# Analytic functions in a neighbourhood of irregular boundary points

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The present paper is a continuation of the previous paper with title "Analytic functions in a lacunary end of a Riemann surface"[1]. We use the same notions and terminologies in the previous one. Let $G$ be an end of a Riemann surface $\in O_g$ (we denote by $O_G$ the class of Riemann surfaces with null boundary) and $G'=G-F$ be a lacunary end and let $p\in\Delta_1(M)$ be a minimal boundary point relative to Martin's topology $M$ over $G$ with irregularity $\delta(p)=\overline{\lim_{\substack{M \\ z\to p}}}\, G(z,p_0)>0$, where $G(z,p_0)$: $p_0\in G'$ is a Green's function of $G'$. Then Theorems 2, 3 and 4 in the previous show that analytic functions in $G'$ of some classes have similar behaviour at $p$ as $p$ is an inner point of $G'$. We shall show these theoremes are valid not only for the above domains but also for any Riemann surface $\notin O_G$. The extensions of Fatou and Beurling's theorems express the behaviour of analytic functions on almost all boundary points but have no effect on the small set, $\{p\in\Delta_1(M): \delta(p)>\delta\}$. The purpose of this paper is to study analytic functions on the small set, to extend theorems in the previous one and to show some examples. Let $G$ be a domain in a Riemann surface $R$. Through this paper we suppose $\partial G$ consists of at most a countably infinite number of analytic curves clustering nowhere in $R$. The following lemma is useful.

LEMMA 5[2]. *Let $R$ be a Riemann surface $\in O_g$ and let $G$ be a domain and $U_i(z)$ $(i=1,2,\cdots,i_0)$ be a harmonic function in $G$ such that $D(U_i(z))<\infty$. Then there exists a sequence of curves $\{\Gamma_n\}$ in $R$ such that $\Gamma_n$ separates a fixed point $p_0$ from the ideal boundary, $\Gamma_n\to$ideal boundary of $R$ and $\int_{\Gamma_n\cap G}\left|\frac{\partial}{\partial n}U_i(z)\right|ds\to 0$ as $n\to\infty$ for any $i$.*

**Generalized Gree's function**[2] (abbreviated by G.G.). Let $R$ be a Riemann surface with an exhaustion $\{R_n\}$ $(n=0,1,2,\cdots)$ and $G$ be a domain in $R$. Let $w_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-(G\cap(R_{n+i}-R_n))$ such that $w_{n,n+i}(z)=0$ on $\partial R_{n+i}-G$ and $=1$ on $G\cap(R_{n+i}-R_n)$. We call $\lim_n \lim_i w_{n,n+i}(z)$ a H.M. (harmonic measure) of the boundary determined by $G$

and denote it by $w(G\cap B, z)$. Let $V(z)$ be a positive harmonic function in $R$ except at most a set of capacity zero where $V(z)=\infty$. If $w(G_\delta\cap B, z)=0$ for any $\delta>0$ and $D(\min(M, V(z))\leqq M\alpha$: $\alpha$ is a const. for any $M$, we call $V(z)$ a G.G., where $G_\delta=\{z\in R: V(z)\geqq\delta\}$. Then it is known

LEMMA 6. 1)[3] *Let $V(z)$ be a non const. G.G. Let $\hat{G}_\delta$ be the symmetric image of $G_\delta$ with respect to $\partial G_\delta=\{z\in R: V(z)=\delta\}$. Identify $\partial G_\delta$ with $\partial\tilde{G}_\delta$. Then we have a Riemann surface $\tilde{G}_\delta$ called a double of $\tilde{G}_\delta$. Then $\tilde{G}_\delta\in O_g$.*

2) *By* 1) *and by Lemma* 5, *we see there exists a const. $\alpha$ such that $D(\min(M, V(z)))=M\alpha$ and $\int_{\partial G_M}\frac{\partial}{\partial n}V(z)\,ds=\alpha$ for any $M$ and $\sup_{z\in R}V(z)=\infty$.*

3) *Let $V(z)$ be a G.G. and let $W(z)$ be a positive harmonic function $\leqq V(z)$[4]. Then $W(z)$ is a G.G.*

4) *A Green's function of $R$ is a G.G. with $D(\min(M, G(z,p_0)))=2\pi M$. Let $p_i$ be a sequence such that $G(z,p_i)\to$a non const. harmonic function $G(z,\{p_i\})$. Then $G(z,\{p_i\})$ is a G.G. with $D(\min(M, G(z,\{p_i\})))\leqq 2\pi M$.*

**G-Martin's topology**[5], *GM*. Let $R$ be a Riemann surface $\notin O_g$ and let $G(z,p_0)$ be a Green's function of $R$. Put $R'=\{z\in R: G(z,p_0)>\delta\}:\delta>0$. Then the doubled surface $\tilde{R}'$ with respect to $\partial R'$ is in $O_g$. Let $G'(z,p_i)$ be a Green's function of $R'$ and let $\{p_i\}$ be a sequence such that $p_i\to$ boundary of $R$ and $G'(z,p_i)$ converges to a harmonic function. Then we say $\{p_i\}$ determines a boundary point $p$ and put $G'(z,p)=\lim G'(z,p_i)$. We denote by $B(R')$ the set of all boundary points. Then $G$-Martin's topology is introduced on $\bar{R}'=R'+B(R')$ as usual with

$$\operatorname{dist}(p_i,p_j)=\sup_{z\in R_0}\left|\frac{G'(z,p_i)}{1+G'(z,p_i)}-\frac{G'(z,p_j)}{1+G'(z,p_j)}\right|: p_i,\ p_i\in\bar{R}'$$

where $R_0$ is a compact set in $R'$.

Then we see $G'(z,p)$: $p\in\bar{R}'$ is a G.G. and $\int_{\partial V_M(p)}\frac{\partial}{\partial n}G'(z,p)\,ds=2\pi:p\in R'$. Where $V_M(p)=\{z\in R': G'(z,p)>M\}$. Let $p$ and $q\in\bar{R}$. Then $\int_{\partial V_M(q)}G'(\zeta,p)\frac{\partial}{\partial n}G'(\zeta,q)\,ds\uparrow$ as $M\to\infty$. We define the value of $G'(z,p)$ at $q$ by $\lim_{M=\infty}\frac{1}{2\pi}\int_{\partial V_M(q)}G'(\zeta,p)\frac{\partial}{\partial n}G'(\zeta,q)\,ds$ also the mass $m(p)$ of $G'(z,p)$ by $\frac{1}{2\pi}\int_{\partial V_M(p)}\frac{\partial}{\partial n}G'(z,p)\,ds$. Then

LEMMA 7. 1) *$G'(p,q)=G'(q,p)$, $G'(p,q)$ is lower semicontinuous on $\bar{R}'\times\bar{R}'$, $G'(p,p)=\infty$, if $G'(z,p)>0$ and $G'(z,p)$ is continuous on $\bar{R}'-p$ for $p\in R'$.*

2) *$m(p)=1$ for $p\in R'$ and $m(p)\geqq\frac{\eta}{2k}$ for $p\in\bar{G}'_\eta\cap B(R')$, $G'_\eta=\{z\in R':G'(z,p_0)>\eta>0\}$, $k=\sup_{z\notin R_0} G'(z,p_0)$, where $R_0$ is a compact set with $R_0\ni p_0$.*

**Energy integral, capacities and transfinite diameters**[5] Let $F$ be a closed set in $R'$. Let $\{R_n\}$ be an exhaustion of $R$ and let $\omega_n(z)$ be a harmonic function in $(R'\cap R_n)-F$ such that $\omega_n(z)=1$ on $F$ except capacity zero, $=0$ on $\partial R'\cap R_n$ and $\frac{\partial}{\partial n}\omega_n(z)=0$ on $(\partial R_n\cap R')-F$. If there exists a const. $M$ such that $D(\omega_n(z))<M$ for any $n$, then $\omega_n(z)$ in mean$\rightarrow$ a function $\omega(F,z)$ called C.P. (capacitary potential). Clearly $\omega(F,z)$ has M.D.I. (minimal Dirichlet integral) among all functions with value 1 on $F$, $=0$ on $\partial R'$ except capacity zero. In this case, $\tilde{R}'$(of $R'$)$\in O_g$, $\omega(F,z)=w(F,z)$. H.M. (harmonic measure of $F$). Let $K$ be a compact set in $R'$. Then evidently there exists a uniquely determined mass $\mu$ on $K$ of unity such that the energy integral $I(\mu)=\frac{1}{4\pi^2}\int G'(p,q)\,d\mu(p)\,d\mu(q)$ is minimal and its potential $U(z)$ has the following properties: $U(z)=M\omega(K,z)$, $I(\mu)=D(M\omega(K,z))=2M$. We define $Cap(K)$ by $1/I(\mu)=1/2\pi M=D(\omega(K,z))/4\pi^2$. We define $Cap(F)$ of a closed set $F\subset\bar{R}'$ by $\sup_{K\subset F} Cap(K)$. Also we define transfinite diameter $D(F)$ by $1/D(F)=\lim_n \inf_{\substack{p_i\in F\\ p_j}} \sum_{\substack{j>i\\ i=1}}^{n} G'(p_i,p_j)/{}_nC_2$. Put $1/D^M(F)=\lim_n \inf_{\substack{p_i\in F\\ p_j}} \sum_{\substack{j>i\\ i=1}}^{n} G'^M(p_i,p_j)/{}_nC_2$ and $D^0(F)=\lim_M D^M(F)$, where $G'^M(p_i,p_j)=\min(M,G'(p_i,p_j))$. Then clearly $D(F)\leqq D^0(F)$.

Let $p\in\bar{R}'$. Then by Green's formula and by Lemma 5 we have

$$G'(q,p)=\frac{1}{2\pi}\int_{\partial V_M(p)} G'(\zeta,q)\frac{\partial}{\partial n}G'(\zeta,p)\,ds:\quad q\notin\bar{V}_M(p)$$

$$M=\frac{1}{2\pi}\int_{\partial V_M(p)} G'(\zeta,q)\frac{\partial}{\partial n}G'(\zeta,p)\,ds\,;\ q\in V_M(p)\,.$$

Put $d\mu_p(\zeta)=\frac{1}{2\pi}\frac{\partial}{\partial n}G'(\zeta,p)\,ds$ on $\partial V_M(p)$. Then $G'^M(z,p)=M\omega(V_M(p),z)$ $=\int G'(\zeta,z)\,d\mu_p(\zeta)$ and $\mu_p=0$ on $B(R')$. Let $p_1,p_2,\cdots,p_n$. Then $G'^M(z,p_i)$ $=\int G'(z,\zeta)\,d\mu_{p_i}(\zeta)$ and

$$\int G'^M(z,p_i)\,d\mu_{p_j}(z)\leqq\int G'(z,p_i)\,d\mu_{p_j}(z)=G'^M(p_j,p_i)\,.$$

Put $\mu=\sum_{i=1}^{n}\mu_{p_i}/n$, then

$$I(\mu)\leqq\frac{1}{n^2}\sum_{\substack{i=1\\j=1}}^{n}G'^{M}(p_i,p_j). \tag{1}$$

LEMMA 8. *Let $A\subset\tilde{A}$ be closed sets in $\bar{R}'$ and suppose there exists a const. $M$ such that $\frac{1}{2\pi}\int\limits_{\partial V_M(p)\cap\tilde{A}}\frac{\partial}{\partial n}G'(z,p)\,ds\geqq\delta_0>0$ for any $p\in A$. Then*

$$1/D^{\circ}(A)\geqq 1/D^{M}(A)\geqq\delta_0^2/\mathring{C}ap(\tilde{A}).$$

PROOF. Let $d\mu_{p_i}=\frac{\partial}{\partial n}G'(z,p_i)\,ds$ on $\partial V_M(p_i)\subset R':p_i\in A$ and let $\mu'_n$ be the restriction $\mu_n=\sum_{i=1}^{n}\mu_{p_i}/n$ on $\tilde{A}\cap R'$. Then $\int d\mu'_n\geqq\delta_0>0$ and by (1)

$$I(\mu'_n)\leqq\frac{1}{n^2}\sum_{\substack{i=1\\j=1}}^{n}G'^{M}(p_i,p_j).$$

By the simmetry of $G^{M}(p_i,p_j)$

$$2\Big(\sum_{\substack{i<j\\i=1}}^{n}G'^{M}(p_i,p_j)\Big)=\sum_{\substack{i=1\\j=1}}^{n}G'^{M}(p_i,p_j)-\sum_{i=1}^{n}G'^{M}(p_i,p_i)\qquad\text{and}$$

$$1/D_n^{M}(A)=\inf_{p_i,p_j\in A}\sum_{\substack{i<j\\i=1}}^{n}G'^{M}(p_i,p_j)/{}_nC_2\geqq\Big(\frac{n}{n-1}\Big)I(\mu'_n)-\frac{M}{n-1}.$$

Now $\mu'_n$ is a mass only on $\tilde{A}\cap R'$ with total mass $\geq\delta_0$. By definition $1/\mathring{C}ap(\tilde{A})$ is the infimum of energy integrals of all distributions on $\tilde{A}\cap R'$ of mass unity. Hence $I(\mu'_n)\geqq\delta_0^2/\mathring{C}ap(\tilde{A})$. Let $n\to\infty$. Then $1/D^{M}(A)\geqq\delta_0^2/\mathring{C}ap(\tilde{A})$.

