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Bi-flows on a network

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Abstract. Flows on a network play an important role in the theory of discrete harmonic functions. In the study of discrete bi-harmonic functions, we encounter a concept of bi-flows. In this paper, we are concerned with minimization problems for bi-flows which are analogous to those for flows.

Key words: discrete potential theory, bi-harmonic Green function, bi-flows on a network.

1. Introduction

In the theory of discrete potential theory on networks, it is well-known that flows have played an important role related to discrete harmonic functions. For example, a minimizing problem related to flows from a node to the ideal boundary with unit strength characterizes the harmonic Green function. In this paper, we introduce an arc-arc incidence matrix $b(y, y')$ of two arcs y and y' and an operator B_r related to it. We say that a function w on arcs is a bi-flow if $B_r w$ is a flow. If u is a bi-harmonic function defined on nodes, then we see that the discrete derivative $w = du$ is a bi-flow. We shall consider two minimizing problems related to bi-flows from a node to the ideal boundary. The optimal solution of each minimizing problem characterizes the bi-harmonic Green function.

We organize this paper as follows: Some properties of b and B_r will be given in Section 3. We define bi-flows as well as weak bi-flows in Section 4. Two minimizing problems related to bi-flows are given in Sections 5 and 6.

2. Preliminaries

Let $N = \{X, Y, K, r\}$ be an infinite network which is connected and locally finite and has no self-loops. Here X is the set of nodes and Y is the set of arcs. The node-arc incidence matrix K is a function on $X \times Y$ and $K(x, y) = -1$ if x is the initial node $x^-(y)$ of y; $K(x, y) = 1$ if x

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is the terminal node $x^+(y)$ of y; otherwise $K(x,y)=0$. The resistance r is a strictly positive function on Y. Let $L(X)$ be the set of all real valued functions on X and let $L_0(X)$ be the set of all $u \in L(X)$ with finite supports. We define $L(Y)$ and $L_0(Y)$ similarly.

For $u \in L(X)$ and $w \in L(Y)$, we define $du \in L(Y)$ and $\partial w \in L(X)$ by

$$du(y) = -r(y)^{-1} \sum_{x \in X} K(x,y)u(x),$$

$$\partial w(x) = \sum_{y \in Y} K(x,y)w(y).$$

Also we define the *Laplacian* $\Delta u \in L(X)$ and the *bi-Laplacian* $\Delta^2 u \in L(X)$ for $u \in L(X)$ by

$$\Delta u = \partial(du), \qquad \Delta^2 u = \Delta(\Delta u).$$

For $y \in Y$, let $e(y) = \{x \in X; K(x,y) \neq 0\} = \{x^+(y), x^-(y)\}$. For $a \in X$, denote by $X(a)$ the set of nodes $x \in X$ such that $K(a,y)K(x,y) \neq 0$ for some $y \in Y$.

We shall study the bi-Laplacian and bi-flows on a network by using an arc-arc incidence function b on $Y \times Y$.

3. An arc-arc incidence function

An arc-arc incidence function b on $Y \times Y$ is defined by

$$b(y,y') = \sum_{z \in X} K(z,y)K(z,y') = \sum_{z \in e(y) \cap e(y')} K(z,y)K(z,y').$$

Proposition 3.1 *The arc-arc incidence function b has the following properties*:

(i) $b(y,y') = b(y',y)$ *for all* $y, y' \in Y$;
(ii) $b(y,y) = 2$;
(iii) $b(y,y') = K(x,y)K(x,y')$ *if y and y' meet only one node x, i.e.,* $e(y) \cap e(y') = \{x\}$;
(iv) $b(y,y') = 0 \quad$ *if* $e(y) \cap e(y') = \emptyset$;

In case $e(y) = e(y')$ and $y \neq y'$,

(v) $b(y,y') = 2$ *if* $x^+(y) = x^+(y')$ *and* $x^-(y) = x^-(y')$;
(vi) $b(y,y') = -2$ *if* $x^+(y) = x^-(y')$ *and* $x^-(y) = x^+(y')$.

Define a linear operator B_r from $L(Y)$ to $L(Y)$ by

$$B_r w(y) = r(y)^{-1} \sum_{y' \in Y} b(y,y')w(y').$$

Lemma 3.1 $B_r w = -d\partial w$ *on* Y.

Proof. A simple calculation shows that

$$\begin{aligned} B_r w(y) &= r(y)^{-1} \sum_{y' \in Y} \Big(\sum_{z \in X} K(z,y)K(z,y') \Big) w(y') \\ &= r(y)^{-1} \sum_{z \in X} K(z,y) \Big(\sum_{y' \in Y} K(z,y')w(y') \Big) \\ &= r(y)^{-1} \sum_{z \in X} K(z,y)\partial w(z) = -d\partial w(y). \end{aligned}$$

□

Define $c(x,z)$ for $x, z \in X$ by

$$c(x,z) = \sum_{y \in Y} r(y)^{-1} K(x,y)K(z,y).$$

Lemma 3.2 (i) $c(x,z) \neq 0$ *if and only if* $z \in X(x)$.
(ii) $\sum_{z \in X} c(x,z) = 0$.
(iii) $\Delta u(x) = -\sum_{z \in X} c(x,z)u(z)$.

