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A construction of special Lagrangian 3-folds via the generalized Weierstrass representation

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Abstract. We show that certain holomorphic loop algebra-valued 1-forms over Riemann surfaces yield minimal Lagrangian immersions into the complex 2-dimensional projective space via the Weierstrass type representation, hence 3-dimensional special Lagrangian submanifolds of $\mathbb{C}^3$. A particular family of 1-forms on $\mathbb{C}$ corresponds to solutions of the third Painlevé equation which are smooth on $(0, +\infty)$.

Key words: special Lagrangian submanifold, harmonic map, Painlevé equation.

1. Introduction

The study of special Lagrangian submanifolds related with integrable systems arose in [13], where Joyce gave an explicit construction of special Lagrangian submanifolds in $\mathbb{C}^3$ with the rich developed theory of harmonic maps relating to integrable systems. In the present paper we provide another construction of special Lagrangian submanifolds in $\mathbb{C}^3$ based on [13], using the generalized Weierstrass representation.

We begin in Section 2 with a brief survey of harmonic maps $\psi : S \to \mathbb{CP}^n$ of a Riemann surface S into the complex projective space $\mathbb{CP}^n$ and a related integrable system. Harmonic maps from a Riemann surface have been well-studied. Each harmonic map $\psi : S \to \mathbb{CP}^n$ defines a sequence $\{\psi_k\}$ of harmonic maps called the *harmonic sequence* [4], [23]. In particular, *superconformal* harmonic maps $\psi : S \to \mathbb{CP}^n$, whose harmonic sequences are periodic with period $n + 1$, correspond to solutions of the *affine Toda equations* for $\mathrm{SU}(n + 1)$ [3].

We can apply this to special Lagrangian submanifolds in $\mathbb{C}^3$. *Special Lagrangian submanifolds* are minimal Lagrangian submanifolds in Calabi-Yau manifolds, originally defined in *calibrated geometry* [11]. A large amount

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of research on them has been motivated by the mirror symmetry conjecture of Strominger, Yau and Zaslow [20]. Section 3 gives a review of special Lagrangian geometry and its relationship with harmonic maps and integrable systems building on [16], [19], and pointed out in [13]. A *special Lagrangian cone* in $\mathbb{C}^{n+1}$ has intersection Σ with the $(2n+1)$-dimensional sphere S^{2n+1} which is a minimal Legendrian submanifold of S^{2n+1}. The image $\pi(\Sigma)$ of the natural projection $\pi : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is minimal Lagrangian, therefore a minimal Lagrangian immersion $\psi : \Sigma \to \mathbb{C}\mathrm{P}^n$ is obtained. The converse statement is also true locally. Adapting Section 2 to the case of $n = 2$, the affine Toda equations for $\mathrm{SU}(3)$ derived from a minimal Lagrangian immersion $\psi : \Sigma \to \mathbb{C}\mathrm{P}^2$ reduce to an equation called the *Tzitzéica equation.*

In Section 4 we reformulate the *generalized Weierstrass representation* [8], a method to construct harmonic maps $\psi : S \to G/K$ of a simply-connected Riemann surface into a compact symmetric space through *loop groups.* Every harmonic map $\psi : S \to G/K$ can be obtained from a class of holomorphic 1-forms on S called *holomorphic potentials.* Here we consider the scheme where the target space is a homogeneous space following [6].

The generalized Weierstrass representation can be applied to the framework in Section 3. We give in Section 5 the condition for holomorphic potentials to yield minimal Lagrangian immersions $\psi : S \to \mathbb{C}\mathrm{P}^2$, hence special Lagrangian cones in $\mathbb{C}^3$.

In Section 6 we specialize to holomorphic potentials of the form

$$\mu = \frac{1}{\lambda}\begin{pmatrix} & & pz^k \\ pz^k & & \\ & qz^l & \end{pmatrix} \mathrm{d}z,$$

where k, l are nonnegative integers and p, q are nonzero complex numbers. They produce a family of special Lagrangian cones over the entire complex plane $\mathbb{C}$ and the corresponding solutions of the Tzitzéica equation are invariant under coordinate changes by rotation. In this situation the Tzitzéica equation reduces to a special case where the complex constants β, γ of the third Painlevé equation for $|z|$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

are zero and our solutions are smooth on $(0, +\infty)$. These special solutions of the third Painlevé equation are known [2], but their relationship to special Lagrangian cones seems to be new.

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2. Harmonic maps and integrable systems

2.1. Harmonic maps into homogeneous spaces

A *harmonic map* $\psi : M \to N$ is a smooth map of Riemannian manifolds which is a critical point of the *energy functional*

$$E(\psi, D) = \frac{1}{2} \int_D |\mathrm{d}\psi(x)|^2 v_M \tag{2.1}$$

for all smooth variations ψ_t of ψ supported in a compact domain D, where v_M is the volume form of M. The map ψ is harmonic if and only if its *tension field*

$$\tau(\psi) = \mathrm{tr}(\nabla \mathrm{d}\psi) \tag{2.2}$$

vanishes identically. Here ∇ denotes the connection on $T^*M \otimes \psi^{-1}TN$ induced by the Levi-Civita connections on M and N. The equation

$$\tau(\psi) = 0$$

is the Euler-Lagrange equation for the variation (2.1).

If N is a reductive homogeneous space, the associated Euler-Lagrange equation can be written more explicitly. We recall some facts on harmonic maps into Lie groups briefly [22]. Let G be a Lie group with its Lie algebra $\mathfrak{g}$ and θ the (left) *Maurer-Cartan form* on G, a left-invariant $\mathfrak{g}$-valued 1-form. It can be verified that θ satisfies the *Maurer-Cartan equation*

$$\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta] = 0, \tag{2.3}$$

here $[\theta \wedge \theta]$ is defined by $[\theta \wedge \theta](X, Y) = 2[\theta(X), \theta(Y)]$ for $X, Y \in T_gG$ and $g \in G$.

Let $\psi : M \to G$ be a smooth map of a Riemannian manifold. A $\mathfrak{g}$-valued 1-form α on M, defined by $\alpha = \psi^*\theta$, also satisfies the Maurer-Cartan equation

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0, \tag{2.4}$$

by pulling back (2.3). This fact shows that the connection $\mathrm{d} + \alpha$ on the trivial principal G-bundle over M is flat. Now ψ is harmonic if and only if

$$\mathrm{d}^*\alpha = 0. \tag{2.5}$$

The above formulation can be extended to the case of a harmonic map into a reductive homogeneous space (cf. [6], [24]). Let $N = G/K$ be a reductive homogeneous space, where G is a Lie group and K is a closed subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{k}$ the Lie algebra of K, and the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. The map $\mathfrak{g} \to T_xN$, defined by

$$\xi \mapsto \frac{d}{dt}(\exp t\xi) \cdot x \bigg|_{t=0}$$

at each $x = gK \in N$, is surjective and then induces an isomorphism $\mathrm{Ad}(g)(\mathfrak{m}) \to T_xN$. It easily leads to a bundle isomorphism $[\mathfrak{m}] \to TN$ of the subbundle $[\mathfrak{m}] = G \times_K \mathfrak{m}$ in the trivial bundle $N \times \mathfrak{g}$. Its inverse map $\beta : TN \to [\mathfrak{m}]$ can be considered as a $\mathfrak{g}$-valued 1-form on N at each $x \in N$, and it is again called the Maurer-Cartan 1-form on N [6]. As in the case of G, one can show that a smooth map $\psi : M \to N$ of a Riemannian manifold is harmonic if and only if

$$\mathrm{d}^*\psi^*\beta = 0. \tag{2.6}$$

We can also interpret the harmonicity of $\psi : M \to N$ in terms of a *frame*, which is a map $F : M \to G$ satisfying $\pi \circ F = \psi$. Here $\pi : G \to N = G/K$ is the coset projection (or the homogeneous projection) [6]. Let $\psi : M \to N$ be a smooth map and $F : M \to G$ a frame of ψ, and define $\alpha = F^{-1}\mathrm{d}F$. The reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ gives a decomposition

$$\alpha = \alpha_0 + \alpha_1,$$

where α_0, α_1 are respectively $\mathfrak{k}$-valued, $\mathfrak{m}$-valued 1-forms on M. Note that

