is the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H . Also for any two graphs G and H the *corona* of H and G is the graph $H \circ G$ formed from one copy of H and $|V(H)|$ copies of G where the i -th vertex of H is adjacent to every vertex in the i -th copy of G .

1.2 Domination

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [3, 4]. A set $S \subseteq V$ is a *dominating set* if each vertex in $V \setminus S$ is adjacent to at least one vertex of S . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. A set $S \subseteq V$ is a *total dominating set* if each vertex in V is adjacent to at least one vertex of S , while the minimum cardinality of a total dominating set is the *total domination number* $\gamma_t(G)$ of G . The reader can see [5] for more details on total domination number.

In [2] Harary and Haynes defined a generalization of domination as follow. Let $k \geq 1$ be an integer. A subset S of V is a *k-tuple dominating set*, abbreviated kDS, of G if for every vertex $v \in V$, $|N_G[v] \cap S| \geq k$, that is, v is in S and has at least $k - 1$ neighbors in S or v is in $V - S$ and has at least k neighbors in S . The *k-tuple domination number* $\gamma_{\times k}(G)$ of G is the minimum cardinality of a kDS of G . Clearly, $\gamma(G) = \gamma_{\times 1}(G) \leq \gamma_{\times k}(G) \leq \gamma_{\times (k+1)}(G)$, while $\gamma_t(G) \leq \gamma_{\times 2}(G)$. For a graph to have a kDS, its minimum degree is at least $k - 1$. Hence for trees, $k \leq 2$. A *k-tuple dominating set* where $k = 2$ is called a *double dominating set* (abbreviated DDS). The concept of *k-tuple domination* has been studied by several authors (see, for example, [11, 12], and elsewhere).

In [6], Henning and Kazemi started the studying of *k-tuple total domination number* in graphs, which is an extension of total domination number. For an integer $k \geq 1$, a subset S of V is a *k-tuple total dominating set* of G , abbreviated kTDS, if for every vertex $v \in V$, $|N(v) \cap S| \geq k$; that is, S is a kTDS if every vertex has at least k neighbors in S . The *k-tuple total domination number* $\gamma_{\times k,t}(G)$ is the minimum cardinality of a kTDS of G . A kTDS S in a graph G is *minimal* if no proper subset of S is a kTDS in G . We remark that $\gamma_t(G) = \gamma_{\times 1,t}(G)$. For a graph to have a kTDS, its minimum degree is at least k . Since every $(k+1)$ TDS is also a kTDS, we note that $\gamma_{\times k,t}(G) \leq \gamma_{\times (k+1),t}(G)$ for all graphs with minimum degree at least $k + 1$. A kTDS of cardinality $\gamma_{\times k,t}(G)$ is called a $\gamma_{\times k,t}(G)$ -set. When $k = 2$, *k-tuple total dominating set* and *k-tuple total domination number* are known respectively as *double total dominating set*, abbreviated DTDS, and *double total domination number*. For more information about the concept of *k-tuple domination* see [6, 7, 8].

1.3 Our goal

In this paper, in Section 2, we will prove that if m is a positive integer, then for which graphs G , $\gamma_F(G) = m$ where $\gamma_F(G)$ is $\gamma_{\times k,t}(G)$ or $\gamma_{\times k}(G)$. Then we will present a necessary and sufficient condition for $\gamma_{\times k,t}(G) = \gamma_{\times (k+1)}(G)$. In Section 3, we will show that if G is a graph of order n with $\delta(G) \geq k + 1 \geq 2$, then $\gamma_{\times k,t}(G)$ has the lower bound $2\gamma_{\times (k+1)}(G) - n$ and characterize graphs that equality holds for them. Finally, in Section 4, two upper bounds for the *k-tuple total domination number* of a graph is given in terms of its order and minimum degree.

The following propositions are useful in the context.

Proposition 1.1. (Henning, Kazemi [6] 2010) Let G be a graph with $\delta(G) \geq k$, and let S be a kTDS of G . Then

- $\max\{\gamma_{\times k}(G), k + 1\} \leq \gamma_{\times k,t}(G) \leq \gamma_{\times (k+1)}(G)$,
- for each spanning subgraph H of a graph G , $\gamma_{\times k,t}(H) \leq \gamma_{\times k,t}(G)$,
- the open neighborhood of each vertex of degree k is a subset of S .