Proof. (i) It is trivial that $z \notin X(x)$ implies $c(x,z) = 0$. If $x = z$, then $K(x,y)K(z,y) \in \{0,1\}$ for all $y \in Y$ and $K(x,y)K(z,y) = 1$ for some $y \in Y$. Therefore $c(x,z) > 0$. Let $z \in X(x) \setminus \{x\}$. Then $K(x,y)K(z,y) \in \{0,-1\}$ for all $y \in Y$ and $K(x,y)K(z,y) = -1$ for some $y \in Y$. Therefore $c(x,z) < 0$.

(ii) Since $\sum_{z \in X} K(z,y) = 0$ for every $y \in Y$, we have

$$\sum_{z \in X} c(x,z) = \sum_{y \in Y} r(y)^{-1} K(x,y) \sum_{z \in X} K(z,y) = 0.$$

$$\text{(iii)} \quad \sum_{z\in X} c(x,z)u(z) = \sum_{z\in X}\sum_{y\in Y} r(y)^{-1}K(x,y)K(z,y)u(z)$$
$$= \sum_{y\in Y} r(y)^{-1}K(x,y)\sum_{z\in X} K(z,y)u(z)$$
$$= -\sum_{y\in Y} K(x,y)du(y) = -\partial du(x) = -\Delta u(x). \qquad \square$$

4. Bi-flows

Let $a, b \in X$. We say that $w \in L(Y)$ is a *flow* from a to b of strength $I[w]$ if the following condition is fulfilled:

$$\partial w(x) = (\varepsilon_b(x) - \varepsilon_a(x))I[w],$$

where $\varepsilon_a(x) = 0$ if $x \neq a$ and $\varepsilon_a(a) = 1$. Denote by $\mathbf{F}(a,b)$ the set of all flows from a to b.

Lemma 4.1 *$B_r w(y) = r(y)^{-1}(K(b,y) - K(a,y))I[w]$ for $w \in \mathbf{F}(a,b)$.*

Proof. We have by Lemma 3.1

$$B_r w(y) = -d\partial w(y) = r(y)^{-1}\sum_{z\in X} K(z,y)(\varepsilon_b(z) - \varepsilon_a(z))I[w]$$
$$= r(y)^{-1}(K(b,y) - K(a,y))I[w]. \qquad \square$$

We say that $w \in L(Y)$ is a *bi-flow* from a to b of strength $J[w]$ if $B_r w \in \mathbf{F}(a,b)$ and $J[w] = I[B_r w]$, i.e.,

$$\partial B_r w(x) = (\varepsilon_b(x) - \varepsilon_a(x))J[w].$$

Denote by $\mathbf{BF}(a,b)$ the set of all bi-flows from a to b.

Assume that $X(a) \cap X(b) = \emptyset$. We say that $w \in L(Y)$ is a *weak bi-flow* from a to b of strength $\tilde{J}[w]$ if

$$\partial B_r w(x) = 0 \quad \text{for all } x \in X \setminus \{X(a) \cup X(b)\},$$
$$\tilde{J}[w] = -\sum_{x\in X(a)} \partial B_r w(x) = \sum_{x\in X(b)} \partial B_r w(x).$$

Denote by $\mathbf{WBF}(a,b)$ the set of all weak bi-flows from a to b.

Denote by $\mathbf{C}$ and $\mathbf{C}_B$ the set of cycles on N and the set of bicycles on N,

$$\mathbf{C} = \{w \in L(Y); \partial w = 0\}, \quad \mathbf{C}_B = \{w \in L(Y); \partial B_r w = 0\}.$$

Denote by $\mathbf{K}_B$ and $\mathbf{H}$ the kernel of B_r and the set of all harmonic functions on X,

$$\mathbf{K}_B = \{w \in L(Y); B_r w = 0\}, \quad \mathbf{H} = \{u \in L(X); \Delta u = 0\}.$$

Lemma 4.2 $\{dh; h \in \mathbf{H}\} \subset \mathbf{C} \subset \mathbf{K}_B \subset \mathbf{C}_B$.

Proof. Let $h \in \mathbf{H}$. Then $\partial(dh) = \Delta h = 0$, so that $dh \in \mathbf{C}$. Let $w \in \mathbf{C}$. Then by Lemma 3.1 $B_r w = -d\partial w = 0$, so that $w \in \mathbf{K}_B$. The inclusion $\mathbf{K}_B \subset \mathbf{C}_B$ is trivial. $\square$

Proposition 4.1 (i) *$\mathbf{C} \subset \mathbf{F}(a,b)$ and $\mathbf{C}_B \subset \mathbf{BF}(a,b)$ for $a, b \in X$.*

(ii) *$\{w \in \mathbf{F}(a,b); I[w] = 0\} = \mathbf{C}$ and $\{w \in \mathbf{BF}(a,b); J[w] = 0\} = \mathbf{C}_B$ for $a, b \in X$.*

(iii) *$\mathbf{F}(a,a) = \mathbf{C}$ and $\mathbf{BF}(a,a) = \mathbf{C}_B$ for $a \in X$.*

(iv) *$\mathbf{F}(a_1,b_1) \cap \mathbf{F}(a_2,b_2) = \mathbf{C}$ and $\mathbf{BF}(a_1,b_1) \cap \mathbf{BF}(a_2,b_2) = \mathbf{C}_B$ for $a_1, a_2, b_1, b_2 \in X$ with $\{a_1, b_1\} \neq \{a_2, b_2\}$.*

Proof. We shall show the assertions for $\mathbf{F}(a,b)$; the assertions for $\mathbf{BF}(a,b)$ can be similarly proved. We easily have (i) and (ii).