$\psi^*\beta = \mathrm{Ad}(F)(\alpha_1)$, and then it can be verified that (2.6) is equivalent to

$$\mathrm{d} * \alpha_1 + [\alpha \wedge *\alpha_1] = 0. \tag{2.7}$$

Suppose that M is a Riemann surface and $\alpha_1 = \alpha_1' + \alpha_1''$ is a decomposition according to types $T^{\mathbb{C}}M = T'M \oplus T''M$. If

$$[\alpha_1' \wedge \alpha_1''] \text{ takes values only in } \mathfrak{k}, \tag{2.8}$$

the condition for $\psi : M \to N$ to be harmonic reduces to

$$\begin{aligned} &\mathrm{d}\alpha_1' + [\alpha_0 \wedge \alpha_1'] = \mathrm{d}\alpha_1'' + [\alpha_0 \wedge \alpha_1''] = 0, \\ &\mathrm{d}\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] + [\alpha_1' \wedge \alpha_1''] = 0. \end{aligned} \tag{2.9}$$

This reduction holds whenever N is a symmetric space since (2.8) holds. Conversely, if M is simply-connected, α_0 respectively α_1 are $\mathfrak{k}$- respectively $\mathfrak{m}$-valued 1-forms on M satisfying (2.8) and (2.9), then there exists a unique harmonic map $\psi = \pi \circ F : M \to N$ where $F : M \to G$ with $F^{-1}\mathrm{d}F = \alpha_0 + \alpha_1$ up to the left translation of G.

2.2. Primitive maps

Suppose that G is a compact semisimple Lie group, $\sigma : G \to G$ is an automorphism with order k, $\zeta = e^{2\pi i/k}$ is the primitive k-th root of unity, and K is the fixed point set of σ. Then the coset space G/K is a reductive homogeneous space, called a *k-symmetric space* [15].

The automorphism $\sigma : G \to G$ induces an automorphism on the Lie algebra $\mathfrak{g}$ of G, which we denote again by σ. Note that the fixed point set of σ coincides with the Lie algebra $\mathfrak{k}$ of K. We can see that σ gives an eigenspace decomposition of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$:

$$\mathfrak{g}^{\mathbb{C}} = \sum_{j \in \mathbb{Z}_k} \mathfrak{g}_j \tag{2.10}$$

where $\mathfrak{g}_j$ is the ζ^j-eigenspace of σ. For the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, it is easy to check

$$\mathfrak{g}_{-j} = \bar{\mathfrak{g}_j}, \ \mathfrak{k}^{\mathbb{C}} = \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{m}^{\mathbb{C}} = \sum_{j \in \mathbb{Z}_k \setminus \{0\}} \mathfrak{g}_j.$$

As K preserves $\mathfrak{g}_j$, we have a decomposition

$$T^{\mathbb{C}}(G/K) = \bigoplus_{j \in \mathbb{Z}_k \setminus \{0\}} [\mathfrak{g}_j]$$

of the complexification of $T(G/K)$. Here $[\mathfrak{g}_j]$ is a subbundle of the trivial $\mathfrak{g}^{\mathbb{C}}$-bundle over G/K where $[\mathfrak{g}_j]_{gK} = \mathrm{Ad}(g)(\mathfrak{g}_j)$. Let S be a Riemann surface, z a complex local coordinate of S and $N = G/K$ a k-symmetric space for $k \geq 3$. A smooth map $\phi : S \to N$ is called *primitive* if

$$\mathrm{d}\phi\left(\frac{\partial}{\partial z}\right) \in [\mathfrak{g}_1],$$

which is equivalent to

$$F^{-1}\frac{\partial F}{\partial z} \in \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{2.11}$$

for a frame $F : S \to G$ of ϕ. We say that a map into a homogeneous space G/H is *equiharmonic* if it is harmonic with respect to any G-invariant metric on G/H. By [1], primitive maps are equiharmonic, and also project to equiharmonic maps under any homogeneous projection.

2.3. The affine Toda equations

The *affine Toda* (or the *Toda lattice*) *equations* are well-known as an integrable system related to soliton theory. The affine Toda equations for $\mathrm{SU}(n+1)$ are the following non-linear partial differential equations:

$$2\frac{\partial^2 w_k}{\partial z \partial \bar{z}} = e^{2(w_{k+1}-w_k)} - e^{2(w_k - w_{k-1})}, \tag{2.12}$$

where $w_k : U \to \mathbb{R}$ for $k \in \mathbb{Z}$ are smooth functions satisfying

$$w_{k+n+1} = w_k,$$
$$w_0 + \cdots + w_n = 0.$$

Here U is an open subset of a Riemann surface S and z is a complex coordinate of U. More generally, they can be given for an arbitrary compact simple Lie group although we will deal with only the case of $\mathrm{SU}(n+1)$ in this paper.

Suppose that $\phi : S \to \mathrm{SU}(n+1)/T$ is a smooth map of a Riemann surface, where T is the standard maximal torus of $\mathrm{SU}(n+1)$ with its Lie algebra $\mathfrak{t}$. A (local) frame $F : U \to \mathrm{SU}(n+1)$ is said to be a *Toda frame* if there exist a complex coordinate z on U and a smooth map $W : U \to i\mathfrak{t}$ such that

$$F^{-1}\frac{\partial F}{\partial z} = \frac{\partial W}{\partial z} + \mathrm{Ad}(\exp W)(B) \tag{2.13}$$

where

$$B = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} \in \mathfrak{su}(n+1)^{\mathbb{C}}.$$

Here W may be written as $W = \mathrm{diag}(w_0, \ldots, w_n)$ with smooth functions $w_k : U \to \mathbb{R}$.

Toda frames can be interpreted as frames of primitive maps $\phi : S \to \mathrm{SU}(n+1)/T$. Defining an automorphism ν on $\mathrm{SU}(n+1)$ by $\nu = \mathrm{Ad}(\mathrm{diag}(1, \zeta, \ldots, \zeta^n))$ with $\zeta = e^{2\pi i/n}$, then there is a decomposition

$$\mathfrak{su}(n+1)^{\mathbb{C}} = \sum_{j \in \mathbb{Z}_n} \mathfrak{g}_j$$

where $\mathfrak{g}_j$ is the ζ^j-eigenspace of ν. Note that $\mathfrak{g}_0$ is the complexification $\mathfrak{t}^{\mathbb{C}}$ of $\mathfrak{t}$. In particular, $\mathfrak{g}_1$ is written as

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} & & & a_{1,n+1} \\ a_{2,1} & & & \\ & \ddots & & \\ & & a_{n+1,n} & \end{pmatrix} \middle| \, a_{1,n+1}, a_{2,1}, \ldots, a_{n+1,n} \in \mathbb{C} \right\}.$$

An element of $\mathfrak{g}_1$ will be called *cyclic* if $a_{1,n+1}, a_{2,1}, \ldots, a_{n+1,n}$ are in $\mathbb{C}\backslash\{0\}$, and we call an element $[\widehat{\xi}]$ of $[\mathfrak{g}_1]$ *cyclic* when there exists a cyclic element ξ of $\mathfrak{g}_1$ such that $[\widehat{\xi}]_{gT} = \mathrm{Ad}(g)(\xi)$.