**Capacity and transfinite diameters of irregular boundary points**[6] $B(R')\cap\bar{G}_\eta$: $G_\eta=\{z\in R': G'(z,p_0)>\eta\}$. Put $F_\eta=\{z\in\bar{R}': G'(z,p_0)\geqq\eta\}$. Then $F_\eta$ is closed in $\bar{R}'$. Let $\{R_n\}$ be an exhaustion of $R$(not of $R'$). Then $\mathring{C}ap(F_\eta\cap(\overline{R'-R_n}))=\lim_{i=\infty}Cap(F_\eta\cap\bar{R}'\cap(\overline{R_{n+i}-R_n}))\leqq\frac{1}{4\pi^2}D(\omega(F_\eta,z))\leqq\frac{1}{\eta^2}D$ $(\min(\eta,G'(z,p_0)))\leqq\frac{2\pi}{\eta}<\infty$. Let $\omega_n(z)$ be C.P. of $F_\eta\cap(R'-R_n)$. Then $\omega_n(z)$ in mean$\to$ a harmonic function $\omega(z)$. Now $\omega(z)=0$ on $\partial R'$ and $\leqq 1$. By $\tilde{R}\in O_g$, $\omega(z)=0$. Hence

$$\mathring{C}ap\Big(F_\eta\cap(R'-R_n)\Big)\downarrow 0\ \text{as}\ n\to\infty. \tag{2}$$

THEOREM 7. *Let $A=F_\xi\cap B(R')$: $\xi>0$. Then $D(A)\leqq D^{\circ}(A)=0$.*

PROOF. Let $v(p_0)$ be a neighbourhood of $p_0$. Then there exists a const. $k$ such that $G'(z,p_0)\leqq k$ in $R'-v(p_0)$. By Green's formula and by Lemma 5

$$\frac{1}{2\pi}\int_{\partial V_M(p)\cap G_{\frac{\xi}{2}}} G'(\zeta,p_0)\frac{\partial}{\partial n}G'(\zeta,p)\,ds = G'(p,p_0)-\frac{1}{2\pi}\int_{\partial V_M(p)-G_{\frac{\xi}{2}}} G'(\zeta,p_0)\frac{\partial}{\partial n}G'(\zeta,p)\,ds.$$

Put $m'(p)=\frac{1}{2\pi}\int_{\partial V_M(p)\cap G_{\frac{\xi}{2}}}\frac{\partial}{\partial n}G'(\zeta,p)\,ds$, then $\frac{1}{2\pi}\int_{\partial V_M(p)-G_{\frac{\xi}{2}}}\frac{\partial}{\partial n}G'(\zeta,p)ds=1-m'(p)$. (3)

Suppose $p\in\bar{G}_\xi$, then by (3) we have

$$m'(p)\geqq\frac{\xi}{2k}\quad\text{for any } p\in\bar{G}_\xi \text{ and for any } M<\infty. \qquad (4)$$

Clearly $\max\limits_{\substack{z\in\partial R_n\cap R'\\ p\in F_\xi\cap B(R)'}} G'(z,p)=M_n<\infty$. Hence for any given number $n$ there exists a number $M_n$ such that $V_M(p)\subset R'-R_n$: $M>M_n$, $p\in F_\xi\cap B(R')$. Hence we have

Proposition. *Let $\xi$ and $n$ be numbers. Then there exists a number $M$ such that $m(p)\geqq\frac{1}{2\pi}\int_{\partial V_M(p)\cap(R'-R_n)\cap G_{\frac{\xi}{2}}}\frac{\partial}{\partial n}G'(\zeta,p)\,ds\geqq\frac{\xi}{2k}$ for $M\geqq M_n$ and for $p\in F_\xi\cap B(R')$.*

Let $\varepsilon>0$ be a given positive number. Then by (2) there exists a number $n$ such that $\mathring{Cap}(F_\eta\cap(R'-R_n))<\varepsilon$: $\eta=\frac{\xi}{2}$. Let $\tilde{A}=F_\eta\cap(\overline{R'-R_n})$ and $A=F_\xi\cap B(R')$. Then by the proposition there exists a number $M'$ such that $\frac{1}{2\pi}\int_{\partial V_M(p)\cap\tilde{A}}\frac{\partial}{\partial n}G'(\zeta,p)\,ds\geqq\frac{\eta}{k}$: $M\geqq M'$ and $p\in A$. Hence by Lemma 8 $1/D^M(A)\geqq\left(\frac{\eta}{k}\right)^2/\varepsilon$. Let $M\to\infty$ and then $\varepsilon\to 0$. Then we have Theorem 7.

Let $\Omega$ be a domain in the $z$-sphere such that $\Omega\notin O_g$. Let $G(z,p)$ be a Green's function of $\Omega$. We shall extend the domain of the definition of $G(z,p)$ to $\bar{\Omega}\times\bar{\Omega}$ by $G(p,q)=\overline{\lim\limits_{\xi\to p}}\,\overline{\lim\limits_{\eta\to q}}\,G(\xi,\eta)$ for $p,q\in\bar{\Omega}\times\bar{\Omega}$. Then we see at once $G(p,q)=G(q,p)$ and $\overset{\circ}{G}(z,p)=G(z,p)$: $z\in\Omega$, $p\in\bar{\Omega}$ (in Lemma 4[1] $G(z,p)$: $p\in\bar{\Omega}$ is defined). Let $F$ be a closed set on $\bar{\Omega}$. Define $D^*(F)$ by $1/D^*(F)=\lim\limits_n\inf\limits_{p_i,p_j\in F}\sum\limits_{\substack{i<j\\ i=1}}^{n}G(p_i,p_j)/{}_nC_2$.

Lemma 9. 1) *Let $\Omega$ be a domain in the $z$-sphere such that $\Omega\notin O_g$. Let $G(z,z')$ be Green's function of $\Omega$. Then there exist consts. $M$ and $\delta$ depending on $\Omega$ such that*

$$G(z,z')\leqq\log\frac{1}{|z-z'|}+M,$$

*for any points z and z' with spherical distance* $<\delta$.

2) *Let F be a closed set on* $\bar{\Omega}$ *such that* $D^*(F)=0$. *Then F is a set of (logarithmic) capacity zero.*

PROOF. By $\Omega\notin O_g$, $C\Omega$ is a set of positive capacity. We can find two closed sets $E_1$ and $E_2$ in $C\Omega$ such that both $E_1$ and $E_2$ are of positive capacity and spherical distance between $E_1$ and $E_2=d>0$. We denote by $[z, z']$ the spherical distance betweed $z$ and $z'$. Put $C(4\delta, z')=\{z: [z, z']\leqq 4\delta\}: \delta\leqq d/8$. We can find a finite number of points, $z_1, z_2, \cdots, z_{i_0}$ such that $\sum_i C(\delta, z_i)\supset$ $z$-sphere, and $C(4\delta, z_i)$ has common points at most one of $E_1$ and $E_2$. Suppose $[z, z']<\delta$. Then there exists $C(4\delta, z)$ such that $C(2\delta, z_i)\ni z$ and $z'$ and $C(4\delta, z_i)\cap E_j=0 (j=1$ or $2)$. Let $\tilde{G}(z, z')$ be Green's function of $CE_j$. Then $\tilde{G}(z, z')\geqq G(z, z')$, $\tilde{G}(z, z')$ is harmonic in $C(4\delta, z_i)$-$z'$ and $\tilde{G}(z, z')$-$\log\dfrac{1}{|z-z'|}$ is continuous on $C(4\delta, z_i)\times C(4\delta, z_i)$. Hence there exists a const. $M(z_i)$ such that $\tilde{G}(z, z')\leqq\log\dfrac{1}{|z-z'|}+M(z_i)$. Hence we have 1) by putting $M=\max_i M(z_i)$

Proof of 2). Let $F_k=F\cap C(2\delta, z_k)$. Then it is sufficient to show $F_k$ is a set of capacity zero. By a conformal mapping we can suppose $z_k=0$ and $\delta\leqq 1/4$. Then we have $\liminf_n \sum_{z_i\in F\, \substack{i<j\\ i=1}}^{n} \log\dfrac{1}{|z_i-z_j|}/{}_nC_2=\infty$ by $D^*(F_k)=D^*(F)=0$. Hence $F_k$ is a set of capacity zero.

**Mass distribution of a generalized Green's function** Let $R$ be a Riemann surface $\notin O_g$. Let $U(z)$ be a positive harmonic function in $R$ and let $G$ be a domain. Let $U_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-((R_{n+i}-R_n)\cap G)$ such that $U_{n,n+i}(z)=0$ on $\partial R_{n+i}-G$, $U_{n,n+i}(z)=U(z)$ on $G\cap(R_{n,n+i}-R_n)$. Put $\lim_n\lim_i U_{n,n+i}(z)={}^{\alpha}_{G}U(z)$. Let $\tilde{U}_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-((R_{n+i}-R_n)\cap G)$ such that $\tilde{U}_{n,n+i}(z)=0$ on $(R_{n+i}-R_n)\cap G$, $=U(z)$ on $\partial R_{n+i}-G$. Put $\lim_n\lim_i \tilde{U}_{n,n+i}(z)={}^{\beta}_{G}U(z)$. Then

LEMMA 10[3]. 1) ${}^{\alpha}_{G}({}^{\alpha}_{G}U(z))={}^{\alpha}_{G}U(z)$ *and* ${}^{\alpha}_{G}U(z)+{}^{\beta}_{G}U(z)=U(z)$.

2) *Let* $U(z)$ *be a harmonic function which is a G.G. with* $D(\min(M, U(z))\leqq Mk\pi$ *and let* $G_\delta=\{z\in R: G(z, p_0)>\delta\}$. *Then* ${}^{\beta}_{G_\delta}U(z)\leqq k\delta/2$ *at* $z=p_0$.

We suppose Martin's top. $M$ is defined on $\bar{R}=R+\Delta(\Delta=\Delta_1+\Delta_0)$. Let $\bar{G}_\delta(M)$ be the closure of $G_\delta$ relative to $M$-top. Let $F_n=\{z\in\bar{R}: M\text{-dist}(z, \bar{G}_\delta(M))\leqq 1/n\}$ and ${}_{F_n}U(z)$ be the lower envelope of superharmonic functions larger than $U(z)$ on $F_n$. Put $U^*_\delta(z)=\lim_n {}_{F_n}U(z)$. Then by Martin's theory $U^*_\delta(z)$ is represented by a canonical distribution $\mu$ on $\bar{G}_\delta(M)\cap\Delta_1$. Clearly

$$U_{\delta}^{*}(z) \geqq {}_{G_{\delta}}^{\alpha}U(z)\,. \tag{5}$$

LEMMA 11. 1) *Let $U(z)$ be a positive harmonic function being a G.G. in R. Then there exists a canonical distribution $\mu$ on $\bigcup_{\delta>0} \overline{G}_{\delta}(M) \cap \Delta_1$ such that*

$$U(z) = \int K(z, p)\, d\mu(p)\,.$$

2) *If there exists a const. $\delta>0$ such that $\overline{G}_{\delta}(M) \cap \Delta_1 = \overline{G}_{\delta'}(M) \cap \Delta_1$ for any $\delta' \leqq \delta$, then there exists a canonical distribution $\mu$ on $\overline{G}_{\delta}(M) \cap \Delta_1$ such that*

$$U(z) = \int K(z, p)\, d\mu(p)\,.$$

Proof of 1) Since $U(z)$ is a G.G. there exists a const. $k$ such that $D(\min(M, U(z))) = k\, M\pi$ for any $M$. By (5) and by Lemma 10

$$\left(U(z) - U_{\delta}^{*}(z)\right) \leqq k\delta/2 \qquad \text{at } z = p_0\,. \tag{6}$$

Let $\delta = \delta_1 > \delta_2 \cdots \downarrow 0$, $U_{\delta_n}^{*}(z)$ and $\mu_n$ be a canonical mass of $U_{\delta_n}^{*}(z)$. Then $\mu_n \uparrow$ and $\mu_n - \mu_{n-1}$ is also canonical on $\overline{G}_{\delta_n}(M) \cap \Delta_1$. Now $U_{\delta_n}^{*}(z) = U_{\delta_1}^{*}(z) + \sum_{i=2}^{n} (U_{\delta_i}^{*}(z) - U_{\delta_{i-1}}^{*}(z))$. Hence by (6) $U(z) = \lim \lim_{n} U_n^{*}(z)$ and $U(z)$ is represented by a canonical distribution $\mu$ on $\bigcup_{\delta>0} \overline{G}_{\delta}(M) \cap \Delta_1$. 2) is evident by 1).