To prove (iii), it suffices to show that $\mathbf{F}(a,a) \subset \mathbf{C}$. Let $w \in \mathbf{F}(a,a)$. Then $\partial w = (\varepsilon_a - \varepsilon_a) I[w] = 0$, so that $w \in \mathbf{C}$.

We shall prove (iv). We need to show that $\mathbf{F}(a_1,b_1) \cap \mathbf{F}(a_2,b_2) \subset \mathbf{C}$. We may assume $a_1 \notin \{a_2, b_2\}$. Using (iii) we may also assume that $a_1 \neq b_1$ and $a_2 \neq b_2$. Let $w \in \mathbf{F}(a_1,b_1) \cap \mathbf{F}(a_2,b_2)$. Then $\partial w(a_1) = -I[w]$ from $w \in \mathbf{F}(a_1,b_1)$ and $\partial w(a_1) = 0$ from $w \in \mathbf{F}(a_2,b_2)$. We have $I[w] = 0$, so that $\partial w = 0$. $\square$

Theorem 4.1 *Assume that $X(a) \cap X(b) = \emptyset$.*

(i) *$\mathbf{BF}(a,b) \subset \mathbf{WBF}(a,b)$ and $J[w] = \tilde{J}[w]$ for $w \in \mathbf{BF}(a,b)$.*

(ii) *$\mathbf{F}(a,b) \subset \mathbf{WBF}(a,b)$ and $\tilde{J}[w] = 0$ for $w \in \mathbf{F}(a,b)$.*

(iii) *$\mathbf{F}(a,b) \cap \mathbf{BF}(a,b) = \mathbf{C}$.*

Proof. It is easy to see that (i) holds. We shall prove (ii). Let $w \in \mathbf{F}(a,b)$. By Lemma 4.1

$$\begin{aligned}\partial B_r w(x) &= \sum_{y \in Y} K(x,y) r(y)^{-1} (K(b,y) - K(a,y)) I[w] \\ &= (c(x,b) - c(x,a)) I[w]. \qquad (1)\end{aligned}$$

For $x \in X \setminus (X(a) \cup X(b))$ we have $\partial B_r w(x) = 0$ by Lemma 3.2 (i). Also Lemma 3.2 (i) and (ii) show that $\sum_{x \in X(a)} \partial B_r w(x) = -\sum_{x \in X(a)} c(x,a) I[w] = 0$. Similarly $\sum_{x \in X(b)} \partial B_r w(x) = 0$.

Next we prove (iii). Lemma 4.2 and Proposition 4.1 (i) show that $\mathbf{C} \subset \mathbf{F}(a,b) \cap \mathbf{BF}(a,b)$. We shall show the converse. Let $w \in \mathbf{F}(a,b) \cap \mathbf{BF}(a,b)$. Let $x \in X(a) \setminus \{a\}$. Then the equation (1) shows that $0 = \partial B_r w(x) = -c(x,a) I[w]$. Lemma 3.2 (i) implies $I[w] = 0$, which means $\partial w = 0$. □

Theorem 4.2 *Suppose that* $X(a) \cup X(b) \neq (X(a) \cap X(b)) \cup \{a,b\}$. *Then* $\mathbf{F}(a,b) \cap \mathbf{BF}(a,b) \subset \mathbf{C} \cap \mathbf{K}_B$.

Proof. It is clear that $(X(a) \cap X(b)) \cup \{a,b\} \subset X(a) \cup X(b)$. By our assumption, there exists $x_0 \in X(a) \cup X(b)$ such that $x_0 \notin (X(a) \cap X(b)) \cup \{a,b\}$. We may assume that $x_0 \in X(a), x_0 \notin X(b)$ and $x_0 \neq a$. Let $w \in \mathbf{F}(a,b) \cap \mathbf{BF}(a,b)$. Since $K(x_0,y)K(b,y) = 0$ for all $y \in Y$, we have by Lemma 4.1

$$\begin{aligned}0 = \partial B_r w(x_0) &= \sum_{y \in Y} K(x_0,y) B_r w(y) \\ &= -I[w] \sum_{y \in Y} r(y)^{-1} K(x_0,y) K(a,y) = -I[w] c(x_0,a).\end{aligned}$$

Lemma 3.2 (i) shows that $c(x_0,a) \neq 0$, and that $I[w] = 0$. Thus $\partial w = 0$ on X. Lemma 4.1 shows that $B_r w = 0$ on Y. □

5. Bi-flows to the ideal boundary

Now we recall some definitions related to the *energy* $H[w]$ of $w \in L(Y)$ and the *Dirichlet sum* $D[u]$ of $u \in L(X)$:

$$\langle w, w' \rangle = \sum_{y \in Y} r(y) w(y) w'(y),$$

$$H[w] = \langle w, w \rangle = \sum_{y \in Y} r(y) w(y)^2,$$

$$L_2(Y; r) = \{w \in L(Y); H[w] < \infty\},$$

$$D[u, u'] = \langle du, du' \rangle = \sum_{y \in Y} r(y) du(y) du'(y),$$

$$D[u] = D[u, u] = H[du] = \sum_{y \in Y} r(y)(du(y))^2,$$

$$\mathbf{D}(N) = \{u \in L(X); D[u] < \infty\}.$$

Lemma 5.1 *$\langle du, du' \rangle = -\sum_{x \in X} u(x) \Delta u'(x)$ for $u \in L_0(X)$ and for $u' \in \mathbf{D}(N)$.*