According to [8], under an appropriate coordinate change, a primitive map $\phi : S \to \mathrm{SU}(n+1)/T$ gives a Toda frame F of ϕ around $p \in S$ such that

$\mathrm{d}\phi(\partial/\partial z)$ is cyclic at p. Moreover, the smooth functions $w_0, \ldots, w_n$, which have the form $W = \mathrm{diag}(w_0, \ldots, w_n)$ in (2.13), are solutions of (2.12).

Given a solution $w_0, \ldots, w_n$ of (2.12) conversely, there exists a local $\mathrm{SU}(n+1)$-valued solution F of (2.13) for $W = \mathrm{diag}(w_0, \ldots, w_n)$, and $\phi = \pi \circ F : U \to \mathrm{SU}(n+1)/T$ is primitive. Therefore, F is a Toda frame of ϕ.

2.4. The harmonic sequences

Solutions of (2.12) can be also constructed from *harmonic sequences* [3]. Let S be a connected Riemann surface, z a local complex coordinate on S and $\psi : S \to \mathbb{C}\mathrm{P}^n$ a harmonic map.

Now suppose that L_0 is a complex line subbundle of the trivial bundle $S \times \mathbb{C}^{n+1}$ determined by $\psi : S \to \mathbb{C}\mathrm{P}^n$, where a section s of L is said to be holomorphic if $\partial s/\partial \bar{z}$ is orthogonal to L with respect to the standard Hermitian inner product $\langle\, , \rangle$ on $\mathbb{C}^{n+1}$. By the bundle maps $(1,0)$- and $(0,1)$-part of $\mathrm{d}\psi$, we can obtain a sequence $\{L_k\}$ of complex line subbundles L_k of $S \times \mathbb{C}^{n+1}$ in which any two adjacent elements are orthogonal, and correspondingly, a sequence $\{\psi_k\}$ of harmonic maps $\psi_k : S \to \mathbb{C}\mathrm{P}^n$ with $\psi_0 = \psi$, which we call the *harmonic sequence* associated to ψ [4], [23].

The harmonic sequence can be locally represented as follows. Let f_0 be a nowhere zero holomorphic local section of L_0, then there exists a sequence of meromorphic local sections f_k of L_k such that

$$\begin{aligned} &\langle f_k, f_l \rangle = 0, \\ &\frac{\partial f_k}{\partial z} = f_{k+1} + \frac{\partial}{\partial z}(\log |f_k|^2) f_k \quad \text{and} \quad \frac{\partial f_k}{\partial \bar{z}} = -\frac{|f_k|^2}{|f_{k-1}|^2} f_{k-1}. \end{aligned} \tag{2.14}$$

From the construction, it follows that adjacent elements f_k and f_{k+1} are orthogonal for all $k \in \mathbb{Z}$. We also point out that the sequence $\{f_k\}$ is unique up to multiplication by nonzero holomorphic functions.

A harmonic map $\psi : S \to \mathbb{C}\mathrm{P}^n$ will be called *superminimal* or (*complex*) *isotropic* if its harmonic sequence is finite. Such maps are well-understood as they arise from holomorphic maps. It is known that every harmonic map $\psi : S \to \mathbb{C}\mathrm{P}^n$ is superminimal if $S = S^2$ [9].

We shall consider the orthogonality of the harmonic sequence. Maps $\psi_k, \psi_l : S \to \mathbb{C}\mathrm{P}^n$ are said to be *orthogonal* when the determined lines in $\mathbb{C}^{n+1}$ are orthogonal at each point of S. It is shown in [4] that if some k consecutive elements in the harmonic sequence are mutually orthogonal then every k consecutive elements in the sequence are mutually orthogonal.

A harmonic map $\psi : S \to \mathbb{C}\mathrm{P}^n$ is called *k-orthogonal* if some (therefore, every) k consecutive maps in the harmonic sequence are mutually orthogonal. Any harmonic map $\psi : S \to \mathbb{C}\mathrm{P}^n$ is 2-orthogonal, and ψ is conformal if and only if ψ is 3-orthogonal. For dimensional reasons, ψ is at most $(n+1)$-orthogonal. Now we will introduce

Definition 2.1 A harmonic map $\psi : S \to \mathbb{C}\mathrm{P}^n$ is *superconformal* if ψ is $(n+1)$-orthogonal, and not superminimal.

The harmonic sequence for a superconformal harmonic map $\psi : S \to \mathbb{C}\mathrm{P}^n$ has periodicity with period $n+1$. Note that every harmonic map $\psi : S \to \mathbb{C}\mathrm{P}^1$ is either superminimal or superconformal, and so is every conformal map $\psi : S \to \mathbb{C}\mathrm{P}^2$; besides, a superconformal harmonic map $\psi : S \to \mathbb{C}\mathrm{P}^n$ for $n \geq 2$ is conformal, so that an immersion.

Suppose that $\psi : S \to \mathbb{C}\mathrm{P}^n$ is superconformal and $\{L_k\}$ is the corresponding sequence of complex line subbundles. Then, it is proved in [3] that the lift $\Phi : S \to \mathrm{SU}(n+1)/T$ defined by

$$\Phi = (L_0, \dots L_n)$$

is primitive, and conversely, a primitive map $\phi : S \to \mathrm{SU}(n+1)/T$ with a cyclic element $\mathrm{d}\psi(\partial/\partial z)$ projects to a superconformal map $\psi = \pi \circ \phi : S \to \mathbb{C}\mathrm{P}^n$, whose lift Φ equals ϕ.

We can see that ψ also gives solutions of (2.12) rather explicitly. Let z be a complex coordinate on a simply-connected open subset U of S, and $\{f_k\}$ a sequence which satisfies (2.14) over U with $\psi = [f_0]$. As the harmonic sequence of ψ is periodic, we can equip $\{f_k\}$ with the periodicity $f_{k+n+1} = f_k$ for any $k \in \mathbb{Z}$, by appropriate coordinate changes. Then it turns out that the functions $w_0, \dots, w_n$ defined by $e^{w_k} = |f_k|$ satisfy (2.12).

3. Special Lagrangian submanifolds in $\mathbb{C}^3$ and harmonic maps

3.1. Special Lagrangian submanifolds

We say that (M, J, ω, Ω) is a *Calabi-Yau manifold* if (M, J, ω) is an n-dimensional ($n \geq 2$) Kähler manifold with a Kähler form ω, and Ω is a nonzero holomorphic n-form on M satisfying

$$\frac{\omega^n}{n!} = (-1)^{n(n-1)/2} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}.$$

This is one of the definitions of Calabi-Yau manifolds. Note that Ω is uniquely determined up to multiplication by U(1) (in the sense of the above definition). Further, it is known that a Calabi-Yau manifold is Ricci-flat, and has its holonomy in SU(n). Here is the simplest example of Calabi-Yau manifolds, which we will deal with in the present paper.