Let $D_1 \supset D_2$ be two domains. Let $U(z)$ be a positive harmonic function in $D_1$. We denote by $\overset{D_1}{\underset{D_2}{I}} U(z)$ the greatest subharmonic function in $D_2$ vanishing on $\partial D_2$ not larger than $U(z)$. Let $V(z)$ be a positive harmonic function in $D_2$ vanishing on $\partial D_2$ except at most a set of capacity zero. We denoteby $\overset{D_1}{\underset{D_2}{E}} V(z)$ the least positive superharmonic function in $D_1$ larger than $V(z)$. Then the following are well known.

$\overset{D_1}{\underset{D_2}{I}} U(z)$ and $\overset{D_1}{\underset{D_2}{E}} V(z)$ (for $\overset{D_1}{\underset{D_2}{E}} V(z) < \infty$) *are harmonic and*

$$\overset{D_1 D_1 D_1}{\underset{D_2 D_2 D_2}{IEI}} U(z) = \overset{D_1}{\underset{D_2}{I}} U(z) \quad \textit{and} \quad \overset{D_1 D_1 D_1}{\underset{D_2 D_2 D_2}{EIE}} V(z) = \overset{D_1}{\underset{D_2}{E}} V(z)$$

*Let $U(z)$ be minimal in $D_1$. Then if $\overset{D_1}{\underset{D_2}{I}} U(z) > 0$, $\overset{D_1 D_1}{\underset{D_2 D_2}{EI}} U(z) = U(z)$ and $\overset{D_1}{\underset{D_2}{I}} U(z)$ is minimal in $D_2$. Let $V(z)$ be minimal in $D_2$. If $\overset{D_1}{\underset{D_2}{E}} V(z) < \infty$, $\overset{D_1 D_1}{\underset{D_2 D_2}{IE}} V(z) = V(z)$ and $\overset{D_1}{\underset{D_2}{E}} V(z)$ is minimal in $D_1$.*

*If $U_n(z) \nearrow U(z)$, $\overset{D_1}{\underset{D_2}{I}} U(z) = \lim_{n} (\overset{D_1}{\underset{D_2}{I}} U_n(z))$.*

**Correspondence between two minimal points** Let $\tilde{R}$ be a Riemann surface $\notin O_g$ and $R$ be a Riemann surface $\subset \tilde{R}$. Let $\{\tilde{R}_n\}$ be an exhaustion of $\tilde{R}$ and $\mathfrak{p}$ be a boundary component of $\tilde{R}$. Suppose Martin's topologies $\widetilde{M}$ and $M$ are defined over $\tilde{R}$ and $R$ respectively. If $p_i \xrightarrow{\alpha} p$ : $\alpha=\widetilde{M}$ or $M$ and $p_i \to \mathfrak{p}$ (considered in $\tilde{R}$), we say a point (relative to $\alpha$-top.) lies over $\mathfrak{p}$. We denote by $\Delta(\alpha)\cap V(\mathfrak{p})$ and $\Delta_1(\alpha)\cap V(\mathfrak{p})$ sets of boundary points, minimal boundary points over $\mathfrak{p}$ respectively. In the present paper boundary components are considered only for $\tilde{R}$ (except special remark). Let $G(z, p_0)$ be a Green's function of $R$. Let $F_\delta(\widetilde{M})=\{z\in\bar{\tilde{R}}: \overline{\lim\limits_{\substack{\zeta\to z\\ \widetilde{M}}}}\, G(\zeta, p_0)\geqq\delta\}$ and $F_\delta(M) =\{z\in\bar{R}: \overline{\lim\limits_{\substack{\zeta\to z\\ M}}}\, G(\zeta, p_0)\geqq\delta\}$. Let A be a set relative to $\widetilde{M}$-top.. We denote by $A\cap\Delta(M)$ the set of point $p$ of $A$ lying over $\Delta(M)$, i.e. there exists a sequence $\{z_i\}$ such that $z_i \xrightarrow{\widetilde{M}} p$ and $z_i \longrightarrow$ boundary of $R$. Then

THEOREM 8. 1)[6] *Let* $z_i \xrightarrow{\widetilde{M}} p\in(\tilde{R}+\Delta_1(\widetilde{M}))\cap F_\delta(\widetilde{M}))\cap\Delta(M)$ *and* $G(z_i, p_0) >\varepsilon_0>0$. *Then* $z_i \xrightarrow{M}$ *a uniquely determined point* $q\in\Delta_1(M)\cap F_\delta(M)$ *and* $K(z, q)=a \underset{R}{\overset{\tilde{R}-p}{I}} \tilde{K}(z, p)$: $a>0$. *We denote* $q$ *by* $\varphi(p)$.

2) *Let* $q\in\Delta_1(M)\cap F_\delta(M)$. *Then there exists a point* $p\in\tilde{R}+\Delta_1(\widetilde{M})$ *such that* $\tilde{K}(z, p)=a' \underset{R}{\overset{\tilde{R}-p}{E}} K(z, q)$; $a'>0$, *clearly* $p=\varphi^{-1}(q)$. *Further*

$$\Delta_1(\widetilde{M})\cap F_\delta(\widetilde{M})\cap V(\mathfrak{p})\approx\Delta_1(M)\cap F_\delta(M)\cap V(\mathfrak{p}),$$
$$F_\delta(\widetilde{M})\cap(\tilde{R}+\Delta_1(M))\cap\Delta(M)\approx\Delta_1(M)\cap F_\delta(M),$$

*where* $\approx$ *means the existence one to one mapping.*

Proof of 1) 1) is proved by L. Naim. Let $\tilde{G}(z, p_0)$ be Green's function of $\tilde{R}$ and $v(p_0)$ be a neighbourhood of $p_0$ and put $M=\sup\limits_{z\notin v(p_0)} \tilde{G}(z, p_0)$. Let $\tilde{K}(z, p)$ and $K(z, q)$ be kernels in $\tilde{R}$ and $R$ respectively. Then if $G(z, p_0)>\varepsilon_0$,

$$\frac{\tilde{G}(z, z_i)}{\varepsilon_0}\geqq\tilde{K}(z, z_i)\geqq\frac{\tilde{G}(z, z_i)}{M}\geqq\frac{G(z, z_i)}{M}\geqq\frac{\varepsilon_0 K(z, z_i)}{M}\geqq\frac{\varepsilon_0 G(z, z_i)}{M^2} \quad (7)$$

Let $z_i \xrightarrow{\widetilde{M}} p$ and let $\{z_i'\}$ be a subsequence of $\{z_i\}$ such that $z_i' \xrightarrow{M} q$. Then by (7) $\underset{R}{\overset{\tilde{R}-p}{I}} \tilde{K}(z, p)>0$. By the minimality of $\underset{R}{\overset{\tilde{R}-p}{I}} \tilde{K}(z, p)$ $\underset{R}{\overset{\tilde{R}-p}{I}} \tilde{K}(z, p)=aK(z, q)$: $a>0$ and $q\in\Delta_1(M)$. Since $\{z_i'\}$ is an arbitrary $M$-convergent subsequence, such point $q$ is uniquely determined. We denote it by $\varphi(p)$. If $p\in F_\delta(\widetilde{M}) \cap V(\mathfrak{p})$, evidently $q\in F_\delta(M)\cap V(\mathfrak{p})$.

Proof of 2) By 1) if $p\in F_\delta(\widetilde{M})\cap\Delta_1(\widetilde{M})\cap V(\mathfrak{p})$, $q\in F_\delta(M)\cap\Delta_1(M)\cap V(\mathfrak{p})$. Conversely let $q\in\Delta_1(M)\cap F_\delta(M)\cap V(\mathfrak{p})$. Then there exists a sequence $\{z_n\}$ such that $z_n\xrightarrow{M} q$ and $G(z_n,p_0)\geqq\delta-\frac{1}{n}$ and $K(z,z_n)\leqq\frac{2G(z,z_n)}{\delta}\leqq\frac{2\widetilde{G}(z,z_n)}{\delta}$ for $\frac{1}{n}\leqq\frac{\delta}{2}$, hence $\overset{\tilde{R}}{\underset{R}{E}}K(z,q)<\infty$. By the minimality of $K(z,q)$, there exists a uniquely determined point $p\in\Delta_1(\widetilde{M})$ such that $\overset{\tilde{R}}{\underset{R}{E}}K(z,q)=a\widetilde{K}(z,p)$: $a>0$, clearly $q=\varphi(p)$. We show $p\in F_\delta(\widetilde{M})$. Let $\Omega_\varepsilon=\{z\in R: G(z,p_0)>\delta-2\varepsilon\}$: $3\varepsilon<\delta$ and let $\{z'_n\}$ be a subsequence of $\{z_n\}$ such that $G'(z,z'_n)$ converges to $G'(z,\{z'_n\})$, where $G'(z,z'_n)$ is a Green's function of $\Omega_\varepsilon$. Then $\frac{\widetilde{K}(z,p)}{a}\geqq K(z,q)\geqq\frac{G'(z,\{z'_n\})}{M}>0$ by $G'(p_0,z_n)=G(z_n,p_0)-(\delta-2\varepsilon)>\varepsilon$ for $\frac{1}{n}<\varepsilon$. Hence

$$\overset{\tilde{R}}{\underset{\Omega_\varepsilon}{I}}\widetilde{K}(z,p)>0\,. \qquad (8)$$

Let $U(z)=\widetilde{K}(z,p)$. Let $V_n(z)$ be a harmonic function in $\Omega_\varepsilon\cap\widetilde{R}_n$ such that $V_n(z)=U(z)$ on $\partial\Omega_\varepsilon\cap\widetilde{R}_n$, $=0$ on $\partial\widetilde{R}_n\cap\Omega_\varepsilon$. Then $V_n(z)\nearrow{}_{C\Omega_\varepsilon}U(z)$ in $\Omega_\varepsilon$. Let $W_n(z)$ be a harmonic function in $\Omega_\varepsilon\cap\widetilde{R}_n$ such that $W_n(z)=0$ on $\partial\Omega_\varepsilon\cap\widetilde{R}_n$, $=U(z)$ on $\partial\widetilde{R}_n\cap\Omega_\varepsilon$. Then $W_n(z)\downarrow\overset{\tilde{R}}{\underset{\Omega_\varepsilon}{I}}U(z)$. On the other hand, $U(z)=V_n(z)+U_n(z)$, $U(z)={}_{C\Omega_\varepsilon}U(z)+\overset{\tilde{R}}{\underset{\Omega_\varepsilon}{I}}U(z)$ and by (8) $U(z)>{}_{C\Omega_\varepsilon}U(z)$. Hence $C\Omega_\varepsilon$ is thin at $p$. Let $v_n(p)=\left\{z\in\tilde{\tilde{R}}:\ \widetilde{M}\text{-}dist(z,p)<\frac{1}{n}\right\}$. Then $Cv_n(p)$ is thin at $p$ and $C(v_n(p)\cap\Omega_\varepsilon)$ is thin at $p$, whence $v_n(p)\cap\Omega_\varepsilon\neq 0$ for any $n$ and $\varepsilon>0$: $\varepsilon<\frac{\delta}{3}$. Let $\varepsilon>\varepsilon_1>\varepsilon_2\cdots\downarrow 0$. We choose $z_n$ in $v_n(p)\cap\Omega_{\varepsilon_n}$, where $\Omega_{\varepsilon_n}=\{z\in R: G(z,p_0)\geqq\delta-2\varepsilon_n\}$. Then $z_n\xrightarrow{\widetilde{M}}p$, $\overline{\lim\limits_n}\,G(z_n,p_0)\geqq\delta$ and $p\in F_\delta(\widetilde{M})$. By the assumption we can find a sequence $\{z_n\}$ such that $z_n\xrightarrow{M}q$, $z_n\longrightarrow\mathfrak{p}$. $G(z_n,p_0)>\varepsilon_0>0$. $G(z,z_n)$ and $\widetilde{G}(z,z_n)$ converge. Then
$K(z,q)\leqq\frac{G(z,\{z_n\})}{\varepsilon_0}\leqq\frac{\widetilde{G}(z,\{z_n\})}{\varepsilon_0}$, $a\widetilde{K}(z,p)=\overset{\tilde{R}}{\underset{R}{E}}K(z,q)\leqq\frac{\widetilde{G}(z,\{z_n\})}{\varepsilon_0}$; $a>0$ and $\widetilde{K}(z,p)$ is bounded outside of a neighbourhood $\mathfrak{v}(\mathfrak{p})$ ($\partial\mathfrak{v}(\mathfrak{p})$ is supposed compact in $\widetilde{R}$). Clearly $p$ lies on a boundary component $\mathfrak{p}'$ of $\widetilde{R}$. Assume $\mathfrak{p}\neq\mathfrak{p}'$. Then $\widetilde{K}(z,p)$ is bounded outside of $\mathfrak{v}(\mathfrak{p}')$ of $\mathfrak{p}'$ such that $\mathfrak{v}(\mathfrak{p})\cap\mathfrak{v}(\mathfrak{p}')=0$. This implies $\sup\limits_{z\in\widetilde{R}}\widetilde{K}(z,p)<\infty$. This is a contradiction. Hence $p$ lies over $\mathfrak{p}$ where $q$ lies. Thus we have 2). The latter part is proved similarly.