Proof.

$$\begin{aligned} \langle du, du' \rangle &= \sum_{y \in Y} r(y) du(y) du'(y) = -\sum_{y \in Y} \sum_{x \in X} K(x, y) u(x) du'(y) \\ &= -\sum_{x \in X} u(x) \sum_{y \in Y} K(x, y) du'(y) = -\sum_{x \in X} u(x) \partial du'(x) \\ &= -\sum_{x \in X} u(x) \Delta u'(x). \end{aligned}$$ □

It is known that $\mathbf{D}(N)$ ($L_2(Y; r)$ resp.) is a Hilbert space with respect to the norm $\|u\|_2 = (D[u] + u(x_0)^2)^{1/2}$ ($H[w]^{1/2}$ resp.) with a fixed node $x_0 \in X$. Denote by $\mathbf{D}_0(N)$ the closure of $L_0(X)$ in the Hilbert space $\mathbf{D}(N)$ (see [3]).

The *Green function* $g_a \in L(X)$ with pole at $a \in X$ is defined as the unique function satisfying the conditions:

$$g_a \in \mathbf{D}_0(N) \quad \text{and} \quad \Delta g_a = -\varepsilon_a \text{ on } X.$$

We know that g_a exists for every a if and only if N is hyperbolic, i.e., $\mathbf{D}_0(N) \neq \mathbf{D}(N)$ (see [2]). Denote by $\mathbf{HD}(N)$ the set of all $u \in \mathbf{D}(N)$ such that $\Delta u = 0$.

Lemma 5.2 $\mathbf{D}_0(N) \cap \mathbf{HD}(N) = \{0\}$ *if and only if* N *is hyperbolic.*

Proof. If N is parabolic, then $1 \in \mathbf{D}(N) = \mathbf{D}_0(N)$, which is also harmonic. This means $1 \in \mathbf{D}_0(N) \cap \mathbf{HD}(N)$.

Conversely, we assume that N is hyperbolic. Let $u \in \mathbf{D}_0(N) \cap \mathbf{HD}(N)$. Then both $u = u + 0$ and $u = 0 + u$ are the Royden decompositions. The uniqueness of the Royden decomposition implies that $u = 0$. □

We say that $w \in L(Y)$ is a *flow* from $a \in X$ to the ideal boundary with strength $I[w]$ if

$$\partial w(x) = -\varepsilon_a(x) I[w].$$

Let $\mathbf{F}(a, \infty)$ be the set of all flows w from a to the ideal boundary. It is well-known that dg_a is characterized as the unique optimal solution to the following extremal problem:

$$d^*(a, \infty) = \inf\{H[w]; w \in \mathbf{F}(a, \infty),\ I[w] = 1\}.$$

We say that $w \in L(Y)$ is a *bi-flow* from $a \in X$ to the ideal boundary with strength $J[w]$ if

$$\partial B_r w(x) = -\varepsilon_a(x) J[w].$$

Notice that

$$J[w] = \Delta \partial w(a).$$

Denote by $\mathbf{BF}(a, \infty)$ the set of all bi-flows from a to the ideal boundary of N.

Analogous to $d^*(a, \infty)$, we consider the following extremal problem:

$$d_B^*(a, \infty) = \inf\{H[w]; w \in \mathbf{BF}(a, \infty),\ \partial w \in \mathbf{D}_0(N),\ J[w] = 1\}. \qquad (*)$$

The *bi-harmonic Green function* $q_a \in L(X)$ with pole at a is defined by

$$q_a(x) = \sum_{z \in X} g_a(z) g_z(x)$$

if the sum converges (see [1], [4]). Notice that

$$\Delta q_a = -g_a \quad \text{and} \quad \Delta^2 q_a = \varepsilon_a \text{ on } X,$$

and that dq_a is a feasible solution to the problem $(*)$.

We proved the following lemma in [6, Theorem 4.2]:

Lemma 5.3 *Let N be parabolic and $u \in \mathbf{D}(N)$. If $\sum_{x\in X} |\Delta u(x)| < \infty$, then $\sum_{x\in X} \Delta u(x) = 0$.*

Corollary 5.1 *If $d_B^*(a,\infty) < \infty$, then N is hyperbolic and $\partial w = -g_a$ for all feasible solution w to the problem $(*)$.*

Proof. Let w be a feasible solution to the problem $(*)$. Then $u = \partial w \in \mathbf{D}_0(N)$ and $\Delta u(x) = -\partial B_r w(x) = \varepsilon_a(x)$. By the above lemma, N must be hyperbolic and $u = -g_a$. □

The next theorem is an extension of [4, Theorem 3.1], which shows that $q_a \in \mathbf{D}(N)$ is equivalent to $q_a \in \mathbf{D}_0(N)$.

Theorem 5.1 *The following are equivalent*:

(i) $q_a \in \mathbf{D}(N)$;
(ii) $q_a \in \mathbf{D}_0(N)$;
(iii) $d_B^*(a,\infty) < \infty$.

In this case dq_a is a unique optimal solution to the problem $(*)$.

Proof. It is obvious that (ii) implies (i). Suppose that $q_a \in \mathbf{D}(N)$. Since dq_a is a feasible solution to the problem $(*)$, it follows that $d_B^*(a,\infty) < \infty$. This shows that (i) implies (iii).