Example 3.1 Let $(\mathbb{C}^n, J)$ be the n-dimensional complex Euclidean space with the standard complex structure J, ω a Kähler form and Ω a holomorphic n-form on $\mathbb{C}^n$ defined by

$$\omega = \frac{i}{2} \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j \quad \text{and} \quad \Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n,$$

where $(z_1, \dots z_n)$ is the standard coordinate system in $\mathbb{C}^n$. Then, (M, J, ω, Ω) is a Calabi-Yau manifold.

Definition 3.2 Let (M, J, ω, Ω) be an n-dimensional Calabi-Yau manifold. A real n-dimensional oriented submanifold N of M is called a *special Lagrangian submanifold* if

$$\operatorname{Re} \Omega|_N = \mathrm{vol}_N, \tag{3.1}$$

where vol_N is the induced volume form on N.

The condition (3.1) is equivalent to

$$\omega|_N = 0 \quad \text{and} \quad \operatorname{Im} \Omega|_N = 0,$$

so special Lagrangian submanifolds can be considered as Lagrangian submanifolds with the additional condition $\operatorname{Im} \Omega|_N = 0$. They are also minimal submanifolds; more generally, *calibrated submanifolds* [11]. For calibrated geometry, see [11], [14]. A good summary of special Lagrangian geometry may be found in [10].

3.2. Special Lagrangian cones and harmonic maps

We shall take immersed submanifolds into account henceforth. A singular submanifold C in $\mathbb{C}^n$ will be called a (real) *cone* if $0 \in C$ and $tC = C$ for any $t \in \mathbb{R}_+$. When C is a linear subspace of $\mathbb{C}^n$, which we call a *trivial* cone, it is nonsingular. Otherwise, each cone has a singular point at the origin.

Let C be a special Lagrangian cone in $\mathbb{C}^{n+1}$. Then the intersection $\Sigma = C \cap S^{2n+1}$ with S^{2n+1} turns out to be a minimal Legendrian submanifold in S^{2n+1}. Conversely, given a minimal Legendrian submanifold Σ of S^{2n+1}, we have a special Lagrangian cone

$$C(\Sigma) = \{tx \mid x \in \Sigma,\ t \in \mathbb{R}_+\} \cup \{0\}$$

in $\mathbb{C}^{n+1}$ (cf. [12]).

Suppose $\pi : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is the natural projection. Then every minimal Legendrian submanifold of S^{2n+1} is mapped by π onto a minimal Lagrangian submanifold of $\mathbb{C}\mathrm{P}^n$. The converse is also true, at least locally (cf. [17]).

We shall describe the above framework for $n = 2$ in terms of harmonic maps. Consider a special Lagrangian cone C in $\mathbb{C}^3$. Now Σ can be regarded as a 2-dimensional oriented submanifold with the induced metric and orientation, and hence a Riemann surface. Let $f : \Sigma \to S^5$ be the inclusion map. Clearly f is conformal and minimal, therefore a harmonic map from a Riemann surface. Furthermore, the composition map $\psi = \pi \circ f : \Sigma \to \mathbb{C}\mathrm{P}^2$ is a minimal Lagrangian immersion. Note that we may take an arbitrary complex local coordinate system of Σ by the fact that the condition (3.1) to be harmonic is invariant under conformal changes, then we obtain a minimal Lagrangian immersion $\psi : \Sigma \to \mathbb{C}\mathrm{P}^2$ from a Riemann surface Σ. The converse discussion enables us to construct a special Lagrangian cone in $\mathbb{C}^3$ from such a map.

We shall give a comment about trivial special Lagrangian cones. Suppose that C is a trivial special Lagrangian cone in $\mathbb{C}^3$. Its intersection Σ with S^5 can be identified with S^2, so that the induced minimal Lagrangian immersion $\psi : \Sigma \to \mathbb{C}\mathrm{P}^2$ is superminimal by the fact in Section 2.4. Conversely, if a minimal immersion $\psi : S \to \mathbb{C}\mathrm{P}^2$ from a Riemann surface S is superminimal Lagrangian, the special Lagrangian cone coming from a lift $f : S \to S^5$ of ψ can be identified with a real 3-dimensional subspace. This is because ψ is conformal then its harmonic sequence has exactly 3 elements, the maximum possible length. Note that the harmonic sequence of $\psi : S \to \mathbb{C}\mathrm{P}^n$ has length $n + 1$ if and only if ψ is *linearly full* (that is, if its image is in no proper complex projective subspace of $\mathbb{C}\mathrm{P}^n$). Then it turns out from [4, Theorem 3.6] that $\psi(S)$ is contained in $\mathbb{R}\mathrm{P}^2$, hence $C(S)$ is equivalent to $\mathbb{R}^3$. Recall that each conformal map $\psi : S \to \mathbb{C}\mathrm{P}^2$ is either superminimal or superconformal, then we deduce that a special Lagrangian cone C in $\mathbb{C}^3$ is nontrivial if and only if the corresponding minimal Lagrangian immersion $\psi : S \to \mathbb{C}\mathrm{P}^2$ is superconformal.

3.3. Superconformal minimal Lagrangian maps into $\mathbb{C}\mathrm{P}^2$

Suppose $\psi : S \to \mathbb{C}\mathrm{P}^2$ is a harmonic map of a Riemann surface S. Then there exist the harmonic sequence $\{\psi_k\}$ of ψ and a sequence $\{f_k\}$ of local sections of the corresponding sequence of bundles $\{L_k\}$. Let $z = x + iy$ be a local complex coordinate on an open subset U of S, $\langle\, ,\, \rangle$ the standard Hermitian inner product on $\mathbb{C}^3$, and ω the Fubini-Study form on $\mathbb{C}\mathrm{P}^2$. The induced form $\psi_k^*\omega$ on S is written as

$$\psi_k^*\omega = \left(\frac{|f_k|^2}{|f_{k-1}|^2} - \frac{|f_{k+1}|^2}{|f_k|^2}\right)\mathrm{d}x \wedge \mathrm{d}y,$$

which shows that each ψ_k has Lagrangian image if only if

$$\frac{|f_k|}{|f_{k-1}|} = \frac{|f_{k+1}|}{|f_k|}. \tag{3.2}$$

Let C be a nontrivial special Lagrangian cone in $\mathbb{C}^3$. As in Section 3.2, we have a superconformal Lagrangian immersion $\psi : S \to \mathbb{C}\mathrm{P}^2$ from a Riemann surface S and a minimal Legendrian immersion $f : S \to S^5$. Then

$$\left\langle f, \frac{\partial f}{\partial x}\right\rangle = 0 \quad \text{and} \quad \left\langle f, \frac{\partial f}{\partial y}\right\rangle = 0$$

because f is perpendicular to the tangent space of S^5, and

$$\omega\left(f, \frac{\partial f}{\partial x}\right) = 0 \quad \text{and} \quad \omega\left(f, \frac{\partial f}{\partial y}\right) = 0$$

since C is a Lagrangian submanifold of $\mathbb{C}^3$. Then

$$\left\langle f, \frac{\partial f}{\partial \bar{z}}\right\rangle = 0, \tag{3.3}$$

so that f is a holomorphic section of the line bundle L_0 coming from $\psi_0 = \psi$. Now there is a sequence $\{f_k\}_{k\in\mathbb{Z}}$ with $f_0 = f$ defined by (2.14) satisfying

$$|f_0| \equiv 1, \tag{3.4}$$

and

$$\frac{1}{|f_{-1}|} = |f_1| \tag{3.5}$$

by (3.2) and (3.4). More specifically, we have

$$\left|\frac{\partial f}{\partial z}\right| = \left|\frac{\partial f}{\partial \bar{z}}\right| \tag{3.6}$$

from (2.14) and (3.5). Note that (3.6) is equivalent to

$$\omega\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 0, \tag{3.7}$$

the other condition for C to be Lagrangian.