Let $R\subset\widetilde{R}\notin O_g$ be Riemann surfaces and let $G(z,p_0)$ be a Green's function of $R$. We suppose Martin's topologies $\widetilde{M}$ and $M$ are defined on $\widetilde{R}$ and $R$. Let $R'=\{z\in R: G(z,p_0)>\xi\}$ and suppose $G$-Martin's top. $GM$ is defined on $R'+B(R')$. Let $w=f(z): z\in R$ be an analytic function in $R$ whose value falls on the $w$-sphere. If the complementary set $Cf(R)$ of $f(R)$ is of positive capacity, we call $f(z)$ a bounded type in $R$. In this paper we consider only functions of bounded type in $R$. Then

THEOREM 9. 1) *Let* $z_i\xrightarrow{M}q\in\varDelta(M)$, $z_i\in G_\delta=\{z\in R: G(z,p_0)>\delta\}$. *Then* $f(z_i)\rightarrow$ *one point denoted by* $f(q)$.

2) *Let* $z_i\xrightarrow{\widetilde{M}}p\in\varDelta_1(\widetilde{M})+\widetilde{R}$, $z\in G_\delta$. *Then* $f(z_i)\longrightarrow f(p)$.

3) *Let* $z_i\xrightarrow{GM}p\in B(R')$: $z\in G_{\delta+\varepsilon}$: $\varepsilon>0$. *Then* $f(z_i)\longrightarrow f(p)$.

4) *Let* $A(\varDelta_1(\widetilde{M})+\widetilde{R},\delta)=\{f(p): p\in(\varDelta_1(\widetilde{M})+\widetilde{R})\cap\overline{G}_\delta(\widetilde{M})\cap\varDelta(M)\}$, $A(\varDelta(M),\delta)=\{f(p): p\in\varDelta(M)\cap\overline{G}_\delta(M)\}$ *and* $A(B(R'),\delta)=\{f(p): p\in B(R')\cap\overline{G}_\delta(GM)\cap\varDelta(M)\}$. *Then* $A(\varDelta_1(\widetilde{M})+\widetilde{R},\delta)\subset A(\varDelta(M),\delta)=A(B(R'),\delta)$: $\delta>\xi$ *and* $A(\varDelta(M),\delta)$ *is a closed set of capacity zero and* $\bigcup_{\delta>0}A(\varDelta(M),\delta)$ *is an* $F_\sigma$ *set of capacity zero.*

PROOF. Let $z_i\in G_\delta$: $\delta>\xi$ and let $G'(z,z_i)$ be a Green's function of $R'$. Then $G(z,z_i)\geqq G'(z,z_i)$. Let $\{z_i'\}$ be a subsequence of $\{z_i\}$ such that $G(z,z_i')$ and $G'(z,z_i')$ converge. Then $G(z,\{z_i'\})\geqq G'(z,\{z_i'\})>0$ and $G'(z,\{z_i'\})=0$ on $\partial R'$ and is a G.G. in $R'$, whence $\sup_{z\in R'}G'(z,\{z_i'\})=\infty$. Assume $f(z)$ does not converge as $z_i\longrightarrow q$. Then there exists two subsequences $\{z_i^k\}$ $(k=1,2)$ of $\{z_i\}$ such that $G(z,z_i^k)\longrightarrow U^k(z)$, $f(z_i^k)\longrightarrow w^k$: $w^1\neq w^2$. Now $\dfrac{G(z,z_i)}{\delta}\geqq K(z,z_i)\geqq\dfrac{G(z,z_i)}{M}$: $M=\sup\limits_{z\notin v(p_0)}G(z,p_0)$ and

$$\delta K(z,q)\leqq U^k(z)\leqq MK(z,q).$$

On the other hand, $U^k(z)\leqq G^w(f(z),w^k)$, where $G^w(w,w^k)$ is a Green's function of $f(R)$ and not necessarily $w^k\in f(R)$ but $\in\overline{f(R)}$. Hence

$$K(z,q)\leqq\frac{1}{\delta}\min(G^w(f(z),w^1),\ G^w(f(z),w^2))\quad\text{and by Lemma 4}$$

$$\infty=\sup_{z\in R}U^k(z)\leqq\frac{M}{\delta}\sup_{z\in R}\min(G^w(w,w^1),G^w(w,w^2))<\infty.$$

This is a contradiction, hence $f(z)\longrightarrow$ uniquely determined point denoted by $f(q)$.

Proof of 2) By Theorem 8, 1) $z\xrightarrow{\widetilde{M}}p\in(\varDelta_1(\widetilde{M})+\widetilde{R})$: $z\in\overline{G}_\delta(\widetilde{M})$ implies $z\xrightarrow{M}q\in\varDelta_1(M)$ and we have 2). 3) is proved similarly as 1).

Proof of 4) Let $w_n \in A(\Delta(M), \delta)$ and $w_n \longrightarrow w^*$. Then there exists $z_n$ such that $z_n \in \Delta(M) \cap \bar{G}_\delta(M)$: $w_n = f(z_n)$. Let $\{R_n\}$ be an exhaustion of $R$. For any $z_n$ we can find $z'_n$ in $(R-R_n) \cap G_{\delta-\frac{1}{n}}$ such that $M$-dist $(z_n, z'_n) \leqq \frac{1}{n}$, $|f(z'_n) - w_n| \leqq \frac{1}{n}$. Consider $K(z, z'_n)$. Then we can find a subsequence $\{z''_n\}$ of $\{z'_n\}$ such that $K(z, z''_n\}$ converges uniformly. This means there exists a point $z^* \in \Delta(M) \cap \bar{G}_\delta(M)$ such that $z''_n \overset{M}{\longrightarrow} z^*$ and $f(z''_n) \longrightarrow f(z^*)$. Clearly $w^* = f(z^*)$. Hence $w^* \in A(\Delta(M), \delta)$ and $A(\Delta(M), \delta)$ is closed. We can choose $\xi$ so that $\xi < \delta$. Since $A(\Delta(M), \delta) = A(B(R'), \delta)$ for $\delta > \xi$ is proved easily, it is sufficient to show $A(B(R'), \delta)$ is a set of capacity zero. By Theorem 7 the transfinite diameter of $B(R') \cap \bar{G}_\delta(GM)$ is zero. Since for any point $w \in A(B(R'), \delta)$ there exists at least a point $z$ in $B(R') \cap \bar{G}_\delta(GM)$ such that $w = f(z)$ and since $G^w(f(z), f(z')) \geqq G'(z, z')$, transfinite diameter $D^*(A(\Delta(M), \delta))$ is zero and by Lemma 9 $A(\Delta(M), \delta)$ is a set of (logarithmic) capacity zero.

We consider the behaviour of $f(z)$ as $z \longrightarrow \Delta(M)$ of $R \subset \tilde{R}$. We define another Riemann surface $R^*$ as follows. We can find a segment $S$ in $R$ such that $f(z)$ is univalent in a neighbourhood $v(S)$ of $S$. Put $S^w = f(S)$. Let $\mathscr{F}$ be a leaf such that $\mathscr{F} = f(R)$ and let $\partial\mathscr{F}$ be its boundary. Let $S(\mathscr{F})$ be a slit in $\mathscr{F}$ with $S(\mathscr{F}) = S^w$. Connect $\mathscr{F} - S(\mathscr{F})$ and $R - S$ crosswise on $S^w (=S)$. Then we have a Riemann surface $R^* = (R-S) + (\mathscr{F} - S(\mathscr{F})) + S$. Put $f(z) =$ projection of $z$ (as $R$ and $R^*$ are considered covering surfaces over the $w$-sphere) in $\mathscr{F} - S(\mathscr{F})$. Then $f(z)$ is analytic in $R^*$. In this case, we also denote by $f(z)$: $z \in R^*$. So long as we consider $f(z)$ near the boundary of $R$, we can use $R^*$ instead of $R$. Let $u(z)$ be a harmonic measure of $\partial\mathscr{F}$ in $R^*$. Then by $R \notin O_g$ $u(z)$ is non const.. Put $U(w) = \sum_i u(z_i)$: $f(z_i) = w$, $z_i \in R^*$. Then by Theorem 1[1)]

$$U(w) \leqq 1 \text{ and } U(w) \text{ is quasisubharmonic in } f(R). \qquad (9)$$

Let $\{R_n\}$ be an exhaustion of $R$. Then for $R_{n_0} \ni p_0$, there exist const.s $N_1$ and $N_2$ such that

$$N_1 G(z, p_0) \leqq U(z) \leqq N_2 G(z, p_0) \qquad \text{in } (R - R_{n_0}). \qquad (10)$$

**Irregularity of minimal points** Irregularity $\delta$ of minimal points relative to $\tilde{M}$ and $M$ top.s are defined by

$$\delta(p, \tilde{M}) = \overline{\lim_{\substack{z \to p \\ \tilde{M}}}} G(z, p_0): p \in \tilde{R} + \Delta_1(\tilde{M}), \quad \delta(q, M) = \overline{\lim_{\substack{z \to q \\ M}}} G(z, p_0): q \in \Delta_1(M).$$

Then by Theorem 8 $\delta(p, \tilde{M}) = \delta(q, M)$: $q = \varphi(p)$. Also put $u(p, \tilde{M}) = \overline{\lim_{\substack{z \to p \\ \tilde{M}}}}$

$u(z); p\in\tilde{R}+\tilde{\Delta}_1(M)$ and $u(q, M)=\overline{\lim_{\substack{z\to q\\ M}}} u(z)$: $q\in\Delta_1(M)$. Then by Theorem 8, 1) $u(p, \widetilde{M})\leqq u(q, M)$. Further $u(p, \widetilde{M})=U(q, M)$ for $p\in\tilde{R}$ and $q=\varphi(p)\in\Delta_1(M)$. In fact let $p\in\tilde{R}$ and $q\in\Delta_1(M)$. Then by Brelot's theorem on a point $p\in\tilde{R}$ there exists only one $M$-point $q$ which is minimal relative to $M$-top., i.e. $z\xrightarrow{\widetilde{M}}p(z\longrightarrow p)$ is equivalent to $z\xrightarrow{M}q$ and we have $u(p, \widetilde{M})=u(q, M)$. We remark $u(z)$ is not harmonic in $R$ but harmonic in $R-S$ and $u(z)$ is the least positive harmonic function in $R-S$ with value $u(z)$ on $S$. Hence $u(z)={}_{CG}u(z)$ for any domain $G\subset R-S$. We define $u(z)$ at $S$ by $u(z)=\overline{\lim_{\zeta\to z}} u(\zeta)$.

THEOREM 10. 1) *Let* $\{z_i\}$ *be a sequence such that* $z_i\xrightarrow{M}q\in\Delta(M)$ *with* $\underline{\lim}\, G(z_i, p_0)>0$. *Then* $f(z_i)\longrightarrow f(q)$ *(by Theorem 9)*: $f(q)\in\overline{f(R)}$ *and for any* $r$ *there exists a uniquely determined connected piece* $\omega_r(q)$ *over* $C(r, f(q))$ $=\{|w-f(q)|<r\}$ *such that* $z_i\in\omega_r(q)$ *for* $i\geqq i(r)$.