We shall show that (iii) implies (ii). We assume that $d_B^*(a,\infty) < \infty$. First we shall prove that there exists an optimal solution to the problem $(*)$. Let $\{w_n\}$ be a minimizing sequence of $(*)$. Then $(w_n + w_m)/2$ is a feasible solution to the problem $(*)$, so that we have

$$\begin{aligned} d_B^*(a,\infty) &\le H[(w_n + w_m)/2] \le H[(w_n + w_m)/2] + H[(w_n - w_m)/2] \\ &= (H[w_n] + H[w_m])/2 \to d_B^*(a,\infty) \end{aligned}$$

as $n, m \to \infty$. Thus $H[w_n - w_m] \to 0$ as $n, m \to \infty$. There exists $w^* \in L_2(Y;r)$ such that $H[w_n - w^*] \to 0$ as $n \to \infty$. Since $\{w_n\}$ converges pointwise to w^* and N is locally finite, we obtain $w^* \in \mathbf{BF}(a,\infty)$ and $J[w^*] = 1$. Also $\partial w_n = -g_a$ implies that

$$\partial w^* = \lim_{n\to\infty} \partial w_n = -g_a \in \mathbf{D}_0(N).$$

Therefore w^* is an optimal solution to the problem $(*)$.

To prove the uniqueness of an optimal solution to the problem $(*)$, let w' be another optimal solution to the problem $(*)$. Then

$$\begin{aligned} d_B^*(a,\infty) &\le H[(w^*+w')/2] \le H[(w^*+w')/2] + H[(w^*-w')/2] \\ &= (H[w^*]+H[w'])/2 = d_B^*(a,\infty), \end{aligned}$$

so that $H[w^*-w']=0$. Hence $w^*=w'$.

For any $\omega \in L_0(Y)\cap \mathbf{C}(N)$ and any $t\in\mathbf{R}$, we see that $w^*+t\omega$ is a feasible solution to the problem $(*)$. Thus

$$d_B^*(a,\infty) \le H[w^*+t\omega] = H[w^*] + 2t\langle w^*,\omega\rangle + t^2 H[\omega],$$

so that $\langle w^*,\omega\rangle = 0$. By the usual way, we see that there exists $u^*\in L(X)$ such that $w^* = du^*$ (see the proof of [6, Theorem 3.2] for details).

Since $D[u^*] = H[w^*] < \infty$, it follows that $u^*\in\mathbf{D}(N)$. Let $u^* = v^* + h$ be the Royden decomposition with $v^*\in\mathbf{D}_0(N)$ and $h\in\mathbf{HD}(N)$. Let $w' = dv^*$. Then w' is a feasible solution to the problem $(*)$, so that

$$D[v^*] + D[h] = D[u^*] = H[w^*] \le H[w'] = D[v^*].$$

This means that $D[h]=0$ and $H[w^*] = H[w']$, i.e., h is a constant function and $w^* = w' = dv^*$.

Let $\{N_n\}$ be an exhaustion of N and $g_a^{(n)}$ the Green function of N_n with pole at a. We have

$$\sum_{z\in X} g_a(z) g_x^{(n)}(z) = -\sum_{z\in X} \big(\Delta v^*(z)\big) g_x^{(n)}(z) = D[v^*, g_x^{(n)}].$$

Since $\{g_x^{(n)}\}_n$ converges to g_x (see [3, Section 3]), it follows that

$$\begin{aligned} \sum_{z\in X} g_a(z) g_x(z) &\le \liminf_{n\to\infty} \sum_{z\in X} g_a(z) g_x^{(n)}(z) = \lim_{n\to\infty} D[v^*, g_x^{(n)}] \\ &= D[v^*, g_x] \le D[v^*]^{1/2} D[g_x]^{1/2} < \infty. \end{aligned}$$

In particular, we obtain $\sum_{z\in X} g_a(z)^2 < \infty$, so that $q_a \in L(X)$ by [4, Theorem 2.3].

Define $f(x)$, $f_n(x)$ and $h_n(x)$ by

$$f(x) = \sum_{z\in X} g_x(z)\Delta v^*(z) = -q_a(x) \in L(X)$$
$$f_n(x) = \sum_{z\in X} g_z^{(n)}(x)\Delta v^*(z)$$
$$h_n = v^* + f_n.$$

Notice that h_n is harmonic on X_n and

$$D[h_n, f_n] = -\sum_{x\in X} (\Delta h_n(x)) f_n(x) = 0,$$

so that $D[v^*] = D[h_n] + D[f_n]$. We see by Lebesgue's dominated convergence theorem that $\{f_n(x)\}$ converges pointwise to $f(x)$ for all $x \in X$. Since $\{D[f_n]\}$ is bounded, we see by [5, Theorem 4.1] that $q_a = -f \in \mathbf{D}_0(N)$.

Let $f' = q_a - v^*$. Then

$$\Delta f' = \Delta q_a - \Delta v^* = -g_a + g_a = 0,$$

so that $f' \in \mathbf{D}_0(N) \cap \mathbf{HD}(N)$. Lemma 5.2 shows $f' = 0$. Therefore $q_a = v^* \in \mathbf{D}_0(N)$ and $dv^* = dq_a$. □

6. Another extremal problem

Analogous to $d^*(a,\infty)$ and $d_B^*(a,\infty)$, we consider the following extremum problem:

$$d_B^{**}(a,\infty) = \inf\{H[w]; w \in \mathbf{BF}(a,\infty),\ J[w] = 1\}. \qquad (**)$$

Clearly $d_B^{**}(a,\infty) \le d_B^*(a,\infty)$.