Therefore, we have seen that superconformal minimal maps $\psi : S \to \mathbb{C}\mathrm{P}^2$ coming from special Lagrangian cones in $\mathbb{C}^3$ are locally characterised by its lifts $f : S \to S^5$ satisfying (3.3) and (3.7).

3.4. Primitive maps for special Lagrangian cones in $\mathbb{C}^3$

We shall now describe the relationship between special Lagrangian cones in $\mathbb{C}^3$ and certain primitive maps [17, Proposition 2]. Let $\psi : S \to \mathbb{C}\mathrm{P}^2$ be superconformal, $\{\psi_k\}$ with $\psi_0 = \psi$ the harmonic sequence of ψ, $\{L_k\}$ the corresponding sequence of complex line bundles, and $\{f_k\}$ the associated sequence of local sections on an open subset U of S satisfying (2.14). As mentioned before, $\{f_k\}$ is unique up to multiplication by nowhere zero holomorphic functions on U. We may assume

$$f_{k+3} = f_k \tag{3.8}$$

for any $k \in \mathbb{Z}$ by a suitable holomorphic coordinate change and $|f_0| \equiv 1$ from now on.

Define the family

$$\mathfrak{F} = \{(p, V) \in S^5 \times \mathrm{Gr}_2(\mathbb{C}^3) \mid p \in V\}. \tag{3.9}$$

Here $\mathrm{Gr}_2(\mathbb{C}^3)$ is the complex Grassmann manifold of complex 2-planes in $\mathbb{C}^3$. It is easy to see that SU(3) acts on $\mathfrak{F}$ transitively, and the isotropy subgroup of $V \in \mathfrak{F}$ is of the form

$$\{\mathrm{diag}(1, e^{i\theta}, e^{-i\theta}) \mid \theta \in [0, 2\pi]\},$$

which we denote by K. Then $\mathfrak{F}$ is isomorphic to $\mathrm{SU}(3)/K$, and therefore identifying $\mathfrak{F}$ with $\mathrm{SU}(3)/K$ allows us to define a map $\Psi : U \to \mathrm{SU}(3)/K$ by

$$\Psi(z) = (f_0(z), V(z)) \tag{3.10}$$

for $\{f_k\}$. Here $V(z)$ is a complex 2-dimensional subspace in $\mathbb{C}^3$ containing $L_1(z)$ and $L_2(z)$, the fibres of L_1 and L_2 over U, where z is some complex coordinate on U. We can also consider $\mathrm{SU}(3)/K$ as a homogeneous space determined by an automorphism $\sigma = \xi \circ \nu$ of SU(3), where ν and ξ are automorphisms of SU(3) defined by

$$\nu(g) = \mathrm{Ad}(T_1)(g) \quad \text{and} \quad \xi(g) = T_2 (g^{-1})^t {T_2}^{-1}$$

for $g \in \mathrm{SU}(3)$. Here $T_1 = \mathrm{diag}(1, \zeta_3, \zeta_3^2)$ for $\zeta_3 = e^{2\pi i/3}$ and

$$T_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}.$$

This framework can be explained from a twistorial viewpoint. For the detail, see [17].

As in section 2.2, there is an eigenvalue decomposition

$$\mathfrak{su}(3)^{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C}) = \sum_{j \in \mathbb{Z}_5} \widehat{\mathfrak{g}}_j$$

where $\widehat{\mathfrak{g}}_j$ is the $(-\zeta_3)^j$-eigenspace of σ. In particular, we have

$$\widehat{\mathfrak{g}}_0 = \left\{ \begin{pmatrix} 0 & & \\ & \alpha & \\ & & -\alpha \end{pmatrix} \middle| \ \alpha \in \mathbb{C} \right\} \quad \text{and} \quad \widehat{\mathfrak{g}}_1 = \left\{ \begin{pmatrix} & & \alpha \\ \alpha & & \\ & \beta & \end{pmatrix} \middle| \ \alpha, \beta \in \mathbb{C} \right\}.$$

Suppose Ψ is primitive. Then

$$\mathrm{d}\Psi\left(\frac{\partial}{\partial z}\right) \in [\widehat{\mathfrak{g}}_1],$$

which is equivalent to

$$F^{-1}\frac{\partial F}{\partial z} \in \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_1 \tag{3.11}$$

for a frame F of Ψ. Note that a frame $F : U \to \mathrm{SU}(3)$ of Ψ may be written as

$$F = (f_0, F_1, F_2),$$

where

$$F_1 = \lambda_1 \frac{f_1}{|f_1|} \quad \text{and} \quad F_2 = \lambda_2 \frac{f_2}{|f_2|}$$

for some function $\lambda_j : U \to \mathbb{C}$ with $|\lambda_j| \equiv 1$ for $j = 1, 2$. A straightforward calculation shows that the condition for F to satisfy (3.11) is

$$\left\langle f_0, \frac{\partial F_2}{\partial z} \right\rangle = \left\langle F_1, \frac{\partial f_0}{\partial z} \right\rangle, \tag{3.12}$$

while

$$\left\langle F_1, \frac{\partial F_1}{\partial z} \right\rangle + \left\langle F_2, \frac{\partial F_2}{\partial z} \right\rangle = 0$$

is the necessary condition to be a frame. Further, (3.12) can be rewritten as

$$\frac{\lambda_2}{|f_2|} = \bar{\lambda}_1 |f_1|$$

and by $|\lambda_1| = |\lambda_2| = 1$,

$$\frac{1}{|f_2|} = |f_1|.$$

Hence ψ is a minimal Lagrangian immersion by (3.2), so we deduce

Proposition 3.3 *Let $\psi : S \to \mathbb{C}\mathrm{P}^2$ be superconformal. If the map $\Psi : U \to SU(3)/K$ defined by* (3.10) *is primitive, then ψ is a minimal Lagrangian immersion.*