Proposition 1.2. (Henning, Kazemi [6] 2010) Let G be a graph of order n with $\delta(G) \geq k$, and let S be a kTDS of G . Then S is minimal if and only if

- a. $S \setminus \{x\}$ is not a k TDS for every $x \in S$, or
- b. for each vertex $x \in S$, there exists a k -element subset $S_x \subseteq S$ such that $x \in S_x$ and $|pn_k(S_x, S)| = |epn_k(S_x, S)| + |ipn_k(S_x, S)| \geq 1$, or
- c. $OB_k(S)$ dominates S .

Proposition 1.3. (*Henning, Kazemi [6] 2010*) Let G be a graph with $\delta(G) \geq k$. Then, $\gamma_{\times k,t}(G) = k + 1$ if and only if $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$ for some graph F .

Proposition 1.4. (*Haynes, Hedetniemi, Slater [3] 1998*) For a graph G with even order n and no isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H .

Proposition 1.5. (*Henning, Yeo, [9] 2007*) If G is a graph of order n and minimum degree $\delta \geq 1$, then

$$\gamma_t(G) \leq \left(\frac{1 + \ln \delta}{\delta} \right) n.$$

2 When $\gamma_{\times k,t}(G) = \gamma_{\times(k+1)}(G)$?

Before we characterize the graphs G that satisfy $\gamma_{\times k,t}(G) = \gamma_{\times(k+1)}(G)$, we need to recognize $\gamma_{\times k,t}(G)$ and $\gamma_{\times(k+1)}(G)$, for any graph G . For finding $\gamma_{\times k,t}(G) = m$, we consider $m \geq k + 2$, by Proposition 1.3.

Theorem 2.1. Let G be a graph with $\delta(G) \geq k \geq 1$. Then for any integer $m \geq k + 2$, $\gamma_{\times k,t}(G) = m$ if and only if $G = K'_m$ or $G = F \circ_k K'_m$, for some graph F and some spanning subgraph K'_m of K_m with $\delta(K'_m) \geq k$ such that m is minimum in the set

$$\{t \mid G = F \circ_k K'_t, \text{ for some } F \text{ and some spanning subgraph } K'_t \text{ of } K_t, \delta(K'_t) \geq k\}. \quad (2.1)$$

Proof. Let S be a $\gamma_{\times k,t}(G)$ -set and let $\gamma_{\times k,t}(G) = m$, for some $m \geq k + 2$. Then $|S| = m$, and every vertex has at least k neighbors in S . Hence $G[S] = K'_m$, for some spanning subgraph K'_m of K_m and $\delta(K'_m) \geq k$. If $|V - S| = m$, then $G = K'_m$. If $|V - S| > m$, let $F = G[V - S]$. Since every vertex in $V - S$ has at least k neighbors in S , we obtain $G = F \circ_k K'_m$. Also by the definition of k -tuple total domination number, clearly m is the minimum of the set given in (2.1).

Conversely, let $G = K'_m$ or $G = F \circ_k K'_m$, for some graph F and some spanning subgraph K'_m of K_m with $\delta(K'_m) \geq k$ such that m is the minimum of the set given in (2.1). Then, since $V(K'_m)$ is a k TDS of G with cardinality m , $\gamma_{\times k,t}(G) \leq m$. Now we prove $\gamma_{\times k,t}(G) = m$. For $m = k + 1$, it holds, by Proposition 1.3. Now let $m \geq k + 2$. If $\gamma_{\times k,t}(G) = m'$, for some $m' < m$, then the previous paragraph shows that for some graph F' and some spanning subgraph $K'_{m'}$ of $K_{m'}$ with $\delta(K'_{m'}) \geq k$, $G = F' \circ_k K'_{m'}$ contradicting with the minimality of m . Therefore $\gamma_{\times k,t}(G) = m$. Q.E.D.