2) *Let* $z_i\xrightarrow{\widetilde{M}}p\in\Delta_1(\widetilde{M})$ *with* $\underline{\lim}\, G(z_i, p_0)>0$. *Then for any* $r>0$, *there exists a uniquely determined connected piece* $\omega_r(p)$ *over* $C(r, f(p))$ *such that* $z_i\in\omega_r(p)$ *for* $i\geqq i(r)$.

3) *Let* $w_0$ *be a point. Then*

$$\sum u(q_i)+\sum u(q_j, M)\leqq 1: \ q_i\in R,\ q_j\in\Delta_1(M),\ f(q_i)=f(q_j)=w_0\,.$$

$$\sum u(p_i)+\sum u(p_j, \widetilde{M})\leqq 1: \ p_i\in R,\ p_j\in\Delta_1(\widetilde{M}),\ f(p_i)=f(p_j)=w_0\,.$$

Proof of 1) Case 1. $f(q)\notin S^w$. We can find $r'<\min(r, \delta)$ (where $\delta$ is the number defined in Lemma 9) such that any connected piece over $C(r', f(q))$ has no common points with $S_w$. We can also suppose $z_i\in R$, $G(z_i, p_0)>\delta'>0$ and by (10) $u(z_i)\geqq\delta''$ and $|f(z_i)-f(q)|<\frac{r'}{2}$ for $i\geqq 1$. Let $\omega$ be a connected piece containing $z_i$. Then since $\omega\cap S=0$, by Lemma 2 we have

$$u(z_i)=\frac{1}{2\pi}\int_{\partial\omega} u(\zeta)\frac{\partial}{\partial n}G^{\omega}(\zeta, z_i)\,ds\,,$$

where $G^{\omega}(\zeta, z_i)$ is a Green's function of $\omega$ and $\partial\omega$ lies over $\partial C(r', f(q))$. Let $G^C(w, w')$ be a Green's function of $C(r', f(q))$. Then $G^C(f(z), f(z_i))=0$ on $\partial\omega$ and $G^C(f(z), f(z_i))\geqq G^{\omega}(z, z_i)\geqq 0$, whence

$$\frac{\partial}{\partial n}G^C(f(z), f(z_i))\geqq\frac{\partial}{\partial n}G^{\omega}(z, z_i)\geqq 0 \text{ on } \partial\omega\,. \tag{11}$$

Now there exists a const. $K$ such that

$$0 \leqq \frac{\partial}{\partial n} G^C(w, w') \leqq K \frac{\partial}{\partial n} G_C(w, f(q))$$

$$\text{on} \qquad \partial C(r', f(q)) : \ |w' - f(q)| < \frac{r'}{2} \tag{12}$$

Suppose $\omega_k (k=1, 2, , , , , , k_0)$ be a connected piece over $C(r', f(q))$ containing at least one $z_i$ of $\{z_i\}$. Then by (11), (12) and $U'(w) \leqq U(w) \leqq 1$ by (9), where $U'(w) = \sum_j u(z_j)\, z_j \in R$ and $f(z_j) = w$. Then

$$k_0 \delta'' \leqq \frac{1}{2\pi} \sum_{k=1}^{k_0} \int_{\partial \omega_k} u(\zeta) \frac{\partial}{\partial n} G^{\omega_k}(\zeta, z_i)\, ds \leqq \sum \frac{1}{2\pi} \int_{\partial \omega_k} u(\zeta) \frac{\partial}{\partial n} G^C(f(\zeta), f(z_i))\, ds$$

$$\leqq \frac{1}{2\pi} \sum \int_{\partial \omega_k} u(\zeta) K \frac{\partial}{\partial n} G^C(f(\zeta), f(q))\, ds \leqq \frac{K}{2\pi} \int_{\partial C} U'(\xi) \frac{\partial}{\partial n} G^C(\xi, f(q))\, ds \leqq K$$

and $k_0 \leqq \frac{K}{\delta''}$. Hence there exists at least one and at most a finite number of connected pieces $\omega_k$ such that $\omega_k$ contains a subsequence of $\{z_i\}$. Let $\omega$ be a connected piece containing a subsequence $\{z_i'\}$ of $\{z_i\}$. Since $r' < \delta$,

$$G^w(w, w') \leqq \log \frac{1}{|w - w'|} + M : \ w, w' \in C(r', f(q)).$$

Hence there exists a const. $L < \infty$ such that $G^w(w, w') < L$ on $\partial C(r', f(q))$ for $|w - f(q)| < \frac{r'}{2}$. Let $G(z, z_i')$ be a Green's function of $R$. Then $G(z, z_i') \leqq G^w(f(z), f(z_i')) \leqq L$ on $\partial\omega$ and $\leqq L$ in $R - \omega$ and $K(z, q) = \lim_i K(z, z_i')$ $\leqq \frac{L}{\delta'}$ in $R - \omega$ by (7). Assume there exists another connected piece $\omega'$ containing a subsequence of $z_i$. Then $K(z, q) \leqq \frac{L}{\delta'}$ in $R$ by $\omega \subset R - \omega'$. On the other hond, $K(z, q) \geqq \frac{G(z, \{z_i'\})}{M}$ and $\sup_{z \in R} K(z, q) = \infty$, where $\{z_i'\}$ is a subsequence of $\{z_i\}$ such that $G(z, z_i') \longrightarrow G(z, \{z_i'\})$. This is a contradiction. Hence there exists uniquely determined connected piece $\omega_{r'}(q)$ containing $z_i$ for $i \geqq i(r')$.

Case 2. $f(q) \in S^w$. Since $f(z)$ is univalent in $v(S)$, we can find $r'(<\delta)$ such that there exists only a connected piece $\omega^*$ and connected pieces $\{\omega_j\}$ over $C(r', f(q))$ such that $\omega^* \cap S \neq 0$, $\omega^*$ is compact in $R$ and $\omega_j \cap S = 0$ for $j = 1, 2, \cdots$. By $z_i \longrightarrow q \in \Delta(M)$, there exists a number $i_0$ such that $z_i \notin \omega^*$ for $i \geqq i_0$. Hence it is sufficient to consider only $\omega_j$. Then we have the same conclusion similarly as case 1. Now $r > r'$, there exists only one connected piece $\omega$ over $C(r, f(q))$ containing $\omega_{r'}(q)$. Clearly $\omega \ni z_i$ for $i \geqq i(r')$. Thus we have 1). We denote it by $\omega_r(q)$. We have 2) by 1) and by Theorem 8.

Proof of 3) Case 1. $w_0 \notin S^w$. In this case we can find $r'$ such that any connected piece over $C(r', w_0)$ has no common point with $S$. Let $q_j$ $(j=1, 2, \cdots)$ be points in $\bigcup_{\delta>0}((R+\Delta_1(M))\cap \bar{G}_\delta(M))$ such that $f(q_j)=w_0$. For any $q_j \in \Delta_1(M)$, there exists $\omega_{r'}(q_j)=\omega_j$ and by definition of $\omega_{r'}(q_j)$, there exists a sequence $\{z_i\}$ such that $z_i \xrightarrow{M} q_j$, $G(z_i, p_0)>\delta'>0$, $|f(z_i)-w_0|<\frac{r'}{2}$, $G^{\omega_j}(z, z_i) \longrightarrow G^{\omega_j}(z, \{z_i\})$, $u(z_i) \longrightarrow u(q_j, M)$ (clearly $>0$). Then by (11), (12) and by Lebesgue's theorem

$$0<u(q_j, M)=\frac{1}{2\pi}\int_{\partial\omega_j} u(\zeta)\frac{\partial}{\partial n} G^{\omega_j}(\zeta, \{z_i\})\, ds, \tag{13}$$

whence $G^{\omega_j}(z, \{z_i\})>0$ and $\leqq M \overset{R}{\underset{\omega_j}{I}} K(z, q_j)$ by (7). Hence $G^{\omega_j}(z, \{z_i\})$ is minimal in $\omega_{r'}(q_j)$.

Suppose $q_j \in R$, then we have at once

$$u(q_j)=\frac{1}{2\pi}\int_{\partial\omega_j} u(\zeta)\frac{\partial}{\partial n} G^{\omega_j}(\zeta, q_j)\, ds \tag{13'}$$

and $G^{\omega_j}(z, q_j)$ is minimal in $\omega_j - q_j$.

Let $\omega$ be a connected piece over $C(r', f(q))$ and let $q_k$ $(k=1, 2, \cdots)$ be a subset of $q_j$ such that $\omega_{r'}(q_k)=\omega$. Then $G^\omega(z, \{z_i\}^k)$ of $q_k$ (or $G^\omega(z, q_k)$) is minimal in $\omega - q_k$ and $\leqq G^C(f(z), w_0)$. Hence

$$\sum G^\omega(z, \{z_i\}^k)+\sum G^\omega(z, q_k)\leqq G^C(f(z), w_0) \quad \text{and}$$

$$\sum u(q_k, M)+\sum u(q_k)\leqq \frac{1}{2\pi}\int_{\partial C} U^\omega(w)\frac{\partial}{\partial n} G^C(w, w_0)\, ds,$$

where $U^\omega(w)=\sum_t u(z_t)$ and $f(z_t)=w_0$, $z_t \in \partial\omega$.

Summing up all connected pieces over $C(r', w_0)$, we have by $U'(W)\leqq U(W)\leqq 1$

$$\sum_j u(q_j, M)+\sum_i u(q_i)\leqq 1,$$

where $f(q_i)=f(q_j)=w_0$, $q_i \in R$, $q_j \in \bigcup_{\delta>0}(\Delta_1(M)\cap G_\delta(M))$.

Case 2. $w_0 \in S^w$. In this case, we use $R^*$ instead of $R$. We can find $r'$ over $C(r', w_0)$ there exist at most two connected pieces $\omega_k$ in $R^*$, which are compact in $R^*$ and $\omega_k \cap S^w \neq 0$ and there exist connected pieces $\omega_m$ in $R$ such that $\omega_m \cap S^w=0$. For $\omega_k$, $G^{\omega k}(z, z_0^k)$ is minimal ($f(z_0^k)=w_0$, $z_0^k \in S$) and (13') holds, for $\omega_m$ (13)or (13') hold. Hence

$\sum_{k=1}^{2} u(z_0^k)+\sum u(q_i)+\sum u(q_j, M)\leqq 1$. Now $u(z_0^1)+u(z_0^2)\geqq u(z_0)=\lim_{\substack{z\to z_0\\ z\in R}} u(z)$ for $z_0\in S$. Put $z_0=q_0$ (considered as a point in $R$). Then

$$\sum u(q_i)+\sum u(q_j, M)\leqq 1 .$$

The latter part is proved by Theorem 8. 1).