Conversely, suppose that ψ has Lagrangian image. By (2.14), we see that

$$\frac{\partial}{\partial \bar{z}} \det(f_0, f_1, f_2) = 0$$

for $\{f_k\}$, so that $D = \det(f_0, f_1, f_2)^{-1/3}$ is a holomorphic function on U. On the other hand,

$$|\det(f_0, f_1, f_2)| = \left| \det\left(f_0, \frac{f_1}{|f_1|}, \frac{f_2}{|f_2|} \right) \right| = 1$$

by $|f_0| \equiv 1$ and (3.2), hence $|D| \equiv 1$. Thus, we can replace f_k by

$$\widehat{f_k} = D f_k.$$

Define $\Psi = (\widehat{f_0}, L_1, L_2)$ and write a frame $\widehat{F}$ of Ψ as

$$\widehat{F} = (\widehat{f_0}, \widehat{F_1}, \widehat{F_2})$$

where

$$\widehat{F_1} = \widehat{\lambda}_1 \frac{\widehat{f_1}}{|\widehat{f_1}|} \quad \text{and} \quad \widehat{F_2} = \widehat{\lambda}_2 \frac{\widehat{f_2}}{|\widehat{f_2}|}$$

for some function $\widehat{\lambda}_j$ on U for $j = 1, 2$. From $|\widehat{f_0}| \equiv 1$ and (3.2), we have

$$\begin{aligned} 1 = \det \widehat{F} &= \widehat{\lambda}_1 \widehat{\lambda}_2 |\widehat{f_1}|^{-1} |\widehat{f_2}|^{-1} \det(\widehat{f_0}, \widehat{f_1}, \widehat{f_2}) \\ &= \widehat{\lambda}_1 \widehat{\lambda}_2 D^3 \det(f_0, f_1, f_2) \\ &= \widehat{\lambda}_1 \widehat{\lambda}_2, \end{aligned} \tag{3.13}$$

and then (3.12) holds for $\widehat{F}$. We summarize

Proposition 3.4 *Let $\psi : S \to \mathbb{C}\mathrm{P}^2$ be a superconformal minimal La-*

grangian immersion. Then there exists some $\{f_k\}$ *which defines* $\Psi : U \to SU(3)/K$ *to be primitive.*

3.5. Special Lagrangian cones in $\mathbb{C}^3$ and the Tzitzéica equation

In Section 2.4 we have shown that a superconformal map $\psi : S \to \mathbb{C}\mathrm{P}^2$ gives solutions of (2.12). For a holomorphic section f_0 of L_0 such that $|f_0| \equiv 1$, (2.12) reduces to

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = e^{-2w} - e^{w}, \tag{3.14}$$

which is called the *Tzitzéica equation* [21]. It is also known as the Bullough-Dodd equation [5].

4. The generalized Weierstrass representation

4.1. Loop groups

A *loop group* is a space of maps from S^1 into a Lie group, to which both algebraic and analytic techniques can be applied. In this section we shall provide several loop groups for the generalized Weierstrass representation. More information can be found in [18].

Suppose that G is a compact semisimple Lie group with its Lie algebra $\mathfrak{g}$, σ is an automorphism of G with order k, K is the σ-fixed subgroup of G. Let σ again denote the automorphism on $\mathfrak{g}$ induced by σ. Then the Lie algebra $\mathfrak{k}$ of K is the fixed subalgebra by σ. Recall that there is the reductive splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

The loop group

$$(\Lambda G^{\mathbb{C}})_\sigma = \left\{ g : S^1 \to G^{\mathbb{C}} \mid g(\zeta\lambda) = \sigma g(\lambda) \text{ for all } \lambda \in S^1 \right\}$$

is called the (*free*) *twisted loop group*. Here $G^{\mathbb{C}}$ is the complexification of G and $\zeta = e^{2\pi i/k}$ is the primitive k-th root of unity. Its Lie algebra, called the *twisted loop algebra*, is given by

$$(\Lambda \mathfrak{g}^{\mathbb{C}})_\sigma = \left\{ X : S^1 \to \mathfrak{g}^{\mathbb{C}} \mid X(\zeta\lambda) = \sigma X(\lambda) \text{ for all } \lambda \in S^1 \right\}$$

with the Sobolev H^s-topology for $s > 1/2$. Let $D = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$ and let H be a subgroup of $K^{\mathbb{C}}$ with Lie algebra $\mathfrak{h}$. We also define subgroups

$$(\Lambda G)_\sigma = \{ g \in (\Lambda G^{\mathbb{C}})_\sigma \mid g(\lambda) \in G \text{ for all } \lambda \in S^1 \}$$

$$(\Lambda_H^+ G^{\mathbb{C}})_\sigma = \{ g \in (\Lambda G^{\mathbb{C}})_\sigma \mid g \text{ extends holomorphically to } D \text{ and } g(0) \in H \}$$

Here we omit H if it equals $K^{\mathbb{C}}$. For $X \in (\Lambda \mathfrak{g}^{\mathbb{C}})_\sigma$, there is a Fourier decomposition

$$X = \sum_{j \in \mathbb{Z}} \lambda^j X_j \quad \text{for } X_j \in \mathfrak{g}_j$$

where $\mathfrak{g}_j$ is the ζ^j-eigenspace of σ. It enables us to represent the Lie algebras of $(\Lambda G)_\sigma$ and $(\Lambda_H^+ G)_\sigma$ as

$$(\Lambda \mathfrak{g})_\sigma = \{ X \in (\Lambda \mathfrak{g}^{\mathbb{C}})_\sigma \mid X_j = \bar{X}_j \text{ for all } j \in \mathbb{Z} \}$$

$$(\Lambda_{\mathfrak{h}}^+ \mathfrak{g}^{\mathbb{C}})_\sigma = \{ X \in (\Lambda \mathfrak{g}^{\mathbb{C}})_\sigma \mid X_j = 0 \text{ for } j > 0 \text{ and } X_0 \in \mathfrak{h} \}$$

where the conjugation is with respect to the real form $\mathfrak{g}$.

Suppose that $G^{\mathbb{C}} = G \cdot \widehat{H}$ is an Iwasawa decomposition of $G^{\mathbb{C}}$, where $\widehat{H}$ is a suitable solvable subgroup. Then, there is the following loop group decomposition [18]

$$\Lambda G^{\mathbb{C}} = \Lambda G \times \Lambda_{\widehat{H}}^+ G^{\mathbb{C}}.$$

For an Iwasawa decomposition $K^{\mathbb{C}} = K \cdot H$, where H is a solvable subgroup of $K^{\mathbb{C}}$ with $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{h}$, the twisted loop group decomposition

$$(\Lambda G^{\mathbb{C}})_\sigma = (\Lambda G)_\sigma \cdot (\Lambda_H^+ G^{\mathbb{C}})_\sigma \tag{4.1}$$

and its Lie algebra decomposition

$$(\Lambda \mathfrak{g}^{\mathbb{C}})_\sigma = (\Lambda \mathfrak{g})_\sigma \cdot (\Lambda_{\mathfrak{h}}^+ \mathfrak{g}^{\mathbb{C}})_\sigma \tag{4.2}$$

has been established in [8].

4.2. Harmonic maps via loops

Harmonic maps into a homogeneous space in Section 2.1 can be described in terms of loop groups, by introducing an S^1-parameter [8], [22]. This parameter is often called a *spectral parameter*.