Theorem 6.1 *Assume that $d_B^{**}(a,\infty) < \infty$. Then there exists a unique optimal solution w^{**} to the problem $(**)$. Also there exists $v^{**} \in \mathbf{D}_0(N)$ such that $w^{**} = dv^{**}$.*

Proof. Let $\{w_n\}$ be a minimizing sequence of $(**)$. Then $(w_n + w_m)/2$ is a feasible solution to the problem $(**)$, so that we have

$$\begin{aligned} d_B^{**}(a,\infty) &\le H[(w_n + w_m)/2] \le H[(w_n + w_m)/2] + H[(w_n - w_m)/2] \\ &= (H[w_n] + H[w_m])/2 \to d_B^{**}(a,\infty) \end{aligned}$$

as $n, m \to \infty$. Thus $H[w_n - w_m] \to 0$ as $n, m \to \infty$. There exists $w^{**} \in L_2(Y; r)$ such that $H[w_n - w^{**}] \to 0$ as $n \to \infty$. Since $\{w_n\}$ converges pointwise to w^{**} and N is locally finite, we obtain $w^{**} \in \mathbf{BF}(a,\infty)$ and $J[w^{**}] = 1$. Therefore w^{**} is an optimal solution to the problem $(**)$.

To prove the uniqueness let w' be another optimal solution to the problem $(**)$. Then

$$\begin{aligned} d_B^{**}(a,\infty) &\le H[(w^{**} + w')/2] \le H[(w^{**} + w')/2] + H[(w^{**} - w')/2] \\ &= (H[w^{**}] + H[w'])/2 = d_B^{**}(a,\infty), \end{aligned}$$

so that $H[w^{**} - w'] = 0$. Hence $w^{**} = w'$.

For any $\omega \in L_0(Y) \cap \mathbf{C}(N)$ and any $t \in \mathbf{R}$, we see that $w^{**} + t\omega$ is a feasible solution to the problem $(**)$. Thus

$$d_B^{**}(a,\infty) \le H[w^{**} + t\omega] = H[w^{**}] + 2t\langle w^{**}, \omega\rangle + t^2 H[\omega],$$

so that $\langle w^{**}, \omega\rangle = 0$. By the usual way, we see that there exists $u^{**} \in L(X)$ such that $w^{**} = du^{**}$. Since $D[u^{**}] = H[w^{**}] < \infty$, $u^{**} \in \mathbf{D}(N)$.

If N is hyperbolic type, then we let $u^{**} = v^{**} + h$ be the Royden decomposition with $v^{**} \in \mathbf{D}_0(N)$ and $h \in \mathbf{HD}(N)$; otherwise let $v^{**} = u^{**} \in \mathbf{D}(N) = \mathbf{D}_0(N)$. Let $w' = dv^{**}$. Then w' is a feasible solution to the problem $(**)$, so that

$$D[v^{**}] + D[h] = D[u^{**}] = H[w^{**}] \le H[w'] = D[v^{**}].$$

This means that $D[h] = 0$ and $H[w^{**}] = H[w']$, i.e., h is a constant function and $w^{**} = w' = dv^{**}$. □

We say that a network N satisfies the condition (LD) if there exists a constant c such that $D[\Delta u] \le cD[u]$ for all $u \in L_0(X)$. We say that a network N is of *bounded degree* if $\sup_{x\in X} \sum_{y\in Y} |K(x,y)| < \infty$.

Next proposition provides a sufficient condition for the condition (LD).

Proposition 6.1 *Assume that $r \equiv 1$ and that N is of bounded degree. Then $D[\Delta u] \leq 8\nu_0^2 D[u]$ for all $u \in \mathbf{D}(N)$, where $\nu_0 = \sup_{x\in X}\sum_{y\in Y} \cdot|K(x,y)|$. Especially N satisfies the condition* (LD).

Proof. First note that a simple calculation shows that

$$\left(\sum_{j=1}^{n}\alpha_j\right)^2 \leq n\sum_{j=1}^{n}\alpha_j^2$$

for $\alpha_1,\ldots,\alpha_n \in \mathbb{R}$.

Let $w = du$ and $v = \Delta u$. Then

$$dv(y) = -\sum_{y'\in Y} b(y,y')w(y') = -\sum_{y'\in Y}\sum_{x\in X} K(x,y)K(x,y')w(y').$$

Since the number of $y' \in Y$ with $\sum_{x\in X} K(x,y)K(x,y')w(y') \neq 0$ is at most $2\nu_0$ for each y, it follows that

$$\begin{aligned}(dv(y))^2 &= \left(\sum_{y'\in Y}\sum_{x\in X} K(x,y)K(x,y')w(y')\right)^2\\ &\leq 2\nu_0\sum_{y'\in Y}\left(\sum_{x\in X} K(x,y)K(x,y')w(y')\right)^2.\end{aligned}$$

Since the number of $x \in X$ with $K(x,y)K(x,y') \neq 0$ is at most two for each $y, y' \in Y$, we have $(\sum_{x\in X} K(x,y)K(x,y'))^2 \leq 2\sum_{x\in X}(K(x,y)K(x,y'))^2$. Using $|K(x,y)K(x,y')|^2 = |K(x,y)K(x,y')|$ we obtain

$$(dv(y))^2 \leq 4\nu_0\sum_{y'\in Y}\left(\sum_{x\in X}|K(x,y)K(x,y')|\right)w(y')^2.$$