**Kindredness of points** Let $p_i\in\Delta_1(\widetilde{M})\cap\bar{G}_\delta(\widetilde{M})$ (or $\in\Delta(M)\cap\bar{G}_\delta(M)$). If there exists a sequence of curves $\{\Gamma_n\}$ $(n=1, 2, \cdots)$ with two endpoints $\{z_n^i\}$ $(i=1, 2)$ such that $z_n^i\xrightarrow{\widetilde{M}}p_i$ and $\inf_{z\in\Gamma_n} G(z, p_0)>\delta_1>0$ $(n=1, 2, \cdots)$ and $\Gamma_n\longrightarrow\Delta(\widetilde{M})$, we say $p_1$ and $p_2$ are chained. If $p_i$ and $p_{i+1}$ $(i=1, 2, \cdots, m-1)$ are chained, we say $p_1$ and $p_m$ are kindred. We see at once $p_1$ and $p_m$ lie on the same boundary component of $R$. Then

THEOREM 11. 1) *Let $q_j\in\Delta(M)\cap\bar{G}_\delta(M)$ $(j=1, 2)$ be kindred, then $f(q_1)=f(q_2)$ and $\omega_r(q_1)=\omega_r(q_2)$, where $\omega_r(q_j)$ is a connected piece over $C(r, f(q_j))$.*

2) *Let $p_j\in\Delta_1(\widetilde{M})\cap\bar{G}_\delta(\widetilde{M})$ be kindred. then $f(p_1)=f(p_2)$ and $\omega_r(p_1)=\omega_r(p_2)$.*

3) *Let $q_1$ and $q_2$ be two points in $\Delta(M)\cap\bar{G}_\delta(M)$ such that there exists a const. $\alpha>0$ and that $K(z, q_1)\geqq\alpha K(z, q_2)$. Then $f(q_1)=f(q_2)$ and $\omega_r(q_1)=\omega_r(q_2)$.*

4) *Let $q_1\in\Delta_0(M)\cap\bar{G}_\delta(M)$ (set of non minimal points) and $\mu$ be its canonical mass of $K(z, q)$. If $\mu$ has a positive mass $\alpha$ at $q_2\in\Delta_1(M)$, then $f(q_1)=f(q_2)$ and $\omega_r(q_1)=\omega_r(q_2)$.*

Proof of 1) Suppose $q_1$ and $q_2$ are chained. Let $\delta^*=\min(\delta, \delta_1)$. Then $f(q_i)$ exists and $\in A(\Delta(M), \delta^*)$. Assume $f(q_1)\neq f(q_2)$. Since $A(\Delta(M), \delta^*)$ is a closed set of capacity zero, we can find an analytic curve $\Gamma$ enclosing only $f(q_1)$ and $\Gamma\cap A(\Delta(M), \delta^*)=0$. Consider $f(\Gamma_n)$. Then since $f(z_n^i)\longrightarrow f(q_i), f(\Gamma_n)$ intersects $\Gamma$ at least one at $\xi_n$. Let $\eta_n$ such that $f(\eta_n)=\xi_n$ $\eta_n\in\Gamma_n$. Then $\eta_n\longrightarrow\Delta(M)$ and $G(\eta_n, p_0)\geqq\delta^*$. We can find a subsequence $\{\eta_n'\}$ of $\{\eta_n\}$ such that $f(\eta_n')\longrightarrow\xi^*$ and $\eta_n'\xrightarrow{M}\eta\in\Delta(M)\cap\bar{G}_{\delta^*}(M)$ and $f(\eta)\in A(\Delta(M), \delta^*)$. This contradicts $\xi^*\in\Gamma$. Hence $f(q_1)=f(q_2)$. Also we see $f(\Gamma_n)\longrightarrow f(q_1)=f(q_2)$. This implies $\omega_r(q_1)\cap\omega_r(q_2)\supset\Gamma_n$ and $\omega_r(q_1)=\omega_r(q_2,)$ because $\partial\omega_r(q_i)$ lies on $\partial C(r, f(q_i))$. Hence we have $f(q_1)=f(q_2)$ and $\omega_r(q_1)=\omega_r(q_2)$ for two kindred points $q_1$ and $q_2$ for any $r>0$.

Proof of 2) is evident by (1) and by Therem 8.

Proof of 3) By Theorem 10 there exist connected pieces $\omega_r(q_1)$ and $\omega_r(q_2)$. Then (see the proof of Theorem 10, 2)) sup $K(z, q_i)<\infty$ in $R-\omega_r(q_i)$: $i\neq 1, 2$. Assume $\omega_r(q_1)\cap\omega_r(q_2)=0$. Then sup $K(z, q_2)<\infty$ in $R$ by the assumption of this theorem. This is a contradiction. Hence $\omega_r(q_1)$

$=\omega_r(q_2)$ for any $r>0$, whence $f(q_1)=f(q_2)$.

Proof of 4) Let $z_i \xrightarrow{M} q_1$. Then there exists a subsequence $\{z_i'\}$ of $\{z_i\}$ such that $G(z, z_i')\longrightarrow G(z, \{z_i'\})$, whence $K(z, q_1)\leqq \dfrac{G(z, \{z_i'\})}{\delta}$. By Lemma 6 $K(z, q_1)$ is a $G.G.$ in $R$ and by Lemma 11 there exists a const. $\delta'>0$ such that $q_2\in\Delta_1(M)\cap\overline{G}_{\delta'}(M)$. Hence by the assumption we have $K(z, q_1)\geqq\alpha K(z, q_2)$: $\alpha>0$ and 4) by 3).

**Application to lacunary domain** Let $\tilde{R}$ be an end of a Riemann surface with relative boundary $\partial\tilde{R}$. Let $F_i$ $(i=1, 2, \cdots)$ be a compact connected set such that $F_i\cap F_j=0$, $F_i$ clusters nowhere in $\tilde{R}+\partial R$ and $R=\tilde{R}-F: F=\sum F_i$ is connected. Then we call $R$ a lacunary end. Let $\mathfrak{p}$ be an ideal boundary component of $\tilde{R}$. Let $\{\mathfrak{v}_n(\mathfrak{p})\}$ be a determining sequence of $\mathfrak{p}$. If there exists $\mathfrak{v}_n(\mathfrak{p})$ such that $\partial\mathfrak{v}_n(\mathfrak{p})$ is a dividing cut and $\inf_{z\in\partial\mathfrak{v}_n} G(z, p_0)>\delta>0$ $(n=1, 2, \cdots)$, we say $F$ is completely thin at $\mathfrak{p}$, where $G(z, p_0)$ is a Green's function of $R$. It is desirable to formulate the behaviour of analytic functions of bounded type in $R$ relative to $M$-top. $\tilde{M}$ over $\tilde{R}$ not to $M$-top over $R$. It is easily seen if $F$ is completely thin at $\mathfrak{p}$, $\delta(p, \tilde{M})\geqq\delta$ for $p\in\Delta_1(\tilde{M})\cap\nabla(\mathfrak{p})$ and any points in $\Delta_1(\tilde{M})\cap\nabla(\mathfrak{p})$ are chained.

THEOREM 12. *Let $w=f(z)$ be an analytic function of bounded type in a lacunary end $R$ of $\tilde{R}$.* 1) *If there exists a number $\delta>0$ such that $\Delta_1(\tilde{M})\cap\overline{G}_\delta(\tilde{M})\cap\nabla(\mathfrak{p})=\Delta_1(\tilde{M})\cap\overline{G}_{\delta'}(\tilde{M})\cap\nabla(\mathfrak{p})$ for any $\delta'\leqq\delta$, then*

$$\bigcap_{\epsilon>0}\bigcup_{n}\overline{(f(G_\epsilon\tilde{M})\cap\mathfrak{v}_n(\mathfrak{p}))}=A=\left\{w=f(p):\ p\in\Delta_1(\tilde{M})\cap\overline{G}_\delta(\tilde{M})\cap\nabla(\mathfrak{p})\right\}$$

2) *If $\bigcup_{\epsilon>0}(\Delta_1(\tilde{M})\cap\overline{G}_\epsilon(\tilde{M})\cap\nabla(\mathfrak{p}))$ consists of a finite number of points $p_i$* $(i=1, 2, \cdots, i_0)$, $\bigcup_{\epsilon>0}\bigcap_{n}\overline{f(G_\epsilon)\cap\mathfrak{v}_n(\mathfrak{p}))}=\bigcup_{i=1}^{i_0} f(p_i)$

3) *If $F$ is completely thin at $\mathfrak{p}$, then $\bigcup_{\epsilon>0}(\Delta_1(\tilde{M})\cap\overline{G}_\epsilon(\tilde{M})\cap\nabla(\mathfrak{p}))$ consists of a finite number of points $p_1, p_2, \cdots, p_{i_0}$ and*

$$\bigcup_{\epsilon>0}\bigcap_{n}(f(\overline{G}_\epsilon(\tilde{M})\cap\mathfrak{v}_n(\mathfrak{p}))=f(p_1)=f(p_2)=\cdots=f(p_{i_0}).$$

REMARK. The former part of 3) is proved under the condition that spherical area of $f(R)<\infty$ in the previous paper. Suppose the spherical area of $f(R)<\infty$. Then we can find a neighbourhood $\mathfrak{v}(\mathfrak{p})$ of $\mathfrak{p}$ such that $f(z)$ is bounded type in $\mathfrak{v}(\mathfrak{p})\cap R$. Hence 3) is an extension of the theorem in the previous one.

Proof of 1) By Theorem 8 $z_i\xrightarrow{\tilde{M}}p\in\Delta_1(\tilde{M})\cap\overline{G}_\delta(\tilde{M})$ implies $z_i\xrightarrow{M}q\in\Delta_1(M)\cap\overline{G}_\delta(M)$: $q=\varphi(p)$. By $f(z_i)\longrightarrow f(p)$ and $\longrightarrow f(q)$ we have $f(p)=f(\varphi(p))$.

Hence if $A\overset{\varphi}{\approx}A'$ we have at once $f(A)=f(A')$. For simplicity put $F_\delta(\alpha)\cap\Delta_1(\alpha)\cap\Gamma(\mathfrak{p})=F_\delta(\alpha)$: $\alpha=\widetilde{M}$ or $M$ and $\bar{G}_\delta(\alpha)\cap\Delta_1(\alpha)\cap\Gamma(\mathfrak{p})=\bar{G}_\delta(\alpha)$. By definition we have

$$\bar{G}_{\delta-\epsilon}(\alpha)\supset F_\delta(\alpha)\supset\bar{G}_\delta(\alpha)\qquad\text{for}\qquad 0<\varepsilon<\frac{\delta}{2}.$$

By $\bar{G}_\delta(\widetilde{M})\subset F_\delta(\widetilde{M})\subset\bar{G}_{\delta-\epsilon}(\widetilde{M})\subset F_{\delta-\epsilon}(\widetilde{M})\subset\bar{G}_{\delta-2\epsilon}(\widetilde{M})=\bar{G}_\delta(\widetilde{M})$

$$\bar{G}_\delta(\widetilde{M})=F_\delta(\widetilde{M})=F_{\delta-\epsilon}(\widetilde{M}). \tag{14}$$

By (14) and Theorem 8

$F_\delta(M)\approx F_\delta(\widetilde{M})=F_{\delta-\epsilon}(\widetilde{M})\approx F_{\delta-\epsilon}(M)\supset\bar{G}_{\delta-\epsilon}(M)\supset F_\delta(M)$ and

$$\bar{G}_{\delta-\epsilon}(M)=F_\delta(M),\qquad 0<\varepsilon<\frac{\delta}{2}. \tag{15}$$

By (14) and (15)

$$f(\bar{G}_\delta(\widetilde{M}))=f(F_\delta(M))=f(\bar{G}_{\delta-\epsilon}(M))=A.$$

Hence it is sufficient to study $f(z)$ relative to $M$-top not $\widetilde{M}$-top. Let $\{z_i\}$ be a sequence such that $z_i\longrightarrow\mathfrak{p}$, $G(z_i,p_0)>\varepsilon>0$, $G(z,z_i)$ converges and $f(z_i)\longrightarrow w_0$. We show $w_0\in A$. We can find a subsequence $\{z_i'\}$ of $\{z_i\}$ such that $z_i'\xrightarrow{M}q\in\Delta(M)\cap\bar{G}_\epsilon(M)\cap\Gamma(\mathfrak{p})$, $K(z,q)$ is representable by a canonical mass $\mu$ on $\Delta_1(M)\cap\Gamma(\mathfrak{p}')$, where $\mathfrak{p}'$ is the ideal boundary component of $R$ (not of $\widetilde{R}$) on which $q$ lies. Now $R$ is a lacunary end. We can find a determining sequence $\mathfrak{v}_n(\mathfrak{p})$ of $\mathfrak{p}$ such that $\partial\mathfrak{v}_n(\mathfrak{p})\cap F=0$ and $\mu=0$ except on $\mathfrak{p}$. Hence $\mu>0$ only on $\Delta_1(M)\cap\Gamma(\mathfrak{p})$. On the other hand, $K(z,q)\leqq\frac{G(z,\{z_i'\})}{\varepsilon}$ and by Lemma 6 $K(z,q)$ is a $G.G.$ in $R$. By Lemma 11 and by (15) $\mu$ is a mass on $\Delta_1(M)\cap F_\delta(M)\cap\Gamma(\mathfrak{p})=\Delta_1(M)\cap\bar{G}_{\delta'}(M)\cap\Gamma(\mathfrak{p})$ for any $\delta'<\delta$. Let $t\in\Delta_1(M)\cap\bar{G}_{\delta'}(M)$, then $K(z,t)\leqq\frac{G^w(f(z),f(t))}{\delta'}$, where $G^w(w,w')$ is a Green's function of $f(R)$ and $\delta'$ is a const. $<\delta$. Hence

$$K(z,q)\leqq\frac{1}{\delta'}\int G^w(f(z),f(t))\,d\mu(t)<\infty\ \text{ by }\int d\mu\leqq1.$$

Since the mapping $w=f(q)$ is continuous relative to $M$-top., there exists a mass $\nu$ such that

$$\int G^w(f(z),f(t))\,d\mu(t)=\int G^w(w,s)\,d\nu(s)\ \text{ and }\ \nu>0\ \text{ on }A.$$

Let $E^*\ K(z,q)$ be the lower envelope of superharmonic functions larger than $K(z,q)$ in $f(R)$. Then $E^*K(z,q)=aG^w(w,f(q))$: $a>0$. Now by

$E^*K(z, q) \leqq \frac{1}{\delta'}\int G^w(w, s)\,d\nu(s)$, $\nu$ has a point mass at $f(q)$ by Lemma 4, whence $f(q)\in A$. Hence $\bigcap_n \overline{f(G_\varepsilon)\cap \mathfrak{v}_n(\mathfrak{p})}) \subset A$ for any $\varepsilon>0$ and we have 1).