Suppose that S is a simply-connected Riemann surface and $\alpha = \alpha_0 + \alpha_1$ is the $\mathfrak{g}$-valued 1-form on S given in Section 2.1. Consider the $\Lambda\mathfrak{g}$-valued 1-form

$$A_\lambda = \lambda^{-1}{\alpha_1}' + \alpha_0 + \lambda{\alpha_1}'' \tag{4.3}$$

on S. Comparing coefficients of λ, we can show that A_λ satisfies

$$\mathrm{d}A_\lambda + \frac{1}{2}[A_\lambda \wedge A_\lambda] = 0 \tag{4.4}$$

for all $\lambda \in S^1$ with if (2.8) and (2.9) hold. Conversely, if (2.8) and (4.4) hold, there exists a map $F_\lambda : S \to G$ satisfying

$$F_\lambda^{-1}\mathrm{d}F_\lambda = A_\lambda,$$

and then $\psi_\lambda = \pi \circ F_\lambda : S \to G/K$ is harmonic for each $\lambda \in S^1$. Introducing a twisted loop group $(\Lambda G)_\sigma$ can simplify the above discussion. It is easy to check that (2.8) is deduced from any $(\Lambda\mathfrak{g})_\sigma$-valued 1-form

$$A = \lambda^{-1}{\alpha_1}' + \alpha_0 + \lambda{\alpha_1}''. \tag{4.5}$$

Therefore, there exists a map $F : S \to (\Lambda G)_\sigma$ with

$$F^{-1}\mathrm{d}F = A \tag{4.6}$$

so that $\psi = \pi \circ F_1 : S \to G/K$ is harmonic, where $F(z)(\lambda) = F_\lambda(z)$. Conversely, if $\psi : S \to G/K$ is a harmonic map preserving (2.8), we can find a map $F : S \to (\Lambda G)_\sigma$ with (4.5) and (4.6) such that $\pi \circ F_1 = \psi$, which we call an *extended lift* of ψ.

4.3. The generalized Weierstrass representation

Suppose S is $\mathbb{C}$ or its simply-connected open subset with a complex coordinate z. Recall that there is an eigenspace decomposition (2.10) of $\mathfrak{g}^{\mathbb{C}}$ and define

$$\Lambda_{-1,\infty} = \big\{X \in (\Lambda\mathfrak{g}^{\mathbb{C}})_\sigma \mid \lambda X \text{ extends holomorphically to } |\lambda| < 1\big\},$$

which is a closed subspace of $(\Lambda\mathfrak{g}^{\mathbb{C}})_\sigma$. Note that $(\Lambda_H^+ G)_\sigma$ acts on $\Lambda_{-1,\infty}$ by the adjoint action. Using a Fourier series decomposition, one can see that

$X \in \Lambda_{-1,\infty}$ if and only if X may be written as

$$X = \sum_{k \geq -1} \lambda^k X_k$$

where $X_k \in \mathfrak{g}_k$ for $k \geq -1$.

Definition 4.1 We call μ a *holomorphic potential* if μ is a $\Lambda_{-1,\infty}$-valued holomorphic 1-form on S.

Clearly, a holomorphic potential μ is of the form

$$\mu = \sum_{k \geq -1} \lambda^k \mu_k,$$

where μ_k are $\mathfrak{g}_k$-valued holomorphic 1-forms on S. Let μ be a holomorphic potential. Then there exists a unique map $L_\mu : S \to (\Lambda G^{\mathbb{C}})_\sigma$ satisfying

$$L_\mu^{-1} \mathrm{d} L_\mu = \mu$$

with an initial condition $L_\mu(0) = e$. By (4.1), L_μ is decomposed into

$$L_\mu = F_\mu B_\mu, \tag{4.7}$$

and then we obtain a map $F_\mu : S \to (\Lambda G)_\sigma$ with $F_\mu(0) = e$. A short calculation shows that F_μ satisfies (4.6), thus F_μ is an extended lift of the harmonic map $\psi = \pi \circ (F_\mu)_1 : S \to G/K$.

5. Constructing special Lagrangian cones in $\mathbb{C}^3$ by the generalized Weierstrass representation

We now give a construction of special Lagrangian cones in $\mathbb{C}^3$ with the generalized Weierstrass representation. Here, let us adopt the same settings as Section 3.4.

Theorem 5.1 *Let z be a complex coordinate on a simply-connected open subset U of $\mathbb{C}$ and μ a holomorphic potential with μ_{-1} whose entries are nowhere vanishing holomorphic functions on U. Suppose that $\widehat{\psi} : U \to SU(3)/K$ is the harmonic map generated from μ through the generalized Weierstrass representation, and $\pi : SU(3)/K \to \mathbb{C}\mathrm{P}^2$ is the* *homogeneous projection. Then* $\psi = \pi \circ \widehat{\psi} : U \to \mathbb{C}\mathrm{P}^2$ *is a superconformal minimal Lagrangian immersion, so that a special Lagrangian cone in* $\mathbb{C}^3$ *and a solution of* (3.14) *are obtained.*

Proof. Let $F_\mu : U \to (\Lambda \mathrm{SU}(3))_\sigma$ denote the extended frame obtained from μ by the generalized Weierstrass representation, then we have

$$F_\mu^{-1} \mathrm{d} F_\mu = \lambda^{-1} {\alpha_1}' + \alpha_0 + \lambda {\alpha_1}''.$$

Since $F_\mu^{-1} \mathrm{d} F_\mu$ is $(\Lambda \mathfrak{su}(3))_\sigma$-valued, α_0 respectively α_1 take values in the 1- respectively ζ_6-eigenspaces of σ where $\zeta_6 = e^{2\pi i/6}$, which are equal to $\widehat{\mathfrak{g}}_0$ and $\widehat{\mathfrak{g}}_1$ in Section 3.4 respectively. Thus (3.11) holds for F_μ. Also, $\widehat{\mathfrak{g}}_0$ and $\widehat{\mathfrak{g}}_1$ are respectively subspaces in the 1- and ζ_3-eigenspaces of ν. Therefore, F_μ also satisfies (2.11) with respect to $\mathrm{SU}(3)/T$, so that ψ turns out to be superconformal minimal Lagrangian from Section 2.4 and Proposition 3.3. □

6. Reduction of the Tzitzéica equation to the third Painlevé equation

Suppose that w is a solution of (3.14) defined on $\mathbb{C}$ and z is the complex coordinate on $\mathbb{C}$. If w depends only on $|z|$, (3.14) reduces to

$$\frac{1}{4}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right) = e^{-2w} - e^{w}, \tag{6.1}$$

where $z = re^{i\theta}$. We can see that (6.1) is transformed into

$$\frac{1}{4}\left(\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{1}{t}\frac{\mathrm{d}u}{\mathrm{d}t}\right) = \frac{4}{9} a^3 e^{-2u} - \frac{4}{9} a^{3/2} \frac{1}{t} e^{u},$$

by the changes

$$e^w = e^u r^{-1/2}$$

and

$$r = at^{2/3}$$

for $a > 0$. Defining $v : (0, +\infty) \to (0, +\infty)$ by $v = e^u$, we have

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} = \frac{1}{v}\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{16}{9}a^{3/2}\frac{v^2}{t} + \frac{16}{9}a^3\frac{1}{v},$$

which is a special case of the third Painlevé equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \tag{6.2}$$

where $\alpha, \beta, \gamma, \delta$ are complex constants. Thus, we have shown that rotationally symmetric solutions of (3.14) are transformed to solutions of (6.2).