Let $Y(x) = \{y \in Y; K(x,y) \neq 0\}$ for $x \in X$. Then $\sum_{x\in X}\sum_{y'\in Y(x)} w(y')^2 = 2\sum_{y\in Y} w(y)^2$. By the above estimation, we have

$$D[\Delta u] = H[dv] = \sum_{y\in Y}(dv(y))^2$$

$$\leq 4\nu_0 \sum_{y\in Y}\sum_{y'\in Y}\Bigl(\sum_{x\in X}|K(x,y)K(x,y')|\Bigr)w(y')^2$$

$$= 4\nu_0 \sum_{y'\in Y}\Bigl(\sum_{x\in X}\Bigl(\sum_{y\in Y}|K(x,y)|\Bigr)|K(x,y')|\Bigr)w(y')^2$$

$$\leq 4\nu_0^2 \sum_{y'\in Y}\sum_{x\in X}|K(x,y')|w(y')^2$$

$$= 4\nu_0^2 \sum_{x\in X}\sum_{y'\in Y(x)} w(y')^2 = 8\nu_0^2\sum_{y\in Y}w(y)^2$$

$$= 8\nu_0^2 D[u]. \qquad \square$$

Lemma 6.1 *Assume that N satisfies the condition* (LD). *If $u \in \mathbf{D}_0(N)$, then $\Delta u \in \mathbf{D}_0(N)$.*

Proof. Let $\{f_n\}$ be a sequence in $L_0(X)$ such that $\|f_n - u\|_2 \to 0$ as $n \to \infty$. Then $\|f_n - f_m\|_2 \to 0$ as $n, m \to \infty$ and $\{D[f_n]\}$ is bounded. By the condition (LD) there exists a constant $c > 0$ such that

$$D[\Delta f_n - \Delta f_m] \leq cD[f_n - f_m] \to 0 \quad (n, m \to \infty).$$

Thus $\|\Delta f_n - \Delta f_m\|_2 \to 0$ as $n, m \to \infty$. Therefore $\{\Delta f_n\}$ is a Cauchy sequence in $\mathbf{D}_0(N)$. We can find $\varphi \in \mathbf{D}_0(N)$ such that $\|\Delta f_n - \varphi\|_2 \to 0$ as $n \to \infty$. Since $\{f_n(x)\}$ converges pointwise to $u(x)$, it follows that $\{\Delta f_n(x)\}$ converges pointwise to $\Delta u(x)$. Since $\{\Delta f_n(x)\}$ also converges pointwise to $\varphi(x)$ and $\{D(\Delta f_n)\}$ is bounded, we see that $\Delta u = \varphi \in \mathbf{D}_0(N)$ by [5, Theorem 4.1]. $\square$

Theorem 6.2 *Assume that N satisfies the condition* (LD). *Then $d_B^{**}(a,\infty) = d_B^{*}(a,\infty)$. If $d_B^{**}(a,\infty) < \infty$, then dq_a is a unique optimal solution to the problem* $(**)$.

Proof. Since $d_B^{**}(a,\infty) \leq d_B^{*}(a,\infty)$, we shall show that $d_B^{**}(a,\infty) \geq d_B^{*}(a,\infty)$. We may assume that $d_B^{**}(a,\infty) < \infty$. Let w^{**} and v^{**} be the same as in Theorem 6.1. By Lemma 6.1, we see that $\Delta v^{**} \in \mathbf{D}_0(N)$. This means that $w^{**} = dv^{**}$ is a feasible solution to the problem $(*)$. We have $d_B^{*}(a,\infty) \leq H[w^{**}] = d_B^{**}(a,\infty)$.

Assume that $d_B^{**}(a,\infty) < \infty$. Then N is hyperbolic by Corollary 5.1. Let $f' = q_a - v^{**}$. Since $q_a \in \mathbf{D}_0(N)$ by Theorem 5.1, it follows that $f' \in \mathbf{D}_0(N)$ and $\Delta f' = \Delta q_a - \Delta v^{**} = -g_a + g_a = 0$, so that $f' \in \mathbf{D}_0(N) \cap \mathbf{HD}(N)$. Hence $f' = 0$. This means that $dq_a = dv^{**}$ is a unique optimal solution to the problem $(**)$. □

7. An example

We show an example of $w \in \mathbf{BF}(a,\infty)$ for the following network:

Example 7.1 Let $X = \{x_n; n \geq 0\}, Y = \{y_n; n \geq 1\}$, $e(y_n) = \{x_{n-1}, x_n\}$ for $n \geq 1$. Let $K(x_n, y_n) = 1$, $K(x_{n-1}, y_n) = -1$ for $n \geq 1$ and $K(x,y) = 0$ for any other pairs. For a strictly positive function r on Y, $N = \{X, Y, K, r\}$ is an infinite network.

