# Kaehler submanifolds with $RS=0$ in a complex projective space

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P. J. Ryan [3] and T. Takahashi [4] has recently studied complex hypersurfaces in a complex space form satisfying the condition

$$R(X, Y)S=0 \tag{0.1}$$

for any vectors $X$ and $Y$ of the hypersurface, where $R$ denotes the curvature tensor, $S$ is the Ricci tensor and $R(X, Y)$ operates on the tensor algebra as a derivation. Ryan proved that these hypersurfaces are Einstein if the ambient space is not complex euclidean, which was generalized by $M$. Kon [1] in the case of Kaehler submanifolds in a complex space form of constant negative holomorphic sectional curvature. On the other hand, Takahashi also verified that such hypersurfaces are cylindrical if the ambient space is complex euclidean.

The purpose of this note is to prove the following

THEOREM. *Let $M$ be an n-dimensional Kaehler submanifold immersed in an $(n+q)$-dimensional complex projective space $PC_{n+q}$. If $M$ satisfies the condition* (0. 1) *and the codimension $q$ is less than $n-1$, then $M$ is Einstein.*

## §1. Kaehler submanifolds in $PC_{n+q}$

Let $M$ be an n-dimensional Kaehler manifold and $\iota$ an isometric and holomorphic immersion of $M$ into an $(n+q)$-dimensional complex projective space $P_{n+q}(c)$ of constant holomorphic sectional curvature $c$. We call such $\iota$ simply a *Kaehler immersion.* Throughout this note, $M$ may be identified with $\iota(M)$, because the argument is local. Let $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+q}$ be a unitary frame field in $P_{n+q}(c)$ in such a way that, restricted to $M$, $e_1, \cdots, e_n$ are tangent to $M$. Its dual coframe field $\omega^1, \cdots, \omega^n, \omega^{n+1}, \cdots, \omega^{n+q}$ consists of complex valued linear differential forms of type (1, 0) on $M$ such that

$$\omega^{\alpha}=0, \tag{1.1}$$

and $\omega^1, \cdots, \omega^n, \omega^{-1}, \cdots, \omega^{-n}$ are linearly independent. Greek indices run over the range $n+1, \cdots, n+q$. The induced Kaehler metric $g$ on $M$ is given by $g=2\sum_i \omega^i \otimes \omega^{-i}$, and $e_1, \cdots, e_n$ is a unitary frame field of $M$ and $\omega^1, \cdots, \omega^n$ is a coframe field of $e_1, \cdots, e_n$. Associated to the frame $e_1, \cdots, e_n, e_{n+1}, \cdots e_{n+q}$, there exist complex valued differential forms $\omega_B{}^A$, which are usually called connection forms on $P_{n+q}(c)$, such that

$$d\omega^A + \sum_B \omega_B{}^A \wedge \omega^B = 0, \qquad \omega_B{}^A + \bar{\omega}_A{}^B = 0, \tag{1.2}$$

$$d\omega_B{}^A + \sum_C \omega_C{}^A \wedge \omega_B{}^C = \Omega_B{}^A, \tag{1.3}$$
$$\Omega_B{}^A = \sum_{C,D} R^A{}_{BC\bar{D}}\, \omega^C \wedge \bar{\omega}^D,$$

where $\Omega_B{}^A$ denotes the curvature form and $R^A{}_{BC\bar{D}}$ denotes the curvature tensor on $P_{n+q}(c)$, which are given by

$$R^A{}_{BC\bar{D}} = \frac{c}{2}(\delta^A_B \delta_{CD} + \delta^A_C \delta_{BD}), \tag{1.4}$$

because $P_{n+q}(c)$ is of constant holomorphic sectional curvature $c$. Here the capital letters run over the range $1, \cdots, n, n+1, \cdots, n+q$.

It follows from (1. 2) and the Cartan's lemma that the exterior derivative of (1. 1) gives

$$\omega_i{}^\alpha = \sum_j h^\alpha_{ij} \omega^j, \qquad h^\alpha_{ij} = h^\alpha_{ji}, \tag{1.5}$$

where the small letters run over the range $1, \cdots, n$. Then the quadratic form $\sum_{i,j} h^\alpha_{ij} \omega^i \otimes \omega^j$ is called the second fundamental form of $M$ in the direction of $e^\alpha$. Since $\omega^\alpha = 0$ again, (1. 2) and (1. 3) become

$$d\omega^i + \sum_j \omega_j{}^i \wedge \omega^j = 0, \tag{1.6}$$

$$d\omega_j{}^i + \sum_k \omega_k{}^i \wedge \omega_j{}^k = \Omega_j{}^i, \tag{1.7}$$
$$\Omega_j{}^i = \sum_{k,l} R^i_{jk\bar{l}}\, \omega^k \wedge \bar{\omega}^l,$$

where $\omega_j{}^i$ (resp. $\Omega_j{}^i$) denotes the connection (resp. curvature) form on $M$, and $R^i{}_{jk\bar{l}}$ denotes the curvature tensor on $M$. It follows form (1. 4), (1. 5) and (1. 7) that we have the equation of Gauss

$$R_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_\alpha h^\alpha_{jk} \bar{h}^\alpha_{il}. \tag{1.8}$$