Now we shall give an example of holomorphic potentials which provide solutions of (3.14) described above (cf. [7]). The notation here is the same as in Section 3.4. Let μ be a holomorphic potential on $\mathbb{C}$ of the form

$$\mu = \frac{1}{\lambda}\begin{pmatrix} & & pz^k \\ pz^k & & \\ & qz^l & \end{pmatrix} \mathrm{d}z,$$

where $k, l \in \mathbb{N} \cup \{0\}$ and $p, q \in \mathbb{C}\backslash\{0\}$. Then it shows that $L : \mathbb{C} \to (\Lambda\mathrm{SL}(3,\mathbb{C}))_\sigma$, the unique solution of the o.d.e.

$$L^{-1}\mathrm{d}L = \mu$$

with an initial condition $L(0) = I$, satisfies

$$L(z)(\lambda) = T^{-1}L(\epsilon z)(\epsilon^m\lambda)T$$

for all $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$. Here $m = (2k + l + 1)/3$ and $T = \mathrm{diag}(1, \epsilon^{(k-l)/3}, \epsilon^{-(k-l)/3})$. The decomposition (4.7) of L gives $F : \mathbb{C} \to (\Lambda\mathrm{SU}(3))_\sigma$ and $B : \mathbb{C} \to (\Lambda^+_H\mathrm{SL}(3,\mathbb{C}))_\sigma$ satisfying

$$F(z)(\lambda) = T^{-1}F(\epsilon z)(\epsilon^m\lambda)T \quad \text{and} \quad B(z)(\lambda) = T^{-1}B(\epsilon z)(\epsilon^m\lambda)T$$

respectively. Note that $B(z)(0)$ may be written in the form

$$B(z)(0) = \mathrm{diag}(1, e^b, e^{-b})$$

for some smooth function $b : \mathbb{C} \to \mathbb{R}$, so that

$$b(z) = b(\epsilon z) \tag{6.3}$$

for any $\epsilon \in \mathbb{C}$ with $|\epsilon| = 1$. Following [3, Theorem 2.5], there exist a map $\eta : \mathbb{C} \to \mathbb{C}$ such that

$$e^{\eta(\hat{z})} = pz^k e^{b(z)} \frac{\partial \varphi}{\partial \hat{z}} \quad \text{and} \quad p^2 q z^{2k+l} \left(\frac{\partial \varphi}{\partial \hat{z}} \right)^3 = 1$$

under an appropriate holomorphic coordinate change $z = \varphi(\hat{z})$, and a function $w(\hat{z}) = 2\log|e^{\eta(\hat{z})}|$ which satisfies (3.14). We can now find

$$\varphi(\hat{z}) = \left(\frac{1}{p^2 q} \right)^{1/3} \left(\frac{c(k,l)}{3} \hat{z} \right)^{3/c(k,l)}$$

as a suitable coordinate change, where $c(k,l) = 2k + l + 3$. It is easy to show from (6.3) that $\hat{b}(\hat{z}) := b(\varphi(\hat{z}))$ depends only on $|\hat{z}|$. Thus we obtain a radial solution of (3.14)

$$w(\hat{z}) = \frac{2(k-l)}{c(k,l)} \log\left(\frac{c(k,l)}{3} |\hat{z}| \right) + 2\hat{b}(\hat{z}) + \frac{2}{3} \log\left| \frac{p}{q} \right|$$

and the corresponding solution of (6.2)

$$v(t) = \left| \frac{p}{q} \right|^{2/3} \left(\frac{c(k,l)}{3} \right)^{2(k-l)/c(k,l)} a^{3(2k-l+3)/2c(k,l)} t^{2k-l+3/c(k,l)} e^{2\hat{b}(at^{2/3})}$$

for $t \in (0, +\infty)$ and $a > 0$.

References

[1] Black M., *Harmonic Maps into Homogeneous Spaces.* Pitman Research Notes in Mathematics Series, 255, Longman, Harlow, New York, 1991.

[2] Bobenko A. I., *Surfaces of constant mean curvature and integrable equations* (Russian). Uspekhi Mat. Nauk **46** (1991), 3–42, 192; translation in Russian Math. Surveys **46** (1991), 1–45.

[3] Bolton J., Pedit F. and Woodward L. M., *Minimal surfaces and the affine Toda field model.* J. Reine Angew. Math. **459** (1995), 119–150.

[4] Bolton J. and Woodward L. M., *Congruence theorems for harmonic maps from a Riemann surface into* $\mathbb{C}\mathrm{P}^n$ *and* S^n. J. London Math. Soc. (2) **45**

(1992), 363–376.
[5] Bullough R. K. and Dodd R. K., *Polynomial conserved densities for the sine-Gordon equations.* Proc. Roy. Soc. London Ser. A **352** (1977), 481–503.
[6] Burstall F. E. and Pedit F., *Harmonic maps via Adler-Kostant-Symes theory.* in: Harmonic maps and integrable systems, edited by A. Fordy and J. Wood, Aspects Math., E23, Vieweg, Braunschweig, 1994, 221–272.
[7] Dorfmeister J., Guest M. A. and Rossman W., *The tt* structure of the quantum cohomology of* $\mathbb{C}\mathrm{P}^1$ *from the viewpoint of differential geometry.* Asian J. Math. **14** (2010), 417–437.
[8] Dorfmeister J., Pedit F. and Wu H., *Weierstrass type representation of harmonic maps into symmetric spaces.* Comm. Anal. Geom. **6** (1998), 633–668.
[9] Eells J. and Wood J. C., *Harmonic maps from surfaces to complex projective spaces.* Adv. in Math. **49** (1983), 217–263.
[10] Gross M., Huybrechts D. and Joyce D. D., *Calabi-Yau manifolds and related geometries.* Lectures from the Summer School held in Nordfjordeid, June 2001, Universitext, Springer-Verlag, Berlin, 2003.
[11] Harvey R. and Lawson H. B., *Calibrated geometries.* Acta Math. **148** (1982), 47–157.
[12] Haskins M., *Special Lagrangian cones.* Amer. J. Math. **126** (2004), 845–871.
[13] Joyce D. D., *Special Lagrangian 3-folds and integrable systems.* in Surveys on geometry and integrable systems, Adv. Stud. Pure Math., 51, Math. Soc. Japan, Tokyo, 2008, 189–233.
[14] Joyce D. D., *Riemannian Holonomy Groups and Calibrated Geometry.* Oxford Graduate Texts in Mathematics, 12. Oxford University Press, Oxford, 2007.
[15] Kowalski O., *Generalized Symmetric Spaces.* Lecture Notes in Mathematics, 805. Springer-Verlag, Berlin-New York, 1980.
[16] Ma H. and Ma Y., *Totally real minimal tori in* $\mathbb{C}\mathrm{P}^2$. Math. Z. **249** (2005), 241–267.
[17] McIntosh I., *Special Lagrangian cones in* $\mathbb{C}^3$ *and primitive harmonic maps.* J. London Math. Soc. (2) **67** (2003), 769–789.
[18] Pressley A. and Segal G., *Loop Groups.* Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986.
[19] Sharipov R. A., *Minimal tori in a five-dimensional sphere in* $\mathbb{C}^3$. Theoret. and Math. Phys. **87** (1991), 363–369.
[20] Strominger A., Yau S.-T. and Zaslow E., *Mirror symmetry is T-duality.*

Nuclear Phys. **B479** (1996), 243–259.

[21] Tzitzéica G., *Sur une nouvelle classe de surfaces.* C.R. Acad. Sci. Paris **150** (1910), 955–956.

[22] Uhlenbeck K., *Harmonic maps into Lie groups: classical solutions of the chiral model.* J. Differential Geom. **30** (1989), 1–50.

[23] Wolfson J., *On minimal surfaces in a Kähler manifold of constant holomorphic sectional curvature.* Trans. Amer. Math. Soc. **290** (1985), 627–646.

[24] Wood J. C., *Harmonic maps into symmetric spaces and integrable systems.* in: Harmonic maps and integrable systems, edited by A. Fordy and J. Wood, Aspects Math., E23, Vieweg, Braunschweig, 1994, 29–55.

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