Proof of 2) Let $\delta=\min_i(\delta(p_i, \widetilde{M}))$. Then $\varDelta_1(\widetilde{M})\cap \overline{G}_\delta(\widetilde{M})\cap \nabla(\mathfrak{p})=\varDelta_1(\widetilde{M})\cap G_{\delta'}(\widetilde{M})\cap \nabla(\mathfrak{p})$ for any $\delta'<\delta$ and $A=\sum f(p_i)$. Thus we have 2).

Proof of 3) Let $p_i$ and $p_j$ in $\varDelta_1(\widetilde{M})\cap \nabla(\mathfrak{p})$. Then $\delta(p_i, \widetilde{M})\geqq\delta>0$, where $\delta$ is the number such that $G(z, p_0)\geqq\delta$ on $\partial\mathfrak{v}_n(\mathfrak{p})$ and $p_i$ and $p_j$ are chained, hence $f(p_i)=f(p_j)$ and $=f(p_1)=\cdots=f(p_{i_0})$. By (10) there exists a number $N$ such that $u(p_i, \widetilde{M})\geqq N\delta(p_i, \widetilde{M})$. Then by Theorem 10

$$\sum^{i_0}\delta(p_i, \widetilde{M})\leqq\frac{1}{N}. \quad \text{Hence } i_0\leqq\frac{1}{N\delta} \text{ and by 2) we have 3).}$$

As a consequence of 3) we have following

COROLLARLY. *Let $\widetilde{R}$ be an end of a Riemann surface $\in O_g$. If $F$ is completely thin at a boundary component $\mathfrak{p}$ of harmonic dimension$=\infty$. Then there exists no analytic function in $\widetilde{R}-F$ of bounded type in $\widetilde{R}-F$.*

We shall give some examples.

EXAMPLE 1. Let $1/2>a_1>b_1>a_2>b_2\cdots\downarrow 0$. Let $S_n^+$ and $S_n^-$ $(n=1, 2, \cdots)$ be slits as follows:

$$S_n^+=\left\{1+a_n\geqq Re\, z\geqq 1+b_n,\ Im\, z=0\right\}$$

$$S_n^-=\left\{-1-b_n\geqq Re\, z\geqq -1-a_n,\ Im\, z=0\right\}.$$

Let $\mathscr{F}_0$ be a circle $|z|<2$ with slits $\sum_1^\infty S_n^+ + \sum_1^\infty S_n^-$. We suppose $a_n, b_n$ are chosen as

1) $$\log\frac{b_n}{a_{n+1}}>\varepsilon_0>0, \quad n=1, 2, \cdots$$

2) $z=\pm 1$ are irreguar points in $\mathscr{F}_0$.

Let $\mathscr{F}_n$ be a whole $z$-plane with slits $S_n^+$ and $S_n^-$. We shall construct an end of a Riemann surface $\in O_g$. We connect $\mathscr{F}_9$ with $\mathscr{F}_n$ $(n=1, 2, \cdots)$ on $S_n^+ + S_n^-$ crosswise. Then we have an end denoted by $\widetilde{R}$ with relative boundary $\partial\widetilde{R}$ lying on $|z|=2$ on $\mathscr{F}_0$. Let $\Gamma_n^+=\{|z-1|=\sqrt{a_{n+1}b_n}\}$, $\Gamma_n^-=\{|z+1|=\sqrt{a_{n+1}b_n}\}$ on $\mathscr{F}_0$ and $D_n=\mathscr{F}_0-\{|z-1|\leqq\sqrt{a_{n+1}b_n}\}-\{|z+1|\leqq\sqrt{a_{n+1}b_n}\}$. Put $\widetilde{R}_n=D_n+\mathscr{F}_1+\cdots+\mathscr{F}_n$. Then $\widetilde{R}_n$ is an $n+1$ sheeted covering surface, $\{\widetilde{R}_n\}$ $(n=1, 2, \cdots)$ is an exhaustion of $\widetilde{R}$, $\partial\widetilde{R}_n=\partial\widetilde{R}+\Gamma_n^+ +\Gamma_n^-$, $\widetilde{R}$ has only one ideal boundary component $\mathfrak{p}$ and $\{\widetilde{R}-\widetilde{R}_n\}$ is an determining sequence of $\mathfrak{p}$. Let $F$ be a connceted closed set of positive capacity in $|z|>3$ and let $F_n$ be a set on $\mathscr{F}_n$ whose projection is $F$. Then $R=\widetilde{R}-\sum F_n$ is a lacunary end. $\tilde{R}$ and $R$ have following properties.

1) $\tilde{R}$ is an end of a Riemann surface $\in O_g$.

Let $G(z, p_0)$ be a Green's function of $R$ and put $G_\delta=\{z\in R: G(z,p_0)>\delta\}$ and $\tilde{M}$ and $M$-top.s over $\tilde{R}$ and $R$ are defined. Then

2) $\varDelta_1(\tilde{M})\cap\varGamma(\mathfrak{p})$ consists of two points $p_1$ and $p_2$ and $\delta(\tilde{M},p_i)>0$.

Let $w=f(z)=$proj. $z(z\in R)$. Then $f(z)$ is bounded type in $R$ and $f(p_i)$ exists: $\sum f(p_i)=\{z=\pm 1\}$ and $p_1$ and $p_2$ are not kindred.

3) Let $\{z_n\}$ be a sequence such that $z_n\in\mathscr{F}_n-F_n$ and proj. $|z_n-1|>\delta'>0$. Then $\lim_n G(z_n, p_0)=0$.

Proof of 1) Let $H_n^+=\{b_n>|z-1|>a_{n+1}\}$ and $H_n^-=\{b_n>|z+1|>a_{n+1}\}$ on $\mathscr{F}_0$. Then $H_n^+ + H_n^-$ separates $\mathfrak{p}$ from $\partial\tilde{R}$ and by $\operatorname{mod} H_n^+=\operatorname{mod} H_n^-$, $\sum_n \operatorname{mod} H_n^+=\infty$ and $\tilde{R}$ is a end of a Riemann surface $\in O_g$.

Proof of 2) Without loss of generality we can suppose $p_0$ lies on $z=3/2$ in $\mathscr{F}_0$. Let $G'(z,p_0)$ be a Green's function of $\mathscr{F}_0$. Put $U(z)=G'(z,p_0)$ and consider $U(z)$ in $\mathscr{F}_0$. Then $U(z)=0$ on $\sum(S_n^+ + S_n^-)$ and subharmonic in $|z|<3/2$. Let $C_n^+=\{|z-1|<\sqrt{a_{n+1}b_n}\}$ and $C_n^-=\{|z+1|<\sqrt{a_{n+1}b_n}$ on $\mathscr{F}_0$ and let $M_n=\max_{z\in C_n^{+-}} U(z)$. Then $M_n=\max_{z\in\partial C_n^{+-}} U(z)$ and $M_n\downarrow$. Assume $M_n\downarrow 0$. Then $U(z)\longrightarrow 0$ as $z\longrightarrow 1$. This means $z=1$ is regular and contradicts 2). Hence $\lim M_n=\delta>0$. By condition 1) and Harnack's theorem there exists a const. $K$ for any positive harmonic function $V(z)$ in $b_n>|z|>a_{n+1}$ such that $\max_{z\in\partial C_n^{+-}} V(z)\leqq K\min_{z\in\partial C_n^{+-}} V(z)$. Hence

$$\min_{z\in\partial C_n^+} G'(z, p_0)\geqq\frac{\delta}{K}\quad\text{similarly}\quad\min_{z\in\partial C_n^-} G'(z, p_0\geqq\frac{\delta}{K}. \tag{1}$$

By Brelot's theorem there exist only a point $q_1$ which is minimal on $z=1$ $(=-1)$ relative to Martin's top. $M'$ over $\mathscr{F}_0$ and there exists a path $\varLambda(q_1)$ $M'$-tending to $q_1$. $\varLambda(q_1)$ intersects $\partial C_n^+(n\geqq n(\varLambda, q_1))$. Hence there exists a sequence $\{z_i\}$ on $\sum_n C_n^+$ such that $z_n\xrightarrow{M'}q_1$: $K'(z,z_n)\longrightarrow K'(z,q_1)$. By (1) $\overset{\tilde{R}}{\underset{\mathscr{F}_0}{E}}K'(z,q_1)<\infty$ and there exists a point $p_1\in\varDelta_1(\tilde{M})\cap\varGamma(\mathfrak{p})$ corresponding to $q_1$. Hence $\varDelta_1(\tilde{M})\cap\varGamma(\mathfrak{p})$ consists of at least two point $p_1$ and $p_2$. Let $p\in\varDelta_1(\tilde{M})\cap\varGamma(\mathfrak{p})$. Then $\varLambda(p)$ corresponding to $p$ must intersect $\partial C_n^+ + \partial C_n^-$. Then there exists a sequence $z_i\xrightarrow{\tilde{M}}p$ and $z_n\in\partial C_n^+$ or $\in\partial C_n^-$. Now $\overset{\tilde{R}}{\underset{\mathscr{F}_0}{I}}K(z,p)\geqq\frac{\lim G'(z,z_i)}{M}>0$, where $M=\max\tilde{G}(z,p_0)$ for $|z|<1$ on $\mathscr{F}_0$ and $\tilde{G}(z,p_0)$ is a Green's function of $\tilde{R}$ and $\overset{R}{\underset{\mathscr{F}_0}{I}}K(z,p_1)=aK'(z,q_1)$ or $K'(z,q_2)$: $a>0$. Hence

$\varDelta_1(\widetilde{M})\cap \mathcal{V}(\mathfrak{p})$ consists of at most two points $p_1$ and $p_2$. Let $G(z,p_0)$ be a Green's function of $R$. Then by $G(z,p_0)\geqq G'(z,p_0)$, $\delta(\widetilde{M},p_i)\geqq\frac{\delta}{K}$. Hence any analytic function of bounded type in $R$ has limit as $z\xrightarrow{\widetilde{M}}p_i$ in $G_{\delta'}=\{z\in R: G(z,p_0)>\delta'\}$. The remaining part of 2) and 3) are the consequence of Theorem 11 and 12.

EXAMPLE 2. Let $1/2>b_0>a_1>b_1>a_2>b_2\cdots\downarrow 0$ and $S_n^+$ and $S_n^-$ be slits:

$$S_n^+=\left\{b_n\leqq Re\,z\leqq a_n,\ \mathrm{Im}\,z=0\right\},\quad S_n^-=\left\{-b_n\geqq Re\,z\geqq -a_n,\ Im\,z=0\right\}$$

Let $w(S_n^{+-},z)$ be a harmonic measure of $S_n^{+-}$ in $|z|<2$. We choose $a_n, b_n$ so that 1) and 2) may satisfied.

1) $$\log(a_n/b_{n+1})>\varepsilon>0\,,\qquad (n=1,2,\cdots)$$

2) $$\sup_{Re\,z=0} w(S_n^{+-},z)\leqq 1/2^{n+3}\,.$$

(clearly $z=0$ is an irregular point in $\{|z|<2\}-\sum S_n^{+-}$).

We shall construct an end $\widetilde{R}$ of a Riemann surface $\in O_g$ and a lacunary end $R$. Let $\mathscr{F}_0$ be a circle $|z|<2$ with slits $\sum_{n=1}^{\infty}S_n^+$.