Let $r_n = r(y_n)$, $R_n = \sum_{k=n+1}^{\infty} r_k$ and $\rho_n = \sum_{k=1}^{n} r_k$. We assume that $\rho := \sum_{n=1}^{\infty} r_n < \infty$. Then it is easy to see that

$$g_{x_k}(x_n) = R_n \ (0 \leq k \leq n), \quad g_{x_k}(x_n) = R_k \ (k > n).$$

Let w be a feasible solution to the problem $(**)$ with $a = x_0$ and let $v = \partial w$. Let $w_n = w(y_n)$ for $n \geq 1$. Let $v_n = v(x_n)$ for $n \geq 0$. We have

$$B_r w(y_n) = \frac{1}{r(y_n)} \sum_{x \in X} K(x, y_n) \partial w(x) = \frac{1}{r_n}(v_n - v_{n-1}),$$

$$\partial B_r w(x_0) = \sum_{y \in Y} K(x_0, y) B_r w(y) = -B_r w(y_1) = -\frac{1}{r_1}(v_1 - v_0),$$

$$\begin{aligned} \partial B_r w(x_n) &= \sum_{y \in Y} K(x_n, y) B_r w(y) = B_r w(y_n) - B_r w(y_{n+1}) \\ &= \frac{1}{r_n}(v_n - v_{n-1}) - \frac{1}{r_{n+1}}(v_{n+1} - v_n). \end{aligned}$$

Since $\partial B_r w(x_0) = -1$ and $\partial B_r w(x_n) = 0$ for $n \geq 1$, it follows that $r_n^{-1}(v_n - v_{n-1}) = 1$. Thus

$$v_n = \rho_n + v_0.$$

From

$$v_n = \sum_{y \in Y} K(x_n, y)w(y) = w_n - w_{n+1} \ (n \geq 1), \quad v_0 = -w_1,$$

it follows that $w_n - w_{n+1} = \rho_n + v_0$, and that

$$w_n = -\sum_{k=1}^{n-1} \rho_k - (n-1)v_0 + w_1 = -\sum_{k=1}^{n-1} \rho_k - nv_0.$$

Let

$$A_n = \sum_{k=1}^{n-1} \rho_k, \quad \alpha = \sum_{n=1}^{\infty} n^2 r_n, \quad \beta = \sum_{n=1}^{\infty} n r_n A_n, \quad \gamma = \sum_{n=1}^{\infty} r_n A_n^2.$$

Then

$$H[w] = \sum_{n=1}^{\infty} r_n w_n^2 = \sum_{n=1}^{\infty} r_n(-A_n - nv_0)^2 = \alpha v_0^2 + 2\beta v_0 + \gamma. \tag{2}$$

Now let w' be a feasible solution to the problem $(*)$. In a similar way we let $w'_n = w'(y_n)$ and $v'_n = v'(x_n) = \partial w'(x_n)$ and obtain

$$\begin{aligned} v'_n &= \rho_n + v'_0, \\ w'_n &= -\sum_{k=1}^{n-1} \rho_k - nv'_0 = -A_n - nv'_0. \end{aligned}$$

Since $v' \in \mathbf{D}_0(N)$, we have $\lim_{n\to\infty} v'_n = 0$, or $v'_0 = -\rho$. Therefore

$$w'_n = -A_n + n\rho. \tag{3}$$

Since $\rho = R_0$ and $\rho_k = R_0 - R_k$ for $k \geq 1$, we have

$$w'_n = -\sum_{k=1}^{n-1}(R_0 - R_k) + nR_0 = \sum_{k=0}^{n-1} R_k. \tag{4}$$

Notice that this is a unique feasible solution to the problem $(*)$. By (3)

$$d_B^*(a,\infty) = H[w'] = \sum_{n=1}^{\infty} r_n(-A_n + n\rho)^2 = \alpha\rho^2 - 2\beta\rho + \gamma.$$

(a) Assume that all of α, β, γ converge. First we note that $\alpha\rho > \beta$. Indeed,

$$A_n = \sum_{k=1}^{n-1} \rho_k = \sum_{k=1}^{n-1} \sum_{j=1}^{k} r_j = \sum_{j=1}^{n-1} (n-j) r_j < n \sum_{j=1}^{n} r_j = n\rho_n,$$

and that

$$\beta = \sum_{n=1}^{\infty} n r_n A_n < \sum_{n=1}^{\infty} n^2 r_n \rho_n < \sum_{n=1}^{\infty} n^2 r_n \rho = \alpha\rho.$$

Now (2) is minimized at $v_0 = -\beta/\alpha$, so that

$$d_B^{**}(a,\infty) = \gamma - \frac{\beta^2}{\alpha}.$$

It follows that

$$d_B^*(a,\infty) - d_B^{**}(a,\infty) = \alpha\rho^2 - 2\beta\rho + \frac{\beta^2}{\alpha} = \alpha\left(\rho - \frac{\beta}{\alpha}\right)^2 > 0.$$

Theorem 6.2 implies that N does not satisfy the condition (LD).

(b) Taking $r_n = n^{-5/3}$ for $n \geq 1$, since $R_n = O(n^{-2/3})$, by (4) we have $w'_n = O(n^{1/3})$, and that $H[w'] = O(\sum_{n=1}^{\infty} n^{-5/3}(n^{1/3})^2) = \infty$. This means $d_B^*(a,\infty) = \infty$. On the other hand the bi-harmonic Green function q_a is given by

$$q_a(x_n) = \sum_{k=0}^{\infty} g_a(x_k) g_{x_k}(x_n) = \sum_{k=0}^{n} R_k R_n + \sum_{k=n+1}^{\infty} R_k^2 = O(n^{-1/3}).$$

Thus $q_a \in L(X)$ does not imply $d_B^*(a,\infty) < \infty$.

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