Now, with respect to these frames, the Ricci form $S$ can be expressed

$$S = \sum_{k,l} (R_{k\bar{l}}\, \omega^k \otimes \bar{\omega}^l + R_{\bar{k}l}\, \bar{\omega}^k \otimes \omega^l),$$

where the Ricci tensor $R_{k\bar{l}}$ is defined by $R_{k\bar{l}} = \sum_i R_{\bar{i}ik\bar{l}}$, and it satisfies $R_{k\bar{l}} = R_{\bar{l}k} = \bar{R}_{l\bar{k}}$. Because of (1. 8), $R_{k\bar{l}}$ is given by

$$R_{k\bar{l}} = \frac{n+1}{2} c\, \delta_{kl} - \sum_{\alpha,i} h^\alpha_{ki} \bar{h}^\alpha_{il}. \tag{1.9}$$

## §2. Proof of Theorem

In this section, let $M$ be an $n$-dimensional kaehler submanifold immersed holomorphically into $P_{n+q}(c)$. We assume that $M$ satisfies the condition (0. 1). In our notations, this condition is equivalent to

$$\sum_k R_{k\bar{j}}\Omega_i{}^k + \sum_k R_{i\bar{k}}\bar{\Omega}_j{}^k = 0.$$

Substituting (1. 7) and (1. 9) into the above equation, we have the equation

$$\begin{aligned}(2.1)\qquad & c\sum_{\alpha,r}(h^\alpha_{ir}\bar{h}^\alpha_{rl}\delta_{jk} - h^\alpha_{kr}\bar{h}^\alpha_{rj}\delta_{il}) \\ &+ 2\sum_{\alpha,\beta,r,s}(h^\beta_{ik}\bar{h}^\beta_{lr}h^\alpha_{rs}\bar{h}^\alpha_{sj} - h^\beta_{kr}\bar{h}^\alpha_{rs}h^\alpha_{si}\bar{h}^\beta_{jl}) = 0.\end{aligned}$$

Let $H^\alpha$ be an $n\times n$ matrix with its components $(h^\alpha_{ij})$. Then, for a suitable choice of the frame $e_1, \cdots, e_n$, a matrix $\sum_\alpha H^\alpha \bar{H}^\alpha$ can be orthogonalized as follows:

$$\sum_\alpha H^\alpha \bar{H}^\alpha = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Since the matrix is a positive semi-definite Hermitian one, the eigenvalues $\lambda_1, \cdots, \lambda_n$ are non-negative real valued functions on $M$. Moreover, we have

$$(2.2)\qquad \sum_{\alpha,i} h^\alpha_{ki}\bar{h}^\alpha_{il} = \lambda_k \delta_{kl}.$$

From the above equation, (2. 1) becomes

$$(2.3)\qquad c(\lambda_i - \lambda_k)\delta_{il}\delta_{jk} + 2(\lambda_j - \lambda_i)\sum_\alpha h^\alpha_{ik}\bar{h}^\alpha_{lj} = 0.$$

It follows from this equation that the equations

$$(2.4)\qquad \begin{cases} (\lambda_i - \lambda_j)\left(\sum_\alpha h^\alpha_{ij}\bar{h}^\alpha_{ij} - \dfrac{c}{2}\right) = 0, \\ (\lambda_i - \lambda_j)\sum_\alpha h^\alpha_{ik}\bar{h}^\alpha_{lj} = 0 \qquad \text{unless } i=l \text{ and } j=k. \end{cases}$$

are obtained.

We may suppose that $\lambda_1, \cdots, \lambda_p$ are all distinct eigenvalues of $\sum_\alpha H^\alpha \bar{H}^\alpha$. Let $n_1, \cdots, n_p$ be the multiplicities of $\lambda_1, \cdots, \lambda_p$ respectively, where $p$ is a function on $M$. If $p=1$ everywhere on $M$, then $M$ is exactly Einstein. Suppose that $p\geqq 2$ at a point $x$ of $M$. Then it follows from (2. 4) that

$$(2.5)\qquad \begin{cases} \sum_\alpha h^\alpha_{ij}\bar{h}^\alpha_{ij} = \dfrac{c}{2} & \text{if } \lambda_i \neq \lambda_j, \\ \sum_\alpha h^\alpha_{ik}\bar{h}^\alpha_{lj} = 0 & \text{if } \lambda_i \neq \lambda_j, \text{ and } (k, l)=(i, j) \text{ or } (j, i). \end{cases}$$

Let $h_{ij}$ be a vector in $C^q$ defined by $h_{ij}=(h_{ij}^{n+1}, \cdots, h_{ij}^{n+q})$. Consider the set $\{h_{ij}; \lambda_i \neq \lambda_j\}$ consisting of $\sum_{r<s}^{p} n_r n_s$ vectors in $C^q$. The equations (2. 5) mean that they are linearly independent. Accordingly, because of $\sum_{r=1}^{p} n_r = n$, we have

$$q \geqq \sum_{r<s}^{p} n_r n_s \geqq n-1,$$

where the second equality holds if $p=2$ and $n_1$ is equal to 1 or $n-1$. This completes the proof.

REMARK. As is well showed at Remark 3. 2 in [2], the product manifold of $P_1(c)$ and $P_{n-1}(c)$ is an $n$-dimensional Kaehler manifold and it is imbeddend into a $(2n-1)$-dimensional complex projective space $P_{2n-1}(c)$. Then $P_1(c) \times P_{n-1}(c)$ satisfies the condition (0. 1), but if $n \geqq 3$, then it is not Einstein. Thes implies that the estimate of the codimension is best possible.

REMARK. The proof in this section can be discussed in the similar manner, though the ambient space is a complex space form of constant negative holomorphic sectional curvature. In this case, the first equation of (2. 5) implies that $M$ is Einstein. This is a brief proof of Kon's result.

## Bibliography


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