Similar to the proof of Theorem 2.1, we can prove next theorem.

Theorem 2.2. Let G be a graph with $\delta(G) \geq k$. Then for any integer $m \geq k + 1$, $\gamma_{\times(k+1)}(G) = m$ if and only if $G = K'_m$ or $G = F \circ_{k+1} K'_m$, for some graph F and some spanning subgraph K'_m of K_m with $\delta(K'_m) \geq k$ such that m is minimum in the set

$$\{t \mid G = F \circ_{k+1} K'_t, \text{ for some } F \text{ and some spanning subgraph } K'_t \text{ of } K_t, \delta(K'_t) \geq k\}. \quad (2.2)$$

The next theorem can be obtained by Propositions 1.2(b), 1.3, and Theorems 2.1, 2.2.

Theorem 2.3. For every graph G with $\delta(G) \geq k$, $\gamma_{\times k,t}(G) = \gamma_{\times(k+1)}(G)$ if and only if G contains a $\gamma_{\times k,t}(G)$ -set S such that $OB_k(S) \subseteq S$.

3 A lower bound

Here we give a lower bound on the k -tuple total domination number of a graph in terms of its order and $(k + 1)$ -tuple domination number.

Theorem 3.1. Let G be a graph of order n with $\delta(G) \geq k + 1 \geq 2$. Then

$$\gamma_{\times k,t}(G) \geq 2\gamma_{\times(k+1)}(G) - n.$$

Proof. Let S be a $\gamma_{\times k,t}(G)$ -set and let A be the set of all isolated vertices in $G[V(G) - S]$. Then $G[V(G) - (S \cup A)]$ contains no isolated vertices. Let D be a minimum dominating set of $G[V(G) - (S \cup A)]$. Then the well-known inequality of Ore [10] implies

$$\begin{aligned} |D| &\leq \frac{|V(G)-(S \cup A)|}{2} \\ &\leq \frac{|V(G)-2S|}{2} \\ &= \frac{n-\gamma_{\times k,t}(G)}{2}. \end{aligned}$$

Since $\delta(G) \geq k + 1$, and S is a k TDS of G , we conclude that $S \cup D$ is a $(k + 1)$ -tuple dominating set of G , and so

$$\begin{aligned} \gamma_{\times(k+1)}(G) &\leq |S \cup D| \\ &= \gamma_{\times k,t}(G) + |D| \\ &\leq \frac{n+\gamma_{\times k,t}(G)}{2}. \end{aligned}$$

Q.E.D.

We notice that $\gamma_{\times k,t}(G) = 2\gamma_{\times(k+1)}(G) - n$ if and only if every inequality in the proof of Theorem 3.1 is an equality. But this holds if and only if $A = \emptyset$, $|S \cup D| = \gamma_{\times(k+1)}(G)$ and $|D| = \frac{n-\gamma_{\times k,t}(G)}{2}$. Thus $\gamma_{\times k,t}(G) = 2\gamma_{\times(k+1)}(G) - n$ if and only if $\delta(G - S) \geq 1$, $OB_k(S) \not\subseteq S$ and (by Proposition 1.4) the components of $G - S$ are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H . Therefore we have proved next theorem.

Theorem 3.2. Let G be a graph of order n with $\delta(G) \geq k + 1 \geq 2$. Then

$$\gamma_{\times k,t}(G) = 2\gamma_{\times(k+1)}(G) - n$$

if and only if $\gamma_{\times k,t}(G) = n$ or there exists a $\gamma_{\times k,t}(G)$ -set S such that $\delta(G - S) \geq 1$, $OB_k(S) \not\subseteq S$ and the components of $G - S$ are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H .

For example, if G is a k -regular graph of order n , then

$$2\gamma_{\times(k+1)}(G) - n = \gamma_{\times k,t}(G) = \gamma_{\times(k+1)}(G) = n,$$

and if G is the complete graph K_n of order at least $k + 2$, then

$$2\gamma_{\times(k+1)}(G) - n < \gamma_{\times k,t}(G) = \gamma_{\times(k+1)}(G) = k + 1.$$

4 Two upper bounds

Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, C)$ is a finite set V of elements, called *vertices*, together with a finite multiset C of arbitrary subsets of V , called *edges*. The *degree* of a vertex v in H , denoted by $\deg(v)$ or $\deg_H(v)$, is the number of edges that contain v . A *k -uniform hypergraph* is a hypergraph in which every edge has size k . Every simple graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs.