$\mathscr{F}_n$ be the whole $z$-plane with slits $\sum_{i=n}^{\infty}S_i^+ + \sum_{i=n+1}^{\infty}S_i^-$ $\qquad (n=\text{odd})$

$\mathscr{F}_n$ be the whole $z$-plane with slits $\sum_{i=n+1}^{\infty}S_i^+ + \sum_{i=n}^{\infty}S_i^-$ $\qquad (n=\text{even})$

Connect $\mathscr{F}_0$ with $\mathscr{F}_1$ on $\sum_{n=1}^{\infty}S_n^+$ crosswise. Connect $\mathscr{F}_n$ and $\mathscr{F}_{n+1}$ on $\sum_{i=n+1}^{\infty}S_i^-$ ($n$=odd) on $\sum_{i=n+1}^{\infty}S_i^+$ ($n$=even). Then we have a Riemann surface $\widetilde{R}$ being a covering surface. Let $F_m$ $(m=1,2,\cdots)$ the part of $\mathscr{F}_m$ over $|z|>1$ and let $R=\widetilde{R}-\sum_{m=1}^{\infty}F_m$. Then $R$ is a lacunary end. Let $\Gamma_n=\{|z|=\sqrt{a_{n+1}b_n}$, $H_n=\{b_n\geqq|z|\geqq a_{n+1}\}$ $(n=0,1,2,\cdots)$. Let $\Gamma_n^m$ be a circle in $\mathscr{F}_m$ whose projection is $\Gamma_n$ and $H_n^m$ be a ring in $\mathscr{F}_m$ whose projection is $H_n$. Let

$D_n^0$ be the part of $\mathscr{F}_0$ over $\qquad 2>|z|>a_n$.

$D_n^m$ be the part of $\mathscr{F}_m$ over $\qquad \infty\geqq|z|>a_n$: $1\leqq m\leqq n-1$.

Put $\widetilde{R}_n=D_n^0+D_n^1+D_n^2+,\cdots+D_n^{n-1}$, Then $\widetilde{R}_n$ (an $n$-sheeted covering surface) has relative boundary $|z|=2$ on $\mathscr{F}_0$ and $\{|z|=a_n\}$ over $\mathscr{F}_1+\mathscr{F}_2+\cdots+\mathscr{F}_{n-1}$ and $\{\widetilde{R}_n\}$ is an exhaustion of $\widetilde{R}$, $\widetilde{R}$ has only one ideal boundary component

$\mathfrak{p}$. $\tilde{R}$ and $R$ have the following properties.

1) $R$ is an end of a Riemann surface $\in O_g$.

2) $\Delta_1(\widetilde{M}) \cap \bar{V}(\mathfrak{p})$ consists of a countably infinite number of points $p_1$, $p_2, \cdots$ with positive irregularity.

3) $p_i$ and $p_{i+1}$ are chained: $i=1, 2, \cdots$

Proof of 1) $H_n$ is a ring with module $\log(a_n/b_{n+1})$ and $\sum_{m=0}^{n} H_n^m$ separates $\partial\tilde{R}$ from $\mathfrak{p}$ and $\sum \frac{1}{n+1} \log \frac{a_n}{b_{n+1}} = \infty$. Hence $\tilde{R}$ is an end of a Riemann surface $\in O_g$. Let $S(z)$ be a positive harmonic function in $a_{n+1} < |z| < b_n$. Then by condition 1) there exists a const. $K$ such that

$$\max_{z \in \Gamma_n} S(z) \leqq K \min_{z \in \Gamma_n} S(z): \quad \Gamma_n = \left\{|z| = \sqrt{a_{n+1}b_n}\right\}.$$

Let $G(z, p_0)$ be a Green's function of $R$ with pole $p$ at $z=3/2$ in $\mathscr{F}_0$. Then there exists a const. $M$ such that $G(z, p_0) \leqq M$ in $R$ over $|z|<1$. Let $V(z)$ be a positive harmonic function in $\left\{|z| < \frac{1}{2}\right\} - \sum_{i=m}^{\infty} S_i^+ - \sum_{i=m}^{\infty} S_i^-$ such that $V(z) \geqq N$ on $|z|=1/2$. Then

$$V(z) \geqq N(1 - \sum_{m}^{\infty} w'(S_i^+, z) - \sum_{m}^{\infty} w'(S_i^-, z)), \tag{1}$$

where $w'(S_i^{+-}, z)$ is *H.M.* of $S_i^{+-}$ relative to $|z| \leqq 1/2$ and $w'(S_i^{+-}, z) \leqq w(S_i^{+-}, z)$. By $\max_{Re\, z=0} \sum_{i=m}^{\infty} (w(S_i^+, z) + w(S_i^-, z)) \leqq 1/2^{m+1}$ we have

$$V(z) \geqq N(1-1/2^{m+1}) \text{ for } Re\, z=0 \text{ and } V(z) \geqq \frac{N}{K}(1-1/2^{m+1}) \text{ on } \sum_{i=1}^{\infty} \Gamma_i \tag{2}$$

Consider $G(z, p_0)$ in $\mathscr{F}_m$ over $\{|z|<1/2\}$. Then there exists a const. $N_m$ such that $G(z, p_0) \geqq N_m$ on $|z|=1/2$. Hence by (2)

$$G(z, p_0) \geqq \frac{N_m}{K}(1-1/2^{m+1}) \text{ for } Re\, z=0 \text{ and on } \sum_{i=1}^{\infty} \Gamma_i. \tag{3}$$

Similarly we have

$$G(z, p_0) \leqq K(M/2^{m+1}) \text{ for } Re\, z=0 \text{ and on } \sum_{i=1}^{\infty} \Gamma_i. \tag{4}$$

Let $G_m$ be the part of $\mathscr{F}_m$ on $\{\sqrt{a_{n+1}b_n} < |z| < \sqrt{a_n b_{n-1}},\ -\pi/2 \leqq \arg z \leqq \pi/2\}$ $G_{m-1}$ be the part of $\mathscr{F}_{m-1}$ on $\{\sqrt{a_{n+1}b_n} < |z| < \sqrt{a_n b_{n-1}},\ -\pi/2 \leqq \arg z \leqq \pi/2\}$ Then $G_m$ and $G_{m-1}$ are connected at $S_n^+$ and $G_m + G_{m-1}$ is bounded by two boundary components $B$ on $\mathscr{F}_m$ and $B'$ on $\mathscr{F}_{m-1}$ for $n \geqq m$, where $B$ is the part of $\mathscr{F}_m$ over $(|z| = \sqrt{a_n b_{n-1}},\ -\pi/2 \leqq \arg z \leqq \pi/2) + (\sqrt{a_{n+1}b_n} < |z| < \sqrt{a_n b_{n-1}}$ $\arg z = \pi/2) + (|z| = \sqrt{a_{n+1}b_n},\ -\pi/2 \leqq \arg z \leqq \pi/2) + (\sqrt{a_{n+1}b_n} < |z| < \sqrt{a_n b_{n-1}}$ $\arg z =$

$-\pi/2)$. and $B'$ is a set on $\mathscr{F}_{m-1}$ whose projection is that of $B$. Then by (3) $G(z, p_0) \geqq \frac{N_m}{K}(1-1/2^{m+1})$ on $B$ and $\geqq \frac{N_{m-1}}{K}(1-1/2^{m+1})$ on $B'$. Hence $G(z, p_0) \geqq \frac{1}{K}(1-1/2^{m+1}) \min(N_m, N_{m-1})$ and similarly $G(z, p_0) \geqq \frac{1}{K}(1-1/2^{m+1})$ $\min(N_m, N_{m+1})$ in the part of $\mathscr{F}_m$ over $\sqrt{a_{n+1}b_n} > |z| > \sqrt{a_n b_{n-1}}$, $\pi/2 \leqq \arg z \leqq 3\pi/2$. Hence $G(z, p_0) \geqq \frac{1}{K}(1-1/2^{m+1}) \min(N_{m-1}, N_m, N_{m+1})$ in $\mathscr{F}_m$ over $|z| < \sqrt{a_m b_{m-1}}$. Now $G_m$ (for $n \leqq m$) is bounded by only one boundary component $B$ on which $G(z, p_0) \geqq \frac{N_m}{K}(1-1/2^{m+1})$. Thus

$$G(z, p_0) \geqq \frac{\min(N_{m-1}, N_m, N_{m+1}}{K}(1-1/2^{m+1}) \text{ in } \mathscr{F}_m \text{ over } |z| < 1/2. \quad (5)$$

For $m$ is even, the same result is obtained.
Similarly we have

$$G(z, p_0) \leqq \frac{KM}{2^m} \text{ in } \mathscr{F}_m \text{ over } |z| < 1. \quad (6)$$

Let $\mathscr{F}'_m = \mathscr{F}_m - F_m$, i.e. unit circle with slits $\sum_m^\infty S_i^{+-} + \sum_{m+1}^\infty S_i^{-+}$ according as $m$=odd or even. Then there exists only one point $q_m$ at $z=0$ which is minimal relative to Martin's top. over $\mathscr{F}'_m$. Let $\Lambda$ be a curve tending to $q_m$. Then $\Lambda$ intersects $\Gamma_n^m$: $n \geqq n(\Lambda)$. There exists a sequence $\{z_i\}$ on $\sum_i \Gamma_i^m$ with $K'(z, z_i) \longrightarrow K'(z, q_m)$, where $K'(z, q_m)$ is a kernel in $\mathscr{F}'_m$. Let $G'(z, p_0)$ be a Green's function of $\mathscr{F}'_m$. Then by (1) it is easily seen lim $G'(z_i, p_0) > 0$ and $\underset{\mathscr{F}'m}{\overset{\tilde{R}}{E}} K'(z, q_m) < \infty$ and there exists apoint $p_m$ in $\Delta_1(\widetilde{M}) \cap V(\mathfrak{p})$ with $\underset{\mathscr{F}'m}{\overset{\tilde{R}}{E}} K'(z, q_m) = aK(z, q_m)$. Clearly by (5) $\delta(\widetilde{M}, p_m) > 0$. By $\mathscr{F}'_m \cap \mathscr{F}'_m = 0$, $q_m \neq q_{m'}$ and $p_m \neq p_{m'}$ for $m \neq m'$. Hence there exist $p_1, p_2, \cdots$ in $\Delta_1(\widetilde{M}) \cap V(\mathfrak{p})$. Conversely let $p \in \Delta_1(\widetilde{M}) \cap V(\mathfrak{p})$ with $\delta(\widetilde{M}, p) > 0$. Then there exists a path $\Lambda$ $\widetilde{M}$-tending to $p$. By (6) there exists a number $k_0$ and an endpart $\Lambda'$ of $\Lambda$ such that $\Lambda'$ has no common points with $\mathscr{F}_k$: $k \geqq k_0$. Now $\sum_{i=1}^n \Gamma_n^i$ separates $\partial\tilde{R}$ from $\mathfrak{p}$ for any $n$ and $\Lambda$ intersects $\sum_{i=1}^{k_0} \Gamma_n^i$ for $n > n(\Lambda)$ and there exists a sequence $\{z_i\}$ and a number $m$ such that $\{z_i\} \subset \sum_{n=1}^\infty \Gamma_n^m$ and $z_i \xrightarrow{\widetilde{M}} p$. By (5) $\lim G'(z_i, p_0) > 0$, $\underset{\mathscr{F}'_m}{\overset{\tilde{R}}{I}} K(z, p) > 0$. Hence $p_m$ corresponds $q_m$. Hence there exists no point with positive irregularity except $p_1, p_2, \cdots$. Let $p_m$,

$p_{m+1} \in \Delta_1(\widetilde{M}) \cap V(\mathfrak{p})$. Then there exist sequences $\{z_i^m\}$, $\{z_i^{m+1}\}$ such that $\{z_i^m\} \subset \sum_n^\infty \Gamma_n^m$, $\{z_i^{m+1}\} \subset \sum_n^\infty \Gamma_n^{m+1}$, $z_i^m \longrightarrow p_m$, $z_i^{m+1} \longrightarrow p_{m+1}$. By (5) $p_m$ and $p_{m+1}$ are chained.

## References

[1] Z. KURAMOCHI: Analytic functions in a lacunary end of a Riemann surface. to appear in Ann. Inst. Fourier, Fas 3, tome 25 (1975).

[2] Z. KURAMOCHI: Mass distributions on the ideal boundaries of abstract Riemann surfaces, 1. Osaka Math. J., 8. 119-137 (1956).

[3] Z. KURAMOCHI: On harmonic functions representable by Poisson's integral, Osaka Math. J. 10, 103-117 (1958).

[4] CONSTANTINESCU und CORNEA: Ideale Ränder Riemannscher Flächen; Springer (1958).

[5] Z. KURAMOCHI: On the existence of functions of Evans's type, J. Fac. Sci. Hokkaido Univ. 19. 1-27 (1965).

[6] L. NAIM: Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel. Ann. Inst. Fourier. 7, 183-281 (1957)

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