For each positive integer k , Henning and Kazemi [6] defined the concept of k -transversal number in hypergraphs as follows: the *k -transversal number* $\tau_k(H)$ of a hypergraph $H = (V, E)$ is the minimum size of a k -transversal in H , that is, a subset $S \subseteq V$ such that contains at least k vertices of every edge in H . Recall that $\tau_1(H)$ is known as *transversal number* $\tau(H)$ of H .

For a graph $G = (V, E)$, we denote by H_G the *open neighborhood hypergraph* of G ; that is, $H_G = (V, C)$ is the hypergraph with vertex set V and with edge set C consisting of the open neighborhoods of vertices of V in G .

We recall the next proposition from [6].

Proposition 4.1. (*Henning, Kazemi [6] 2010*) If G is a graph with minimum degree at least k and H_G is the open neighborhood hypergraph of G , then $\gamma_{\times k,t}(G) = \tau_k(H_G)$.

Perhaps much of the recent interest in total domination in graphs arises from the fact that total domination in graphs can be translated to the problem of finding transversals in hypergraphs. Proposition 4.1 shows that the problem of finding k -tuple total dominating sets in graphs can be translated to the problem of finding k -transversals in hypergraphs. The main advantage of considering hypergraphs rather than graphs is that the structure is easier to handle—for example, we can often restrict our attention to uniform hypergraphs where every edge has the same size.

Let $H = (V, C)$ be a hypergraph and let $V_k \subseteq V$ be a subset of vertices with cardinality k . We define $d_k(V_k)$ as the number of edges e of H such that $V_k \subseteq e$, and also

$$\Delta_k(H) = \max\{d_k(V_k) \mid V_k \subseteq V\}.$$

Notice that $d_1(V_1) = \deg_H(v)$, where $V_1 = \{v\}$, and $\Delta_1(H)$ is the same maximum degree $\Delta(H)$ among the vertices of H . The *components* of a hypergraph H are the minimal sets of vertices in H such that no edge in H contains vertices from different components. A hypergraph is called *connected* if it contains one component. For disjoint subsets X and Y of vertices in a hypergraph H , we define $H(X, Y)$ to be the hypergraph obtained from H by deleting the vertices in X and all edges incident with X and by deleting the vertices in Y and shrinking all edges incident with Y . Furthermore, if $Y = \emptyset$, we denote $H(X, Y)$ simply by $H - X$ or by $H - v$ if $X = \{v\}$.

In the next two theorems we will present two upper bounds on the k -tuple total domination number of a graph G in terms of its order, minimum degree and k . The proof of the following is along similar lines to the proof presented in Proposition 1.5 by Henning and Yeo for the total domination number, and thus we omit it.

Theorem 4.2. Let G be a graph of order n with $\delta(G) \geq k \geq 1$. If for every two vertices x and y of G , $N_G(x) \cap N_G(y) = \emptyset$ or $|N_G(x) \cap N_G(y)| \geq k$, then

$$\gamma_{\times k,t}(G) \leq \left(\frac{1 + \ln \delta}{\delta} \right) \binom{n}{k}.$$

Before we present the second upper bound, we define some needed notations. Suppose that G is a graph with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and every vertex v_i has degree d_i . For any integer m , we define

$$d_m^\wedge = \frac{1}{n} \sum_{i=1}^n \binom{d_i}{m}.$$

By these notations, we present next theorem, which its proof is along similar lines to the presented by Chang [1] for the k -tuple domination number, and thus we omit the proof.

Theorem 4.3. If k is a positive integer and G is a graph of order n with $\delta > k \geq 1$, then

$$\gamma_{\times k, t}(G) \leq \left(\frac{\ln(\delta - k) + \ln d_k^\wedge + 1}{\delta - k} \right) n.$$

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